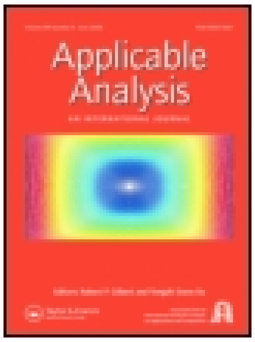

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Nguyen Hoang

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Regularity properties of viscosity solution of nonconvex Hamilton–Jacobi equations

Nguyen Hoang

Department of Mathematics, College of Education, Hue University, Hue, Vietnam

ABSTRACT
Some properties of characteristic curves in connection with the viscosity solution of the Hamilton–Jacobi equation (H, σ) defined by the Hopf formula $u(t,x) = \max_{q\in\mathbb{R}^n}\{\langle x, q\rangle - \sigma^*(q) - tH(q)\}$ are studied. We are concerned with the points at which the solution $u(t,x)$ is differentiable, and the strip of the form $\mathcal{R} = (0,\theta) \times \mathbb{R}^n$ in the domain Ω where $u(t,x)$ is of class $C^1(\mathcal{R})$.

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1. Introduction

Consider the Cauchy problem for the Hamilton–Jacobi equation (H, σ) :

$$u_t + H(D_x u) = 0\,, \ \ (t,x) \in \Omega = (0,T) \times \mathbb{R}^n, \tag{1.1}$$

$$u(0,x) = \sigma(x)\,, \ \ x \in \mathbb{R}^n. \tag{1.2}$$

It is well-known that, due to the nonlinearity in the Hamiltonian $H(p)$ in general, smooth solutions of the problem exist in a narrow neighborhood of the hyperplane $t = 0$ no matter how smooth the given data are. The studies of global solutions (i.e. the solutions defined on the whole domain Ω) of the Cauchy problem began in the decade of 1950's with the notion of Lipschitz solution. By the definition, it is a locally Lipschitz function $u(t,x)$ that satisfies Equation (1.1) almost everywhere on Ω and $u(0,x) = \sigma(x),\ x \in \mathbb{R}^n$.

In 1965, Hopf [1] established two well-known formulas for representations of Lipschitz solutions of the Hamilton–Jacobi equations (H, σ) which depend on the Fenchel conjugates H^* and σ^*, respectively. If $H = H(p)$ is convex and superlinear and σ is Lipschitz on $\mathbb{R}^n$, a solution of the equation is given by

$$u(t,x) = \min_{y\in\mathbb{R}^n} \left\{\sigma(y) + tH^*\Big(\frac{x-y}{t}\Big)\right\}. \tag{1.3}$$

In the case where $H = H(p)$ is continuous and $\sigma(x)$ is a convex and Lipschitz function,

$$u(t,x) = \max_{q\in\mathbb{R}^n}\{\langle x, q\rangle - \sigma^*(q) - tH(q)\}. \tag{1.4}$$

CONTACT Nguyen Hoang nguyenhoanghue@gmail.com, nguyenhoang@hueuni.edu.vn

These formulas are called Hopf formulas. When $n = 1$, the representation formula (1.3) was proved by Lax [2] in 1958, and so this formula is also called Hopf-Lax formula.

In general, the Lipschitz solutions of the problem are not unique, and thus one must restrict to consider the solutions in some specific classes of functions.

In 1983, Crandall and Lions in [3] first introduced the notion of viscosity solutions of Hamilton–Jacobi equations together with several existence and uniqueness theorems. By the definition, a viscosity solution is a continuous function u satisfying the differential inequalities associated with the equation. The viscosity solution plays a fundamental role in the study of Hamilton–Jacobi equations as well as their related problems such as calculus of variation, optimal control theory and differential games.

In 1984, Bardi and Evans [4] proved that the function $u(t, x)$ defined by (1.3) as well as (1.4) is a viscosity solution of the corresponding problem (H, σ).

Note that the representation formula (1.3) and its generalization for the case where $H = H(t, x, p)$ were widely studied under an essential assumption that $H(t, x, p)$ is a convex function with respect to p; see [5,6] and references therein. Indeed, one can prove that the value function of a calculus of variation problem or an optimal control problem is a viscosity solution of the associated dynamic programming equation where the Hamiltonian is convex in the gradient variable p. Many important results on the theoretical aspect as well as applications have been obtained in the literature. In particular, regularity properties and propagation of singularities of viscosity solutions in the case of convex Hamiltonians have been intensively studied; see [6–9] and the references therein.

On the other hand, in the theory of differential games, the Hamiltonians of the associated dynamic programming equations are neither convex nor concave in general (see [10,11]). Nevertheless, not many researchers were interested in this case, even for simple Hamiltonian $H = H(p)$. In [12], Bardi and Faggian explicitly presented lower and upper estimates of the form 'maxmin' and 'minmax' for the viscosity solutions where either the Hamiltonians or the initial data are not necessarily convex, but can be expressed as the sum of a convex and a concave function. Recently, Evans [11,13] establishes a general representation formula for nonconvex Cauchy problem (H, σ) by the methods of 'generalized envelopes' and 'adjoint and compensated compactness'.

This paper is devoted to studying some regularity properties of the viscosity solution $u(t, x)$ given by Hopf formula (1.4). Our method is to investigate the set of maximizers $\ell(t, x)$ in the formula (1.4) along characteristic curves. We examine the differentiability of $u(t, x)$ on the characteristic curves and define some strips of the form $\mathcal{R} = (0, \theta) \times \mathbb{R}^n \subset \Omega$ so that $u(t, x) \in C^1(\mathcal{R})$. Some results similar to the ones of Hopf-Lax formula (1.3) in [14] are obtained. This study continues our effort initiated in [15] to fill the gap between the viscosity solution and the classical solution for nonconvex Hamiltonians.

Note that in [8] the authors introduced and studied the backward and forward problems and proved that under some conditions, a viscosity solution $u = u(t, x)$ of a Cauchy problem is of class $C^1((0, T) \times \mathbb{R}^n)$ if $u(t, x)$ is both a backward and forward solution of the problem.

The structure of the paper is as follows. In Section 2, we present some necessary notions and properties of the Hopf formula and viscosity solutions. In Section 3, we suggest a classification of characteristic curves at each point of the domain and then study the differentiability properties of the Hopf formula $u(t, x)$ on these curves. In the last section, we establish various conditions based on the characteristic curves so that $u(t, x)$ defined by (1.4) is continuously differentiable on the strip of the form $(0, t_0) \times \mathbb{R}^n$. Several illustrative examples are also given.

We use the following notations. Let T be a positive number, $\Omega = (0, T) \times \mathbb{R}^n$; $|\,.\,|$ and $\langle .,. \rangle$ be the Euclidean norm and the scalar product in $\mathbb{R}^n$, respectively, and let $B'(x_0, r)$ be the closed ball centered at x_0 with radius r. For a function $u(t, x)$ defined on Ω, we denote $D_x u = u_x = (u_{x_1}, \ldots, u_{x_n})$ and $Du = (u_t, D_x u)$.

2. The Hopf formula and viscosity solution

We now consider the Cauchy problem for the Hamilton–Jacobi equation:

$$u_t + H(D_x u) = 0\,,\ (t,x) \in \Omega = (0,T) \times \mathbb{R}^n,\ T > 0, \tag{2.1}$$
$$u(0,x) = \sigma(x)\,,\ x \in \mathbb{R}^n, \tag{2.2}$$

where the Hamiltonian $H(p)$ is a continuous function and $\sigma(x)$ is a convex function on $\mathbb{R}^n$.

Let σ^* be the Fenchel conjugate of σ. We denote by

$$D = \operatorname{dom}\sigma^* = \{y \in \mathbb{R}^n \mid \sigma^*(y) < +\infty\}$$

the effective domain of the convex function σ^*.

We assume a compatible condition for $H(p)$ and $\sigma(x)$ as follows:

(Hf1): *For every* $(t_0, x_0) \in [0,T) \times \mathbb{R}^n$, *there exist positive constants* r *and* N *such that*

$$\langle x, p\rangle - \sigma^*(p) - tH(p) < \max_{|q|\le N}\{\langle x, q\rangle - \sigma^*(q) - tH(q)\}$$

whenever $(t,x) \in [0,T) \times \mathbb{R}^n$, $|t - t_0| + |x - x_0| < r$ *and* $|p| > N$.

Let

$$\varphi(t,x,q) = \langle x, q\rangle - \sigma^*(q) - tH(q),\ (t,x) \in \Omega,\ q \in \mathbb{R}^n. \tag{2.3}$$

For each $(t,x) \in \Omega$, denote

$$\ell(t,x) = \{q \in \mathbb{R}^n \mid \varphi(t,x,q) = \max_{p\in\mathbb{R}^n} \varphi(t,x,p)\}. \tag{2.4}$$

Remark 2.1: In virtue of (Hf1) $\ell(t,x) \neq \emptyset$, for all $(t,x) \in \Omega$. Moreover, the multi-valued function

$$\Omega \ni (t,x) \mapsto \ell(t,x) \subset \mathbb{R}^n$$

is upper semi-continuous on Ω; see [16].

First, we briefly recall definitions of Fréchet semidifferentials of a function and viscosity solution.

Definition 2.2: Let $u = u(t,x) : \ \Omega \to \mathbb{R}$ be a function and let $(t_0, x_0) \in \Omega$. For $(h,k), (p,q) \in \mathbb{R} \times \mathbb{R}^n$, we denote

$$\tau(p,q,h,k) = \frac{u(t_0 + h, x_0 + k) - u(t_0, x_0) - ph - \langle q, k\rangle}{\sqrt{|h|^2 + |k|^2}},$$
$$D^+u(t_0,x_0) = \{(p,q) \in \mathbb{R}^{n+1} \mid \limsup_{(h,k)\to(0,0)} \tau(p,q,h,k) \le 0\}$$
$$D^-u(t_0,x_0) = \{(p,q) \in \mathbb{R}^{n+1} \mid \liminf_{(h,k)\to(0,0)} \tau(p,q,h,k) \ge 0\},$$

where $p \in \mathbb{R},\ q \in \mathbb{R}^n$.

Then $D^+u(t_0,x_0)$ (resp. $D^-u(t_0,x_0)$) is called the *superdifferential* (resp. *subdifferential*) of $u(t,x)$ at (t_0,x_0).

Definition 2.3: A continuous function $u : [0,T) \times \mathbb{R}^n \to \mathbb{R}$ is called a *viscosity subsolution* (resp. *viscosity supersolution*) of the Cauchy problem (2.1)–(2.2) on $\Omega = (0,T) \times \mathbb{R}^n$, provided that the following hold:

(i) $u(0,x) = \sigma(x)$ for all $x \in \mathbb{R}^n$;

(ii) For each $(t_0, x_0) \in \Omega$ and $(p, q) \in D^+u(t_0, x_0)$, one has

$$p + H(q) \leq 0,$$

(resp. for each $(t_0, x_0) \in \Omega$ and $(p, q) \in D^-u(t_0, x_0)$, one has

$$p + H(q) \geq 0).$$

A continuous function $u : [0, T) \times \mathbb{R}^n \to \mathbb{R}$ is called a *viscosity solution* of the problem (2.1)–(2.2) if it is a viscosity sub- and supersolution of the problem.

Note that, there are several propositions which are equivalent to this definition, e.g. the notion of C^1-test function is used instead of semidifferentials; see [3].

From now on, the *Hopf formula* for the problem (2.1)–(2.2) is the function defined by

$$u(t, x) = \max_{q \in \mathbb{R}^n} \{\langle x, q \rangle - \sigma^*(q) - tH(q)\}. \tag{2.5}$$

We collect here some properties of the Hopf formula $u(t, x)$ for further presentation.

Theorem 2.4: *Assume* (Hf1). *Then we have the following:*

(a) *$u(t, x)$ is a convex function on Ω and it is a Lipschitz solution of the problem* (2.1)–(2.2).
(b) *$u(t, x)$ is a viscosity solution of the problem* (2.1)–(2.2).
(c) *$u(t, x)$ is differentiable at $(t, x) \in \Omega$ if and only if, the set $\ell(t, x)$ defined by* (2.4) *is a singleton. Then $(u_t(t, x), u_x(t, x)) = (-H(q), q)$, $\{q\} = \ell(t, x)$. Moreover, the function $u(t, x)$ is differentiable at (t_0, x_0) in both variables if and only if, $v(x) = u(t_0, x)$ is differentiable at x_0. Consequently, $u(t, x)$ is continuously differentiable in an open set $\mathcal{V} \subset \Omega$ if $\ell(t, x)$ is a singleton for all $(t, x) \in \mathcal{V}$.*

***Proof*:** For the proof of (a) see [1,16]. There are several ways to prove (b). The reader can find the first proof in [4]; see also [17,18].

For the proof of (c), first note that, if $v(x) = u(t_0, x)$ is differentiable at x_0 then $\ell(t_0, x_0) = \{q\}$ is a singleton (see [19, p.112]). Conversely, if $\ell(t_0, x_0)$ is a singleton, say $\{q\}$, then all partial derivatives of $u(t, x)$ at (t_0, x_0) exist and $u_x(t_0, x_0) = q$, $u_t(t_0, x_0) = -H(q)$. Since the function $u(t, x)$ is convex, then it is differentiable at this point. Besides, if it is differentiable on $\mathcal{V}$ then it is continuously differentiable on this open set by a property of convex functions. □

Definition 2.5: We call a point $(t_0, x_0) \in \Omega$ *regular* for $u(t, x)$ if the function is differentiable at this point. If $u(t, x)$ is not differentiable at a point $(t_1, x_1) \in \Omega$, then this point is said to be *singular* for the function.

Consequently, by Theorem 2.4, we see that $(t_0, x_0) \in \Omega$ is regular for the Hopf formula $u(t, x)$ if and only if $\ell(t_0, x_0)$ is a singleton.

We conclude the section by introducing the notion of semiconcavity and uniform convexity that will be used later.

Definition 2.6: Let $\mathcal{O}$ be a convex subset of $\mathbb{R}^m$ and let $v : \mathcal{O} \to \mathbb{R}$ be a continuous function.

(a) We say that the function v is *semiconcave* with linear modulus if there exists a constant $C \geq 0$ such that

$$\lambda v(x) + (1 - \lambda)v(y) - v(\lambda x + (1 - \lambda)y) \leq \lambda(1 - \lambda)\frac{C}{2}|x - y|^2$$

for any x, y in $\mathcal{O}$ and for any $\lambda \in [0, 1]$. The number C is called a *semiconcavity constant* of v. Alternatively, the continuous function $w : \mathcal{O} \to \mathbb{R}$ is called *semiconvex* if the function $v = -w$ is semiconcave.

(b) We call the function v *uniformly convex* with constant $\Lambda > 0$ if the function $v_1(x) = v(x) - \frac{\Lambda}{2}|x|^2$, $x \in \mathcal{O}$ is a convex function.

Remark 2.7:

(i) The semiconcavity of a function was first studied to solve the problem of uniqueness of Lipschitz solution of Hamilton–Jacobi equations. A comprehensive presentation of the theory of semiconcave functions can be found in the interesting monograph [6].

(ii) In article [20], the authors presented and studied the notion of σ-smoothness (resp. ρ-convexity) of a function. When considering a special case for σ (resp. ρ), one obtains the notion of semiconcavity (resp. uniform convexity). The following proposition is extracted from Proposition 2.6 and its corollaries from above-mentioned article. See also [14].

Proposition 2.8: *Given a function $v : \mathbb{R}^m \to \mathbb{R}$. Then we have*

(i) *If v is a uniformly convex function with a constant $\Lambda > 0$, then the Fenchel conjugate function v^* is a semiconcave function with a semiconcavity constant $\frac{1}{\Lambda} > 0$.*

(ii) *If v is a semiconcave function with a semiconcavity constant $C^* > 0$, then v^* is a uniformly convex function with a constant $\frac{1}{C^*}$.*

3. A classification of characteristics

In this section, we focus on the study of the relationship between the Hopf formula and characteristics. To this aim, let us recall the Cauchy method of characteristics for the problem (2.1)–(2.2). See [6] for example.

First, by the routine, we assume that $H(p)$ and $\sigma(x)$ are of class $C^2(\mathbb{R}^n)$.

The system of characteristic differential equations of the problem (2.1)–(2.2) is as follows

$$\dot{x} = H_p\,; \qquad \dot{v} = \langle H_p, p\rangle - H\,; \qquad \dot{p} = 0, \tag{3.1}$$

with initial conditions

$$x(0) = y\,; \qquad v(0) = \sigma(y)\,; \qquad p(0) = \sigma_y(y)\,, \quad y \in \mathbb{R}^n. \tag{3.2}$$

Then a characteristic strip of the Cauchy problem (2.1)–(2.2) (i.e. a solution of the system of differential equations (3.1)–(3.2)) is defined by

$$\begin{cases} x = x(t,y) = y + tH_p(\sigma_y(y)), \\ v = v(t,y) = \sigma(y) + t\{\langle H_p(\tau, \sigma_y(y)), \sigma_y(y)\rangle - H(\sigma_y(y))\}, \\ p = p(t,y) \ = \ \sigma_y(y). \end{cases} \tag{3.3}$$

The first component of solution (3.3) is called a characteristic curve (briefly, characteristics) emanating from y, i.e. the straight line defined by

$$\mathcal{C} :\ x = x(t,y) = y + tH_p(\sigma_y(y)),\ t \in [0,T]. \tag{3.4}$$

Let $t_0 \in (0,T]$. If for any $t \in (0,t_0)$ such that $x(t,\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism, then $u(t,x) = v(t, x^{-1}(t,x))$ is a C^2 solution of the problem on the region $(0,t_0) \times \mathbb{R}^n$.

From now on, we make an additional assumption on H and σ.

(Hf2): *Assume that H and σ are functions of class $C^1(\mathbb{R}^n)$.*

Note that, in this case, the characteristic strip (3.3) is also defined.

Let $(t_0, x_0) \in \Omega$. Denote by $\ell^*(t_0, x_0)$ the set of all $y \in \mathbb{R}^n$ such that there is a characteristic curve emanating from y and passing the point (t_0, x_0). We have $\ell(t_0, x_0) \subset \sigma_y(\ell^*(t_0, x_0))$; see [15]. Therefore $\ell^*(t_0, x_0) \neq \emptyset$ since $\ell(t_0, x_0) \neq \emptyset$ by (Hf1).

Proposition 3.1: *Let $(t_0, x_0) \in \Omega$. Then a characteristic curve passing (t_0, x_0) has the form*

$$x = x(t, y) = x_0 + (t - t_0)H_p(\sigma_y(y)), \; t \in [0, T] \tag{3.5}$$

for some $y \in \ell^*(t_0, x_0)$.

***Proof*:** Let $\mathcal{C}$: $x = x(t, y) = y + tH_p(\sigma_y(y))$ be a characteristic curve passing (t_0, x_0). By the definition, $y \in \ell^*(t_0, x_0)$. Then we have

$$x_0 = y + t_0 H_p(\sigma_y(y)).$$

Therefore,

$$x = x_0 - t_0 H_p(\sigma_y(y)) + tH_p(\sigma_y(y)) = x_0 + (t - t_0)H_p(\sigma_y(y)).$$

Conversely, let $\mathcal{C}_1 : x = x(t, y) = x_0 + (t - t_0)H_p(\sigma_y(y))$ for $y \in \ell^*(t_0, x_0)$ be some curve passing (t_0, x_0). Then we can rewrite $\mathcal{C}_1$ as:

$$x = x_0 - t_0 H_p(\sigma_y(y)) + tH_p(\sigma_y(y)) = x_0 + (t - t_0)H_p(\sigma_y(y)). \tag{3.6}$$

On the other hand, let $\mathcal{C}_2$:

$$x = y + tH_p(\sigma_y(y)) \tag{3.7}$$

be a characteristic curve also passing (t_0, x_0). Besides that, both $\mathcal{C}_1$, $\mathcal{C}_2$ are integral curves of the ODE $x' = H_p(\sigma_y(y))$, thus they must coincide. This proves the proposition. □

Remark 3.2: Suppose that $\sigma_y(y) = p_0 \in \ell(t_0, x_0)$ for some $(t_0, x_0) \in \Omega$ then y belongs to the subgradient of the convex function σ^* at $p_0 : y \in \partial\sigma^*(p_0)$. Moreover, from (3.6) and (3.7), we have $y = x_0 - t_0 H_p(p_0)$.

Now, let $\mathcal{C}$ be a characteristic curve passing a point (t_0, x_0). Then $\mathcal{C}$ can be written as

$$x = x(t, y) = x_0 + (t - t_0)H_p(\sigma_y(y)), \; t \in [0, T].$$

We say that the characteristic curve $\mathcal{C}$ is of the type (I) at the point $(t_0, x_0) \in \Omega$, if $\sigma_y(y) = p_0 \in \ell(t_0, x_0)$. If $\sigma_y(y) \in \sigma_y(\ell^*(t_0, x_0)) \setminus \ell(t_0, x_0)$ then $\mathcal{C}$ is said of the type (II) at the point (t_0, x_0).

Note that, in [14] we also got a similar classification of characteristic curves at a point (t_0, x_0) based on their initial points.

The following lemma is helpful in studying Fenchel conjugate function of a C^1- convex function.

Lemma 3.3 (see [14]): *Let v be a convex function and $D = \text{dom } v \subset \mathbb{R}^n$. Suppose that there exist p, $p_0 \in D$, $p \neq p_0$ and $y \in \partial v(p_0)$ such that*

$$\langle y, p - p_0 \rangle = v(p) - v(p_0).$$

Then for all z in the straight line segment $[p, p_0]$, we have

$$v(z) = \langle y, z \rangle - \langle y, p_0 \rangle + v(p_0).$$

Moreover, $y \in \partial v(z)$ for all $z \in [p, p_0]$.

***Proof*:** For the convenience of the reader, we reproduce the proof here. Take $z = \lambda p + (1-\lambda)p_0 \in [p, p_0],\ \lambda \in [0,1]$. Then we have

$$v(z) \le \lambda v(p) + (1-\lambda)v(p_0) = \lambda(v(p) - v(p_0)) + v(p_0).$$

From the hypotheses, we have

$$\begin{aligned} v(z) &\le \lambda\langle y, p - p_0\rangle + v(p_0) \\ &\le \langle y, \lambda p + (1-\lambda)p_0 - p_0\rangle + v(p_0). \end{aligned}$$

On the other hand, since $y \in \partial v(p_0)$, then

$$\langle y, \lambda p + (1-\lambda)p_0 - p_0\rangle \le v(z) - v(p_0).$$

Thus

$$v(z) = \langle y, z\rangle - \langle y, p_0\rangle + v(p_0).$$

Next, let $z \in [p, p_0]$. For any $x \in D$, we have

$$\begin{aligned} v(x) - v(z) =&\ v(x) - \langle y, z\rangle + \langle y, p_0\rangle - v(p_0) \\ &= v(x) - v(p_0) - \langle y, z - p_0\rangle \\ &\ge \langle x - p_0, y\rangle - \langle z - p_0, y\rangle \\ &\ge \langle x - z, y\rangle. \end{aligned}$$

This gives us that $y \in \partial v(z)$. □

Now we present some properties of characteristic curves of the type (I) at (t_0, x_0) given by the following theorem.

Theorem 3.4: *Assume* (Hf1), (Hf2). *Let* $(t_0, x_0) \in \Omega = (0, T) \times \mathbb{R}^n,\ p_0 = \sigma_y(y_0) \in \ell(t_0, x_0)$ *and let*

$$\mathcal{C} : x = x(t) = x_0 + (t - t_0)H_p(p_0), (t, x) \in \Omega \tag{3.8}$$

be a characteristic curve of the type (I) at (t_0, x_0). *Then we have the following:*

(i) $p_0 \in \ell(t, x)$ *for all* $(t, x) \in \mathcal{C},\ 0 \le t \le t_0$. *Moreover,* $\ell(t, x) \subset \ell(t_0, x_0)$.
(ii) *The set* $\ell(t, x) = \{p_0\}$ *for all* $(t, x) \in \mathcal{C},\ 0 \le t < t_0$.

As a consequence, if the characteristic curve $\mathcal{C} : x = x(t)$ is of the type (I) at (t_0, x_0) then it is of the type (I) at any point $(t_1, x(t_1)),\ t_1 \le t_0$ and the Hopf formula is differentiable on a piece of the curve $\mathcal{C}$ corresponding to $t \in [0, t_0)$.

***Proof*:** Take an arbitrary $p \in \mathbb{R}^n$ and denote by

$$\eta(t, p) = \varphi(t, x, p) - \varphi(t, x, p_0),\ (t, x) \in \mathcal{C},\ t \in [0, t_0],$$

where $\varphi(t, x, p) = \langle x, p\rangle - \sigma^*(p) - tH(p)$. Then

$$\eta(t, p) = \langle x(t), p - p_0\rangle - (\sigma^*(p) - \sigma^*(p_0)) - t(H(p) - H(p_0)) \tag{3.9}$$

for $(t, x) \in \mathcal{C}$.

First, we will check that $\eta(t, p) \le 0$ for all $t \in [0, t_0]$.

It is obvious that, $\eta(t_0, p) \le 0$. On the other hand, from (3.9) and Remark 3.2, we have

$$\eta(0,p) = \langle y_0, p - p_0 \rangle - (\sigma^*(p) - \sigma^*(p_0)),$$

where $y_0 \in \partial\sigma^*(p_0)$. By a property of subgradient of convex functions, we have

$$\eta(0,p) = \langle y_0, p - p_0 \rangle - (\sigma^*(p) - \sigma^*(p_0)) \le 0. \tag{3.10}$$

As a result, we have $\eta(0,p) \le 0$ and $\eta(t_0,p) \le 0$, for any $p \in \mathbb{R}^n$.

Since $x = x(t) = x_0 + (t - t_0)H_p(p_0)$, then from (3.9), we also have

$$\eta'_t(t,p) = \langle H_p(p_0), p - p_0 \rangle - (H(p) - H(p_0)) = c \ \text{(const)}, \ \forall t \in [0, t_0]. \tag{3.11}$$

Now we start to prove (i). Fix $(t_1, x_1) \in \mathcal{C}$ where $0 \le t_1 \le t_0$ and $x_1 = x(t_1)$. For any $p \in \mathbb{R}^n$, as shown above, we get

+ If $\eta'_t(t,p) = c > 0$ then $\eta(t_1,p) < \eta(t_0,p) \le 0$.

+ If $\eta'_t(t,p) = c \le 0$ then $\eta(t_1,p) \le \eta(0,p) \le 0$.

Thus we obtain that for all $p \in \mathbb{R}^n, \varphi(t_1,x_1,p) \le \varphi(t_1,x_1,p_0)$. Consequently, $p_0 \in \ell(t_1,x_1)$ for any $(t_1,x_1) \in \mathcal{C},\ t_1 \in [0,t_0]$.

Next, we check that $\ell(t,x) \subset \ell(t_0,x_0), t \in [0,t_0]$. To this end, take $p \in \mathbb{R}^n \setminus \ell(t_0,x_0)$. If $\eta'_t(t,p) = c \ge 0$, we have

$$\eta(t,p) \le \eta(t_0,p) < 0,$$

and if $\eta'_t(t,p) = c < 0$, then

$$\eta(t,p) < \eta(0,p) = \langle y, p - p_0 \rangle - (\sigma^*(p) - \sigma^*(p_0)) \le 0,\ t \in [0,t_0).$$

Therefore, in any case, $\eta(t,p) < 0$. This means that $p \notin \ell(t,x)$ and the inclusion $\ell(t,x) \subset \ell(t_0,x_0)$ has been checked.

The proof of (i) is then complete.

The next step is to prove (ii). Let $(t_1,x_1) \in \mathcal{C}$ where $t_1 \in [0,t_0)$. Take $p \in \ell(t_1,x_1)$. Then we have

$$\eta(t_1,p) = \varphi(t_1,x_1,p) - \varphi(t_1,x_1,p_0) = 0. \tag{3.12}$$

As in (3.11), we have $\eta'_t(t,p) = c\ \text{(const)}, \forall t \in [0,t_0]$.

If $c > 0$ then $\eta(t_1,p) < \eta(t_0,p) \le 0$ and if $c < 0$ then $\eta(t_1,p) < \eta(0,p) \le 0$. These yield a contradiction to the equality (3.12).

Now we consider the case $\eta'_t(t,p) = 0$, or

$$\langle H_p(p_0), p - p_0 \rangle - (H(p) - H(p_0)) = 0,\ \forall t \in [0,t_0]. \tag{3.13}$$

From the equality (3.12), we have

$$\langle x_0, p - p_0 \rangle - (\sigma^*(p) - \sigma^*(p_0)) = t_0(H(p) - H(p_0)). \tag{3.14}$$

Subtracting both sides of (3.14) by $\langle t_0 H_p(p_0), p - p_0 \rangle$, and noticing that $y_0 = x_0 - t_0 H_p(p_0)$, we get

$$\langle y_0, p - p_0 \rangle - (\sigma^*(p) - \sigma^*(p_0)) = (H(p) - H(p_0)) - \langle H_p(p_0), p - p_0 \rangle. \tag{3.15}$$

Thus

$$\langle y_0, p - p_0 \rangle - (\sigma^*(p) - \sigma^*(p_0)) = 0.$$

As mentioned before, since $p_0 = \sigma_y(y_0)$, then $y_0 \in \partial\sigma^*(p_0)$. If $p \neq p_0$ we see that the straight line segment $[p, p_0]$ is contained in $\mathcal{D} = \{z \in \text{dom}\sigma^* \mid \partial\sigma^*(z) \neq \emptyset\}$. Applying Lemma 3.3, we see that the function σ^* is not strictly convex on the set $[p, p_0]$. This is a contradiction, since $\sigma(x)$ is of class $C^1(\mathbb{R}^n)$, then σ^* is essentially strictly convex on $D = \text{dom}\sigma^*$. In particular, σ^* is a strictly convex function on $[p, p_0]$ (see [21, Theorem 26.3]). Thus $p = p_0$ and consequently, $\ell(t, x) = \{p_0\}$ for all $(t, x) \in \mathcal{C},\ 0 \leq t < t_0$. □

The reachable gradient can be considered as the intermediate notion between the gradient and the sub/superdifferential of a locally Lipschitz function; see, e.g. [6]. It is useful to study the differentiability of a function. We use Theorem 3.4 to establish a relationship between $\ell(t_0, x_0)$ and the set of reachable gradients.

Let us recall the set $D^*v(t_0, x_0)$ of *reachable gradients* of a function $v(t, x)$ defined on Ω at (t_0, x_0) as follows:

Given $(p, q) \in \mathbb{R}^{n+1}$. We say that $(p, q) \in D^*v(t_0, x_0)$ if and only if there exists a sequence $(t_k, x_k)_k \subset \Omega \setminus \{(t_0, x_0)\}$ such that $v(t, x)$ is differentiable at (t_k, x_k) and

$$(t_k, x_k) \to (t_0, x_0),\ (v_t(t_k, x_k), D_x v(t_k, x_k)) \to (p, q) \ \text{ as } k \to \infty.$$

If $v(t, x)$ is a locally Lipschitz function, then $D^*v(t, x) \neq \emptyset$, and it is a compact set ([6, p.54]).

Now let $u(t, x)$ be the Hopf formula and let $(t_0, x_0) \in \Omega$. We denote by

$$\mathcal{H}(t_0, x_0) = \{(-H(q), q) \mid q \in \ell(t_0, x_0)\}. \tag{3.16}$$

Then a relationship between $D^*u(t_0, x_0)$ and the set $\ell(t_0, x_0)$ is given by the following theorem.

Theorem 3.5 (cf. [14, p.273]): *Assume* (Hf1), (Hf2). *Let $u(t, x)$ be the Hopf formula for the problem* (2.1)–(2.2). *Then for all $(t_0, x_0) \in \Omega$, we have*

$$D^*u(t_0, x_0) = \mathcal{H}(t_0, x_0).$$

Proof: Let (p_0, q_0) be an element of $\mathcal{H}(t_0, x_0)$, then $p_0 = -H(q_0)$ for some $q_0 \in \ell(t_0, x_0)$. Let $\mathcal{C} : x = x(t)$ be the characteristic curve of the type (I) at (t_0, x_0) defined as in Theorem 3.4. By this theorem, all points $(t, x) \in \mathcal{C},\ t \in [0, t_0)$ are regular. Put $t_k = t_0 - 1/k,\ x_k = x(t_k)$ then $\mathcal{C} \ni (t_k, x_k) \to (t_0, x_0)$. By Theorem 2.4, (c) one has $(u_t(t_k, x_k), D_x u(t_k, x_k)) = (-H(q_0), q_0)$. Thus,

$$(-H(q_0), q_0) = \lim_{k\to\infty} (u_t(t_k, x_k), D_x u(t_k, x_k)) \in D^*u(t_0, x_0)$$

and therefore, $\mathcal{H}(t_0, x_0) \subset D^*u(t_0, x_0)$.

On the other hand, let $(p, q) \in D^*u(t_0, x_0)$ and $(t_k, x_k)_k \subset \Omega \setminus \{(t_0, x_0)\}$ such that $u(t, x)$ is differentiable at (t_k, x_k) and

$$(t_k, x_k) \to (t_0, x_0),\ (u_t(t_k, x_k), D_x u(t_k, x_k)) \to (p, q) \ \text{ as } k \to \infty.$$

Since $(u_t(t_k, x_k), D_x u(t_k, x_k)) = (-H(q_k), q_k)$ for $q_k \in \ell(t_k, x_k)$, and the multi-valued function $\ell(t, x)$ is u.s.c, then letting $k \to \infty$, we see that $q \in \ell(t_0, x_0)$ and $p = \lim_{k\to\infty} -H(q_k) = -H(q)$. Thus $(p, q) \in \mathcal{H}(t_0, x_0)$.

The proof of Theorem 3.5 is then complete. □

Remark 3.6: A general result for the correspondence between $D^*u(t, x)$ and the set of minimizers of a problem of calculus of variation $(CV)_{t,x}$ is established for convex Hamiltonian $H(t, x, p)$ in p in [6, Theorem 6.4.9, p.167].

4. Existence of a strip of differentiability of the Hopf formula

Let $v(t,x)$ be a continuous function on $\Omega = (0,T) \times \mathbb{R}^n,\ T > 0$. Suppose that there exists $t_0 \in (0,T)$ such that $v \in C^1((0,t_0) \times \mathbb{R}^n)$.

Denote $\theta = \sup\{t \in (0,T) \mid v \in C^1((0,t) \times \mathbb{R}^n)\}$. Then $\mathcal{R} = (0,\theta) \times \mathbb{R}^n$ is the largest strip in $(0,T) \times \mathbb{R}^n$ on which the function $v(t,x)$ is continuously differentiable. We call $\mathcal{R}$ the *strip of differentiability* of the function $v(t,x)$.

First, we present a result on the existence of strips of the form $\mathcal{R}_* = (0,t_*) \times \mathbb{R}^n \subset \Omega$ such that the viscosity solution $u(t,x)$ defined by the Hopf formula is continuously differentiable on $\mathcal{R}_*$.

Theorem 4.1: *Assume* (Hf1). *Suppose that the Hamiltonian $H = H(p)$ is a semiconvex function with a semiconvexity constant $\gamma > 0$. In addition, let σ be a semiconcave function with a semiconcavity constant $\mu^{-1} > 0$. Then there exists $t_* \in (0,T)$ such that for all $t_0 \in (0,t_*)$, the function $v(x) = u(t_0,x)$ is semiconcave, where $u(t,x)$ is the Hopf formula defined by* (2.5).

***Proof*:** We follow an argument in the proof of Theorem 3.5.3 (iv) [6] with a suitable adjustment to this case. See also [14].

By the assumption and Proposition 2.8, we first note that the Fenchel conjugate function σ^* is a uniformly convex function with constant $\mu > 0$. By the definition, the function $\sigma^*(p) - \frac{\mu}{2}|p|^2$ is convex. Then, for all $a, b \in \mathbb{R}^n$ we obtain

$$\sigma^*(a) + \sigma^*(b) - 2\sigma^*\left(\frac{a+b}{2}\right) \geq \frac{\mu}{2}(|a|^2 + |b|^2 - 2|\frac{a+b}{2}|^2) = \frac{\mu}{4}|a-b|^2. \tag{4.1}$$

Now, take $t_* \in (0,T)$ such that $0 < \gamma t_* \leq \frac{\mu}{2}$. Let $t_0 \in (0,t_*), x, y \in \mathbb{R}^n$, pick out $p \in \ell(t_0,x), q \in \ell(t_0,y)$. Using the inequality (4.1) and the Cauchy inequality of the form $2\langle x-y, p-q\rangle \leq \frac{\mu}{2}|p-q|^2 + \frac{2}{\mu}|x-y|^2$, we have

$$\begin{aligned}
&u(t_0,x) + u(t_0,y) - 2u(t_0, \frac{x+y}{2}) \\
&\quad \leq \langle x,p\rangle - \sigma^*(p) - t_0H(p) + \langle y,q\rangle - \sigma^*(q) \\
&\qquad - t_0H(q) - 2\Big(\langle \frac{x+y}{2}, \frac{p+q}{2}\rangle - \sigma^*(\frac{p+q}{2}) - t_0H(\frac{p+q}{2})\Big) \\
&\quad \leq 2\Big(\sigma^*(\frac{p+q}{2}) - \frac{\sigma^*(p)+\sigma^*(q)}{2}\Big) + \frac{1}{2}\langle x-y, p-q\rangle + 2t_0(H(\frac{p+q}{2}) - \frac{H(p)+H(q)}{2}) \\
&\quad \leq -\frac{\mu}{4}|p-q|^2 + \frac{1}{4}(2\langle x-y,p-q\rangle) + 2t_0(\frac{\gamma}{8}|p-q|^2) \\
&\quad \leq -\frac{\mu}{4}|p-q|^2 + t_0(\frac{\gamma}{4}|p-q|^2 + \frac{1}{4}(\frac{\mu}{2}|p-q|^2 + \frac{2}{\mu}|x-y|^2) \\
&\quad \leq \frac{1}{4}(\gamma t_0 - \frac{\mu}{2})|p-q|^2 + \frac{1}{2\mu}|x-y|^2.
\end{aligned}$$

Since $\gamma t_0 - \frac{\mu}{2} < 0$, we get

$$u(t_0,x) + u(t_0,y) - 2u(t_0,\frac{x+y}{2}) \leq \frac{1}{2\mu}|x-y|^2, x, y \in \mathbb{R}^n.$$

Therefore, the function $v(x) = u(t_0,x)$ is a semiconcave function. □

Corollary 4.2: *Suppose that all assumptions of Theorem 4.1 hold. Then the function $u(t,x)$ defined by the Hopf formula is of class $C^1((0,t_*) \times \mathbb{R}^n)$, where $0 < \gamma t_* \leq \frac{\mu}{2}$.*

***Proof*:** Let $(t_0,x_0) \in (0,t_*) \times \mathbb{R}^n$. By Theorem 4.1, the function $v(x) = u(t_0,x)$ is semiconcave on $\mathbb{R}^n$. Moreover, $v(x)$ is also a convex function. By Theorem 3.3.7 [6], the function $v(x) = u(t_0,x)$ is of class

$C^1(\mathbb{R}^n)$. Thus, $\ell(t_0, x_0)$ is a singleton and then $u(t,x)$ as a function of two variables is differentiable at (t_0, x_0). Besides, $u(t,x)$ is a convex function, therefore $u(t,x)$ is of class $C^1((0,t_*) \times \mathbb{R}^n)$. □

Corollary 4.3: *Assume* (Hf1). *In addition, suppose that H and σ belong to the class $C^{1,1}(\mathbb{R}^n)$. Then the strip of differentiability of the function $u(t,x)$ defined by the Hopf formula $u(t,x)$ is ever nonempty.*

***Proof*:** Since $H, \sigma \in C^{1,1}(\mathbb{R}^n)$, then H and σ are both semiconvex and semiconcave functions with linear modulus (see [6, Proposition 2.1.2]). By Theorem 4.1 and Corollary 4.2, the function $u(t,x)$ is of class $C^1((0,t_1) \times \mathbb{R}^n)$ for some $t_1 \in (0,T)$. By the definition, the strip of differentiability of $u(t,x)$ is not empty. □

Further we need an additional assumption for the problem (2.1)–(2.2) as follows.

(Hf3): *Either $\sigma(x)$ is Lipschitz or $H_p(p)$ is bounded on $\mathbb{R}^n$.*

Being inspired by the proof of Lemma 6.5.1 [6], we can obtain the following lemma which is useful in studying the regularity of the Hopf formula.

Lemma 4.4: *Assume* (Hf1), (Hf2) *and* (Hf3). *Let $(t_0, x_0) \in [0,T) \times \mathbb{R}^n$. Moreover, suppose that there exists $t_* \in (t_0, T)$ such that $\ell(t_*, y) = \{p(y)\}$ is a singleton, for all $y \in \mathbb{R}^n$. Then there exist $x_* \in \mathbb{R}^n$ and a characteristic curve $\mathcal{C}$ of the type (I) at (t_*, x_*) : $x = x_* + (t - t_*)H_p(p(x_*))$, that goes through (t_0, x_0).*

***Proof*:** Following Remark 2.1, the multi-valued function $y \mapsto \ell(t_*, y)$ is upper semi-continuous. By the assumption, $\ell(t_*, y) = \{p(y)\}$, thus the single-valued function $y \mapsto p(y)$ is continuous on $\mathbb{R}^n$.

For all $y \in \mathbb{R}^n$, let

$$\Lambda(y) = x_0 - (t_0 - t_*)H_p(p(y)),$$

then the function Λ is also continuous on $\mathbb{R}^n$.

First, suppose that $\sigma(x)$ is Lipschitz on $\mathbb{R}^n$, then $D = \operatorname{dom} \sigma^*$ is bounded. Hence, $D \subset B'(0, M)$ for some positive number M. Let $N = (t_* - t_0) \sup_{|p| \le M} |H_p(p)|$.

Note that, if $y \in B'(x_0, N)$ then

$$|\Lambda(y) - x_0| \le (t_* - t_0)|H_p(p(y))| \le N.$$

Therefore, Λ is a continuous function from the closed ball $B'(x_0, N)$ into itself. By Brouwer theorem, Λ has a fixed point $x_* \in B'(x_0, N)$, i.e. $\Lambda(x_*) = x_*$, hence,

$$x_0 = x_* + (t_0 - t_*)H_p(p(x_*)).$$

In other words, there exists a characteristic curve $\mathcal{C}$ of the type (I) at (t_*, x_*) described as in Proposition 3.1 that passes (t_0, x_0).

Next, if $\sup_{p \in \mathbb{R}^n} |H_p(p)| < \infty$ then we take $N = (t_* - t_0) \sup_{p \in \mathbb{R}^n} |H_p(p)|$ and argue as above. The lemma is then proved. □

Remark 4.5: By the Cauchy method of characteristics and by the assumptions that H and σ are of class $C^2(\mathbb{R}^n)$, the unique C^2-solution $u(t,x)$ of the problem (2.1)–(2.2) exists in a narrow neighborhood of the hyperplane $t = 0$ where characteristic curves do not meet. Nevertheless, if $u(t,x)$ given by the Hopf formula is differentiable in some open set containing $(t_0, x_0) \in \Omega$, then several characteristic curves may cross at (t_0, x_0) as in the following example.

Consider the following problem

$$u_t - \left(1 + |u_x|^2\right)^{\frac{1}{2}} = 0, \; t > 0, \; x \in \mathbb{R},$$

$$u(0,x) = \frac{x^2}{2}, \; x \in \mathbb{R}.$$

The Hopf formula of this problem is

$$u(t,x) = \max_{y\in\mathbb{R}}\{xy - \frac{y^2}{2} + t(1+y^2)^{\frac{1}{2}}\}.$$

By a computation, we recognize that $\ell(t,x)$ is a singleton for all points in the region $\mathcal{R}^* = ((0,+\infty)\times\mathbb{R}) \setminus \{(t,0)|\ t \geq 1\}$. Thus, the solution $u(t,x)$ is continuously differentiable in this region. Using the method of characteristics, we see that when $t > 1$, the characteristic curves intersect. Concretely, two curves of the form $x(t,y) = y - \dfrac{ty}{\sqrt{1+y^2}}$ starting from $y_0 = 1$ and $y_1 = 2$ meet each other at the point $\big(\frac{\sqrt{10}}{2\sqrt{2}-\sqrt{5}}, \frac{2(\sqrt{2}-\sqrt{5})}{2\sqrt{2}-\sqrt{5}}\big) \in \mathcal{R}^*$, but the differentiability of the solution $u(t,x)$ is also preserved in some neighborhood of this point.

However, if the Hopf formula $u(t,x)$ is differentiable on a whole strip of the form $(0,t_0)\times\mathbb{R}^n$ then the situation is different. More specific, we have the following theorem as a necessary condition.

Theorem 4.6: *Assume* (Hf1), (Hf2) *and* (Hf3). *Suppose that $u(t,x)$ is differentiable on the strip $\mathcal{R} = (0,T_0)\times\mathbb{R}^n$, $T_0 \leq T$. Then at any point $(t_0,x_0) \in \mathcal{R}$, there are no characteristic curves crossing each other.*

Proof: On the contrary, suppose that two distinct characteristic curves $\mathcal{C}_i : x = x_i(t) = y_i + tH_p(\sigma_y(y_i))$, $i = 1,2$, $y_1 \neq y_2$ meet at (t_0,x_0). If both $\mathcal{C}_i, i = 1,2$ are of the type (I) at (t_0,x_0), then $\{p_1,p_2\} \subset \ell(t_0,x_0)$, where $p_1 = \sigma_y(y_1) \neq \sigma_y(y_2) = p_2$. By Theorem 2.4, the function $u(t,x)$ is not differentiable at (t_0,x_0). This contradicts to the hypothesis. Therefore, at least one $\mathcal{C}_i, i = 1,2$, say, $\mathcal{C}_1$ is of the type (II) at (t_0,x_0). Let

$$t_+ = \inf\{t \in [0,t_0] \mid \mathcal{C}_1 \text{ is of the type (II) at } (t,x_1(t)\}.$$

Consider the point (t_+,x_+) where $x_+ = x_1(t_+)$. Since $u(t,x)$ is differentiable at (t_0,x), $x \in \mathbb{R}^n$, then $\ell(t_0,x)$ is a singleton for all $x \in \mathbb{R}^n$. Applying Lemma 4.4, there exists a point $(t_0,x_*) \in \mathcal{R}$ and a characteristic curve $\mathcal{C}' : x = x_* + (t-t_0)H_p(p(x_*))$ of the type (I) at (t_0,x_*) that passes (t_+,x_+).

We first note that $0 < t_+ \leq t_0$. Indeed, if $t_+ = 0$ then $\mathcal{C}_1 = \mathcal{C}'$ since there is a unique characteristic curve starting at $(0,y_1)$. Then $(t_0,x_0) = (t_0,x_*)$. This is a contradiction by the type of $\mathcal{C}_1$ and $\mathcal{C}'$ at (t_0,x_0). Next, we consider the following cases:

(i) If $H_p(\sigma_y(y_1)) = H_p(p(x_*))$ then $\mathcal{C}_1 = \mathcal{C}'$; i.e. $x_0 = x_*$. This means that the characteristic curve $\mathcal{C}_1$ is of the type (I) at (t_0,x_0), and we get a contradiction.

(ii) If $H_p(\sigma_y(y_1)) \neq H_p(p(x_*))$ (i.e. $\mathcal{C}_1 \neq \mathcal{C}'$), then $\mathcal{C}_1$ is of the type (I) at all points $(t,x_1(t))$, $0 < t < t_+$.

Let $x_n = x_1(t_n)$ where $t_n = t_+ - \frac{1}{n}$, then $p(x_n) = \sigma_y(y_1)$. Since $u(t,x)$ is differentiable at (t_n,x_n), thus

$$(u_t(t_n,x),u_x(t_n,x_n)) = (-H_p(p(x_n)),p(x_n)) = (-H_p(\sigma_y(y_1)),\sigma_y(y_1))$$

for $n \in \mathbb{N}$. Letting $n \to \infty$, we see that $(-H_p(\sigma_y(y_1)),\sigma_y(y_1)) \in D^*u(t_+,x_+)$. On the other hand, $(-H_p(p(x_*)),p(x_*)) \in D^*u(t_+,x_+)$. Since $\sigma_y(y_1) \neq p(x_*)$ thus $D^*u(t_+,x_+)$ is not a singleton. It follows that $u(t,x)$ is not differentiable at (t_+,x_+), which also contradicts the hypothesis of the theorem.

The proof Theorem 4.6 is now complete. □

Next, we present some sufficient conditions so that there exists a strip of the form $(0,t_*)\times\mathbb{R}^n$ on which the function $u(t,x)$ is differentiable. Note that the similar results were previously established in [14] for the Hopf-Lax formula (1.3).

It is known that in the Cauchy problem (H, σ), the characteristic curves bring the initial data with them to construct a solution at any point of the domain. The next two theorems show that, if the initial datum is of class C^1 and this smoothness is preserved at some terminal time $t = t_*$, then the solution $u(t,x)$ defined by the Hopf formula is of class C^1 in the strip $(0, t_*] \times \mathbb{R}^n$. The first result is concerned with noncrossing characteristics condition, i.e. $\ell^*(t_*, x)$ is a singleton.

Theorem 4.7: *Assume* (Hf1), (Hf2). *Let $u(t,x)$ be the viscosity solution of the problem* (2.1)–(2.2) *defined by the Hopf formula* (2.5). *Suppose that there exists $t_* \in (0,T)$ such that at any point (t_*, x), $x \in \mathbb{R}^n$ there is at most a characteristic curve passing through, (i.e. the mapping: $\mathbb{R}^n \ni y \mapsto x(t_*, y) = y + t_* H_p(\sigma_y(y))$ is injective). Then $u(t,x)$ is continuously differentiable in the strip* $(0, t_*] \times \mathbb{R}^n$.

***Proof*:** First, by the assumption, the set $\ell^*(t_*, x)$ is a singleton for all $x \in \mathbb{R}^n$. Moreover, $\ell(t_*, x) \subset \sigma_y(\ell^*(t_*, x))$, $x \in \mathbb{R}^n$, then $\ell(t_*, x)$ is also a singleton.

Next, take an arbitrary point $(t_0, x_0) \in (0, t_*) \times \mathbb{R}^n$. We check that $\ell(t_0, x_0)$ is a singleton. Let $\mathcal{C}$:

$$x = x_0 + (t - t_0) H_p(p_0)$$

where $p_0 = \sigma_y(y_0) \in \ell(t_0, x_0)$, be a characteristic curve going through (t_0, x_0) that is defined as in Proposition 3.1.

Let (t_*, x_*) be the intersection point of the straight line $\mathcal{C}$ and plane $P^{t_*} : t = t_*$. Since $\ell(t_*, x_*) \neq \emptyset$, and by the assumption, $\mathcal{C}$ is the unique characteristic curve that starts at $(0, y_0)$ and passes through (t_*, x_*). Therefore, $\mathcal{C}$ can be rewritten as follows:

$$x = x_* + (t - t_*) H_p(p_*),$$

where $p_* \in \ell(t_*, x_*)$.

On the other hand, $\ell(t_*, x_*) \subset \sigma_y(\ell^*(t_*, x_*))$ and $\ell^*(t_*, x_*) = \{y_*\}$ is a singleton, so is $\ell(t_*, x_*)$. Consequently, $\mathcal{C}$ is of the type (I) at (t_*, x_*) and $\ell(t,x) = \{p_*\}$ for all $(t,x) \in \mathcal{C}$, $t < t_*$; particularly $\ell(t_0, x_0) = \{p_*\} = \{p_0\}$ by Theorem 3.4.

Thus, $\ell(t,x)$ is a singleton for all $(t,x) \in (0, t_*] \times \mathbb{R}^n$. Following Theorem 2.4, we obtain that $u(t,x)$ is of class $C^1((0, t_*] \times \mathbb{R}^n)$. □

The next theorem concerns with the single-valuedness of the set of maximizers $\ell(t_*, x)$ while $\ell^*(t_*, x)$ may not be a singleton.

Theorem 4.8: *Assume* (Hf1), (Hf2) *and* (Hf3). *If $\ell(t_*, x)$ is a singleton for every point of the plane $P^{t_*} = \{(t_*, x) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n\}$, for some $t_* \in (0,T)$, then the function $u(t,x)$ defined by the Hopf formula* (2.5) *is continuously differentiable in the strip* $(0, t_*] \times \mathbb{R}^n$.

***Proof*:** Let $(t_0, x_0) \in (0, t_*] \times \mathbb{R}^n$. By Lemma 4.4, there exists a characteristic curve $\mathcal{C}$ of the type (I) at (t_*, x_*) passing (t_0, x_0). Since $\ell(t_*, x_*)$ is a singleton, so is $\ell(t_0, x_0)$ by Theorem 3.4. Applying Theorem 2.4, we see that $u(t,x)$ is continuously differentiable in $(0, t_*] \times \mathbb{R}^n$. □

As a direct consequence of the above theorem, we have the following:

Corollary 4.9: *Assume* (Hf1), (Hf2) *and* (Hf3). *If $g(x) = u(T,x)$ is of class $C^1(\mathbb{R}^n)$ where $u(t,x)$ defined by the Hopf formula* (2.5), *then $u(t,x)$ is of class $C^1((0,T] \times \mathbb{R}^n)$, i.e. $u(t,x)$ is a classical solution of the problem* (2.1)–(2.2).

We note that the key hypotheses of above theorems are equivalent to the fact that, there is a unique characteristic curve of the type (I) at a regular point (t_*, x_*), $x_* \in \mathbb{R}^n$, $t_* > t_0$ for $u(t,x)$ that goes through the point (t_0, x_0). This makes the point (t_0, x_0) regular.

In general, suppose that $u(t,x)$ is differentiable at $(t_0, x_0) \in (0, t_*) \times \mathbb{R}^n$ and $\mathcal{C}_0$ is a unique characteristic curve of the type (I) at (t_0, x_0) that cuts P^{t_*} at (t_*, x_*). Then there may exist other characteristic curves of the type (I) or (II) at the point (t_*, x_*), that is $\ell^*(t_*, x_*)$ need not be a singleton. Even neither is $\ell(t_*, x_*)$; see Remark 4.5. In other words, the function $v(x) = u(t_*, x)$

may not be differentiable on $\mathbb{R}^n$. Nevertheless, when considering the type of characteristic curves, we have:

Theorem 4.10: *Assume* (Hf1), (Hf2). *Let $u(t,x)$ be the viscosity solution of the problem* (2.1)–(2.2) *defined by the Hopf formula* (2.5) *and let $t_* \in (0,T)$. Suppose that all characteristic curves passing (t_*,x), $x \in \mathbb{R}^n$ are of the type (I) at this point. Then $u(t,x)$ is continuously differentiable in the open strip $(0,t_*) \times \mathbb{R}^n$.*

Proof: We argue similarly to the proof of Theorem 4.7. Let $(t_0,x_0) \in (0,t_*) \times \mathbb{R}^n$ and let $\mathcal{C}$:

$$x = x_0 + (t - t_0)H_p(p_0)$$

where $p_0 = \sigma_y(y_0) \in \ell(t_0,x_0)$ be any characteristic curve of the type (I) at the point (t_0,x_0).

Let (t_*,x_*) be the intersection point of $\mathcal{C}$ and plane P^{t_*} : $t = t_*$. Then we have

$$x_* = x_0 + (t_* - t_0)H_p(p_0).$$

Therefore, we can rewrite $\mathcal{C}$ as

$$x = x(t) = x_* - (t_* - t_0)H_p(p_0) + (t - t_0)H_p(p_0) = x_* + (t - t_*)H_p(p_0).$$

Thus $\mathcal{C}$ is also a characteristic curve that passes through the point (t_*,x_*). By the assumption, $\mathcal{C}$ is of the type (I) at this point, so each $\ell(t,x)$, $(t,x) \in \mathcal{C}$, $0 \le t < t_*$ is a singleton by Theorem 3.4. In particular, $\ell(t_0,x_0)$ is a singleton. Applying Theorem 2.4 again, we come to the conclusion of the theorem. □

Example: Let

$$u_t - \ln(1 + u_x^2) = 0,\ t \in (0,2),\ x \in \mathbb{R},$$
$$u(0,x) = \begin{cases} \frac{x^2}{2}, & |x| \le 1 \\ x \operatorname{sgn} x - \frac{1}{2}, & |x| > 1 \end{cases}.$$

The viscosity solution of this problem defined by the Hopf formula is

$$u(t,x) = \max_{|y|\le 1} \{xy - \frac{y^2}{2} + t\ln(1 + y^2)\}.$$

Let $\varphi(t,x,y) = xy - \frac{y^2}{2} + t\ln(1+y^2)$, then $\varphi_y(t,x,y) = x - y + \frac{2ty}{1+y^2}$.

A simple computation shows that at point $(t_0,x_0) = (2,\frac{2}{5})$, we have $\varphi_y(2,\frac{2}{5},y) = 0 \Leftrightarrow y_1 = 2$; $y_2 = \frac{-4+\sqrt{11}}{5}$, $y_3 = \frac{-4-\sqrt{11}}{5}$ and the function $\varphi(t_0,x_0,y)$ attains its maximum at $y_1 = 2$.

There are three characteristic curves that go through the point $(2,\frac{2}{5})$ as follows:

$\mathcal{C}_1 : x = 2 - \frac{4t}{5}$, starting at y=2 and

$\mathcal{C}_i = y_i - \frac{2y_it}{1+y_i^2}$, $i = 2, 3$, starting at $y_2 = \frac{-4+\sqrt{11}}{5}$, $y_3 = \frac{-4-\sqrt{11}}{5}$.

We see that $\mathcal{C}_1$ is the characteristic curve of the type (I) at $(2,\frac{2}{5})$ and $\mathcal{C}_2$, $\mathcal{C}_3$ are the characteristic curves of the type (II) at this point since $\ell(2,\frac{2}{5}) = \{\sigma'(y_1)\} = \{2\}$ and $\sigma_y(y_i) \notin \ell(2,\frac{2}{5})$, $i = 2, 3$. Note that, $(2,\frac{2}{5})$ is a regular point of $u(t,x)$.

Now let $(t_1,x_1) = (t_1,0)$ and let the characteristic curve $\mathcal{C}_1'$ starting from $y \in \mathbb{R}$ go through $(t_1,0)$. Then y is a solution of the equation $y - \frac{2t_1y}{1+y^2} = 0$.

If $0 \le t_1 \le \frac{1}{2}$ then $(t_1,0)$ is a regular point of $u(t,x)$ and $\mathcal{C}_1'$: $x = 0$ is of the type (I) at $(t_1,0)$.

If $t_1 > \frac{1}{2}$ then $(t_1, 0)$ is singular, since $\ell(t_1, 0) = \{y_2, y_3\}$, where $y_2 = \sqrt{2t_1 - 1}$, $y_3 = -\sqrt{2t_1 - 1}$. In this case, the characteristic curves C_2' and C_3' starting at y_2 and y_3 are of the type (I), and C_1' is of the type (II) at $(t_1, 0)$.

Let $t_* = \frac{1}{2}$. We have $\varphi(\frac{1}{2}, x, y) = xy - \frac{y^2}{2} + \frac{1}{2}\ln(1 + y^2)$, then $\varphi_y'(\frac{1}{2}, x, y) = x - y + \frac{y}{1+y^2}$ and $\varphi_y''(\frac{1}{\sqrt{2}}, x, y) = -y^2 \frac{3+y^2}{(1+y^2)^2} < 0$, $y \neq 0$. Therefore, $\ell(\frac{1}{2}, x)$ is a singleton for all $x \in \mathbb{R}$. Applying Theorem 4.7, we see that the solution $u(t, x)$ is continuously differentiable on the strip $(0, \frac{1}{2}) \times \mathbb{R}^n$.

At last, the segment $x = 0$; $t \in (\frac{1}{2}, 2]$ is a set of singular points for $u(t, x)$. So the singularities of $u(t, x)$ propagate to the boundary.

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