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SOME DIFFERENTIAL PROPERTIES OF A HOPF-TYPE FORMULA FOR HAMILTON - JACOBI EQUATIONS

Nguyen Hoang

Department of Mathematics, College of Education
Hue University, 32 Le Loi, Hue, Vietnam
e-mail: nguyenhoanghue@gmail.com or: nguyenhoang@hueuni.edu.vn

Abstract

A Hopf-type formula of the Cauchy problem for Hamilton - Jacobi equations (H, σ) is defined by $u(t,x) = \max_{q\in\mathbb{R}^n} \{\langle x, q\rangle - \sigma^*(q) - \int_0^t H(\tau, q)d\tau\}$. We investigate the points on the domain Ω where the function $u(t,x)$ is differentiable, and the strip of the form $(0, t_0) \times \mathbb{R}^n$ of Ω where the function $u(t,x)$ is continuously differentiable. Moreover, we present a simple propagation of singularity in forward of $u(t,x)$.

1 Introduction

Consider the Cauchy problem for Hamilton - Jacobi equation (H, σ)

$$\frac{\partial u}{\partial t} + H(t, D_x u) = 0\,,\ (t,x) \in \Omega = (0,T) \times \mathbb{R}^n, \tag{1.1}$$

$$u(0,x) = \sigma(x)\,,\ x \in \mathbb{R}^n. \tag{1.2}$$

If the Hamiltonian $H = H(p)$ is convex and superlinear, σ is Lipschitz on $\mathbb{R}^n$, then the function

$$u(t,x) = \min_{y\in\mathbb{R}^n} \Big\{\sigma(y) + tH^*\big(\frac{x-y}{t}\big)\Big\}, \tag{1.3}$$

is called the Hopf-Lax formula for the problem (H, σ).

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If $H = H(p)$ is only a continuous function, $\sigma(x)$ is a convex and Lipschitz function, then the Hopf formula of the problem (H, σ) is

$$u(t,x) = \max_{q\in\mathbb{R}^n}\{\langle x, q\rangle - \sigma^*(q) - tH(q)\}, \tag{1.4}$$

see [1, 4, 5]. Here * denotes the Fenchel conjugate.

It is well-known that both formulas (1.3) and (1.4) are Lipschitz solutions as well as viscosity solutions of the problem (H, σ) where $H = H(p)$ under the corresponding assumptions stated as above, see [1, 2, 4].

If $H = H(t, p)$ is continuous and σ is convex, then a generalization of formula (1.4) called Hopf-type formula is

$$u(t,x) = \max_{q\in\mathbb{R}^n}\{\langle x, q\rangle - \sigma^*(q) - \int_0^t H(\tau, q)d\tau\}. \tag{1.5}$$

Ones prove that $u(t, x)$ is a locally Lipschitz continuous function satisfying the initial condition (1.2) in $\mathbb{R}^n$, and equation (1.1) at almost all points in the domain Ω, i.e. a Lipschitz solution, but in general, it is not a viscosity solution, see [5, 10]. Recently, in [7] we prove that the formula (1.5) defines a viscosity solution of the problem for a specific class of Hamiltonians $H = H(t, p)$.

In this paper we first analyze properties of characteristics of the Cauchy problem in connection with formula (1.5) where $H = H(t, p)$. We introduce a classification of characteristic curves at each point of the domain and then study differential properties of Hopf-type formula $u(t, x)$ on these curves. Next, we present various conditions based on the characteristics so that $u(t, x)$ defined by (1.5) is continuously differentiable on the strip $(0, t_0) \times \mathbb{R}^n$. Finally, we show that the singularities of the solution $u(t, x)$ may propagate forward from t-time t_0 to the boundary of the domain.

This paper can be considered as a continuation of [6] to the case where dimension of state variable n is greater than 1, see also [8]. Our method is to exploit the relationship between Hopf-type formula and characteristics where the role of the set of maximizers is essential.

We use the following notations. For a positive number T, denote $\Omega = (0, T) \times \mathbb{R}^n$. Let $|\,.\,|$ and $\langle ., . \rangle$ be the Euclidean norm and the scalar product in $\mathbb{R}^n$, respectively. For a function $u : \Omega \to \mathbb{R}$, we denote by $D_x u$ the gradient of u with respect to variable x, i.e., $D_x u = (u_{x_1}, \ldots, u_{x_n})$, and let $B'(x_0, r)$ be the closed ball centered at x_0 with radius r.

2 The differentiability of Hopf-type formula and Characteristics

We now consider the Cauchy problem for Hamilton - Jacobi equation of the form:

$$\frac{\partial u}{\partial t} + H(t, D_x u) = 0\,,\ (t,x) \in \Omega = (0,T) \times \mathbb{R}^n, \tag{2.1}$$

$$u(0,x) = \sigma(x)\,,\ x \in \mathbb{R}^n, \tag{2.2}$$

where the Hamiltonian $H(t,p)$ is of class $C([0,T] \times \mathbb{R}^n)$, and $\sigma(x) \in C(\mathbb{R}^n)$ is a convex function.

Let σ^* be the Fenchel conjugate of σ, i.e., $\sigma^*(y) = \max_{x \in \mathbb{R}^n}\{\langle x, y\rangle - \sigma(x)\}$. We denote by $D = \operatorname{dom} \sigma^* = \{y \in \mathbb{R}^n \mid \sigma^*(y) < +\infty\}$ the effective domain of the convex function σ^*.

In [10] we assumed a compatible condition for $H(t,p)$ and $\sigma(x)$ as follows.

(A1): *For every $(t_0, x_0) \in [0,T) \times \mathbb{R}^n$, there exist positive constants r and N such that*

$$\langle x, p\rangle - \sigma^*(p) - \int_0^t H(\tau, p)d\tau < \max_{|q| \le N}\{\langle x, q\rangle - \sigma^*(q) - \int_0^t H(\tau, q)d\tau\},$$

whenever $(t,x) \in [0,T) \times \mathbb{R}^n$, $|t - t_0| + |x - x_0| < r$ and $|p| > N$.

From now on, we denote

$$u(t,x) = \max_{q \in \mathbb{R}^n}\{\langle x, q\rangle - \sigma^*(q) - \int_0^t H(\tau, q)d\tau\}. \tag{2.3}$$

and

$$\varphi(t,x,q) = \langle x, q\rangle - \sigma^*(q) - \int_0^t H(\tau, q)d\tau,\ (t,x) \in \Omega,\ q \in \mathbb{R}^n. \tag{2.4}$$

For each $(t,x) \in \Omega$, let $\ell(t,x)$ be the set of all $p \in \mathbb{R}^n$ at which the maximum of the function $\varphi(t,x,\cdot)$ is attained. In virtue of (A1), $\ell(t,x) \neq \emptyset$.

Remark. If $\sigma(x)$ is convex and Lipschitz on $\mathbb{R}^n$ then $\operatorname{dom} \sigma^*$ is bounded, hence condition (A1) is clearly satisfied. Thus (A1) can be considered as a generalization of the hypotheses used earlier, see [1, 4].

The following theorem is necessary for further presentation.

Theorem 2.1. *[10] Assume (A1). Then the function $u(t,x)$ defined by (2.3) is a locally Lipschitz function satisfying equation (2.1) a.e. in Ω and $u(0,x) = \sigma(x),\ x \in \mathbb{R}^n$. Furthermore, $u(t,x)$ is of class $C^1(V)$ in some open $V \subset \Omega$ if and only if, for every $(t,x) \in V$, $\ell(t,x)$ is a singleton.*

Remark 2.2. If $\ell(t_0, x_0) = \{p\}$ is a singleton, then all partial derivatives of $u(t, x)$ at (t_0, x_0) exist and $u_x(t_0, x_0) = p,\ u_t(t_0, x_0) = -H(t_0, p)$ see ([11], p. 112). Moreover, we have:

Theorem 2.3. *Assume (A1). Let $(t_0, x_0) \in \Omega$ such that $\ell(t_0, x_0)$ is a singleton. Then the function $u(t, x)$ defined by (2.3) is differentiable at (t_0, x_0).*

Proof. By assumption, $\ell(t_0, x_0) = \{p\}$, put $p_t = -H(t_0, p)$. For $(h, k) \in \mathbb{R} \times \mathbb{R}^n$ let

$$\alpha = \limsup_{(h,k)\to(0,0)} \frac{u(t_0 + h, x_0 + k) - u(t_0, x_0) - p_t h - \langle p, k\rangle}{\sqrt{h^2 + |k|^2}}.$$

Then there exists a sequence $(h_m, k_m)_m \to 0$ such that $\lim_{m\to\infty} \Phi_m = \alpha$, where

$$\Phi_m = \frac{u(t_0 + h_m, x_0 + k_m) - u(t_0, x_0) - p_t h_m - \langle p, k_m\rangle}{\sqrt{h_m^2 + |k_m|^2}}.$$

For each $m \in \mathbb{N}$, we choose $p_m \in \ell(t_0 + h_m, x_0 + k_m)$ then

$$\begin{aligned} \Phi_m &\le \frac{\varphi(t_0 + h_m, x_0 + k_m, p_m) - \varphi(t_0, x_0, p_m) - p_t h_m - \langle p, k_m\rangle}{\sqrt{h_m^2 + |k_m|^2}} \\ &\le \frac{-h_m(p_t + H(\tau_m, p_m)) - \langle p_m - p, k_m\rangle}{\sqrt{h_m^2 + |k_m|^2}}, \end{aligned}$$

for some τ_m lying between t_0 and $t_0 + h_m$; $\varphi(t, x, p)$ is given by (2.4).

Taking into account the assumption (A1), it is easy to see that, for (h_m, k_m) small enough, the sequence $(p_m)_m$ is bounded, then we can choose a subsequence also denoted by $(p_m)_m$ such that $p_m \to p_0$ as $m \to \infty$. Since the set-valued mapping $(t, x) \mapsto \ell(t, x)$ is upper semicontinuous, see [10], then $p_0 \in \ell(t_0, x_0)$, that is $p_0 = p$.

Now, letting $m \to \infty$ we have

$$\alpha = \lim_{m\to\infty} \Phi_m \le \lim_{m\to\infty} \frac{-h_m(p_t + H(\tau_m, p_m)) - \langle p_m - p, k_m\rangle}{\sqrt{h_m^2 + |k_m|^2}} = 0.$$

On the other hand, let

$$\beta = \liminf_{(h,k)\to(0,0)} \frac{u(t_0 + h, x_0 + k) - u(t_0, x_0) - p_t h - \langle p, k\rangle}{\sqrt{h^2 + |k|^2}}.$$

We have, for $p \in \ell(t_0, x_0)$

$$\begin{aligned} u(t_0 + h, x_0 + k) - u(t_0, x_0) &\ge \varphi(t_0 + h, x_0 + k, p) - \varphi(t_0, x_0, p) \\ &\ge -hH(\tau^*, p) + \langle p, k\rangle, \end{aligned}$$

where τ^* lies between t_0 and $t_0 + h$. Therefore

$$\beta \geq \liminf_{(h,k)\to(0,0)} \frac{-h(-p_t - H(\tau^*, p))}{\sqrt{h^2 + |k|^2}} = 0.$$

Thus,

$$\lim_{(h,k)\to(0,0)} \frac{u(t_0 + h, x_0 + k) - u(t_0, x_0) - p_t h - \langle p, k\rangle}{\sqrt{h^2 + |k|^2}} = 0,$$

which shows that $u(t, x)$ is differentiable at (t_0, x_0).

The proof of the theorem is then complete. □

Next, we investigate the differentiability of Hopf-type formula $u(t, x)$ on the characteristics. First, let us recall the Cauchy method of characteristics for Problem (2.1) - (2.2). Note that, to use the method of characteristics, the given data are assumed at least to be of class C^1.

From now on, we thus suppose that $H(t, p)$ and $\sigma(x)$ are of class C^1.

The characteristic differential equations of Problem (2.1) - (2.2) is as follows

$$\dot{x} = H_p \; ; \qquad \dot{v} = \langle H_p, p\rangle - H \; ; \qquad \dot{p} = 0, \tag{2.5}$$

with initial conditions

$$x(0) = y \; ; \qquad v(0) = \sigma(y) \; ; \qquad p(0) = \sigma_y(y) \; ; \quad y \in \mathbb{R}^n. \tag{2.6}$$

A solution of the system of differential equations (2.5) - (2.6) is defined by

$$\begin{cases} x = x(t, y) = y + \displaystyle\int_0^t H_p(\tau, \sigma_y(y))d\tau, \\ v = v(t, y) = \sigma(y) + \displaystyle\int_0^t \langle H_p(\tau, \sigma_y(y)), \sigma_y(y)\rangle d\tau - \int_0^t H(\tau, \sigma_y(y))d\tau, \\ p = p(t, y) \; = \; \sigma_y(y). \end{cases} \tag{2.7}$$

This solution is called a characteristic strip of Problem (2.1) - (2.2).

The first component of solution (2.7) is called a characteristic curve (briefly, characteristics) emanating from $(0, y)$ i.e. the curve defined by

$$\mathcal{C} : \; x = x(t, y) = y + \int_0^t H_p(\tau, \sigma_y(y))d\tau, \; t \in [0, T]. \tag{2.8}$$

Let $(t_0, x_0) \in \Omega$. Denote by $\ell^*(t_0, x_0)$ the set of all $y \in \mathbb{R}^n$ such that there is a characteristic curve emanating from $(0, y)$ and passing the point (t_0, x_0). We have $\ell(t_0, x_0) \subset \sigma_y(\ell^*(t_0, x_0))$, see [6]. Therefore $\ell^*(t_0, x_0) \neq \emptyset$.

Proposition 2.4. *Let $(t_0, x_0) \in \Omega$. Then a characteristic curve passing (t_0, x_0) has form*

$$x = x(t, y) = x_0 + \int_{t_0}^{t} H_p(\tau, \sigma_y(y))d\tau, \;\; t \in [0, T], \tag{2.9}$$

for some $y \in \ell^*(t_0, x_0)$.

Proof. Take $y \in \ell^*(t_0, x_0)$ and let $\mathcal{C}: \; x = x(t, y) = y + \int_0^t H_p(\tau, \sigma_y(y))d\tau$ be a characteristic curve emanating from $(0, y)$. Since $\mathcal{C}$ goes through (t_0, x_0) we have

$$x_0 = y + \int_0^{t_0} H_p(\tau, \sigma_y(y))d\tau \tag{2.10}$$

Therefore, the equation of $\mathcal{C}$ can be written as

$$x = x_0 - \int_0^{t_0} H_p(\tau, \sigma_y(y))d\tau + \int_0^{t} H_p(\tau, \sigma_y(y))d\tau = x_0 + \int_{t_0}^{t} H_p(\tau, \sigma_y(y))d\tau.$$

Conversely, let $\mathcal{C}_1 : x = x(t, y) = x_0 + \int_{t_0}^t H_p(\tau, \sigma_y(y))d\tau$ where $y \in \ell^*(t_0, x_0)$ be some curve passing (t_0, x_0). Then we can rewrite $\mathcal{C}_1$ as:

$$x = x_0 - \int_0^{t_0} H_p(\tau, \sigma_y(y))d\tau + \int_0^{t} H_p(\tau, \sigma_y(y))d\tau.$$

On the other hand, let $\mathcal{C}_2$ defined by (2.8)

$$x = y + \int_0^{t} H_p(\tau, \sigma_y(y))d\tau$$

be a characteristic curve also passing (t_0, x_0). Besides that, both $\mathcal{C}_1$, $\mathcal{C}_2$ are integral curves of the ODE $\dot{x} = H_p(t, \sigma_y(y))$, thus they must coincide. This proves the proposition. □

Remark 2.5. Suppose that $p_0 = \sigma_y(y) \in \ell(t_0, x_0)$ for some $y \in \ell^*(t_0, x_0)$. Then y is in the subgradient of convex function σ^* at p_0, i.e., $y \in \partial\sigma^*(p_0)$. Moreover, from (2.8) and (2.10), we have $y = x_0 - \int_0^{t_0} H_p(\tau, p_0)d\tau$.

Now, let $\mathcal{C}$ be a characteristic curve passing (t_0, x_0) that is written as

$$x = x(t, y) = x_0 + \int_{t_0}^{t} H_p(\tau, \sigma_y(y))d\tau$$

We say that the characteristic curve $\mathcal{C}$ is of the *type (I)* at the point $(t_0, x_0) \in \Omega$, if $\sigma_y(y) = p \in \ell(t_0, x_0)$. If $\sigma_y(y) \in \sigma_y(\ell^*(t_0, x_0)) \setminus \ell(t_0, x_0)$ then $\mathcal{C}$ is said to be of *type (II)* at this point.

In the sequel, we need an additional condition for the Hamiltonian $H = H(t,p)$.

(A2): The Hamiltonian $H(t,p)$ has one of the following forms:

a) $H(t,p) = g(t)h(p) + k(t)$ for some functions $g,\ h,\ k$ where $g(t)$ does not change its sign for all $t \in (0,T)$.
b) $H(t,\cdot)$ is a convex function for all $t \in (0,T)$.
c) $H(t,\cdot)$ is a concave function for all $t \in (0,T)$.

Remark 2.6. 1. In particular, if $H(t,p) = H(p)$ then the condition (A2) - a) is obviously satisfied.

2. In [7] we proved that if the assumptions (A1) and (A2) are satisfied, then the function $u(t,x)$ defined by Hopf-type formula (2.3) is a viscosity solution of Problem (2.1) - (2.2). Moreover, if $\sigma(x)$ is Lipschitz on $\mathbb{R}^n$ then $u(t,x)$ is a semiconvex function.

We introduce the following lemma which is necessary in the sequel, see [8].

Lemma 2.7. *Let $v : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $D = \mathrm{dom} v \subset \mathbb{R}^n$. Suppose that there exist $p,\, p_0 \in D,\ p \neq p_0$ and $y \in \partial v(p_0)$ such that*

$$\langle y, p - p_0 \rangle = v(p) - v(p_0).$$

Then for all z in the straight line segment $[p, p_0]$ we have

$$v(z) = \langle y, z \rangle - \langle y, p_0 \rangle + v(p_0).$$

Moreover, $y \in \partial v(z)$ for all $z \in [p, p_0]$.

Now some properties of characteristic curves passing a point (t_0, x_0) are given by the following theorems.

Theorem 2.8. *Assume (A1) and (A2). Let $(t_0, x_0) \in (0,T) \times \mathbb{R}^n,\ p_0 = \sigma_y(y) \in \ell(t_0, x_0)$ and let*

$$\mathcal{C} : x = x(t) = x_0 + \int_{t_0}^{t} H_p(\tau, p_0) d\tau,\ t \in [0,T], \tag{2.11}$$

be a characteristic curve of type (I) at (t_0, x_0). Then for all $(t_1, x_1) \in \mathcal{C},\ 0 \le t_1 \le t_0$ one has $p_0 \in \ell(t_1, x_1)$. Moreover, $\ell(t_1, x_1) \subset \ell(t_0, x_0)$.

Proof. Fix $(t_1, x_1) \in \mathcal{C},\ 0 \le t_1 \le t_0$. Take an arbitrary element $p \in \mathbb{R}^n$. Let

$$\eta(t,p) = \varphi(t,x,p) - \varphi(t,x,p_0),\ (t,x) \in \mathcal{C},\ t \in [0,t_0], \tag{2.12}$$

where $\varphi(t,x,p) = \langle x, p \rangle - \sigma^*(p) - \int_0^t H(\tau,p) d\tau$.
To prove that $p_0 \in \ell(t_1, x_1)$ it suffices to show that $\eta(t_1, p) \le 0$.

It is obviously that, $\eta(t_0, p) \leq 0$. We rewrite $\eta(t, p)$ to obtain

$$\eta(t,p) = \langle x(t), p - p_0\rangle - (\sigma^*(p) - \sigma^*(p_0)) - \int_0^t (H(\tau, p) - H(\tau, p_0))d\tau \quad (2.13)$$

for $(t, x) \in \mathcal{C}$.

By Remark 2.5, $x(0) = y \in \partial\sigma^*(p_0)$ and a property of subgradient of convex function, we have

$$\eta(0, p) = \langle y, p - p_0\rangle - (\sigma^*(p) - \sigma^*(p_0)) \leq 0. \quad (2.14)$$

As a result, we have $\eta(0, p) \leq 0$ and $\eta(t_0, p) \leq 0$.

From (2.11)-(2.13) we also have

$$\eta'(t, p) = \langle H_p(t, p_0), p - p_0\rangle - (H(t, p) - H(t, p_0)),\ \ t \in [0, t_0].$$

Next, we consider the following cases:

Case 1. Assume $H(t, p) = g(t)h(p) + k(t)$, and $g(t)$ does not change its sign in $(0, T)$. Then

$$\begin{aligned}\eta'(t, p) =&\langle g(t)h_p(p_0), p - p_0\rangle - g(t)(h(p) - h(p_0))\\ =&\big(\langle h_p(p_0), p - p_0\rangle - (h(p) - h(p_0))\big)g(t) = \lambda g(t),\end{aligned}$$

where $\lambda = \langle h_p(p_0), p - p_0\rangle - (h(p) - h(p_0))$ is a constant. Therefore, $\eta'(t, p)$ does not change its sign on $[0, t_0]$.

Case 2. Assume $H(t, \cdot)$ is convex. By a property of convex function, we have

$$\langle H_p(t, p_0), p - p_0\rangle \leq H(t, p) - H(t, p_0).$$

Therefore $\eta'(t, p) \leq 0$, for all $t \in [0, t_0]$.

Case 3. Assume $H(t, \cdot)$ is concave. Then $-H(t, \cdot)$ is convex. Arguing as in Case 2, we have $\eta'(t, p) \geq 0$, for all $t \in [0, t_0]$.

Combining the three cases above, we have, for all $t \in [0, t_0]$, $\eta'(t, p)$ does not change its sign on $[0, t_0]$. Thus,

(i) If $\eta'(t, p) \geq 0, t \in [0, t_0]$, then $\eta(t_1, p) \leq \eta(t_0, p) \leq 0$.

(ii) If $\eta'(t, p) \leq 0, t \in [0, t_0]$, then $\eta(t_1, p) \leq \eta(0, p) \leq 0$.

Consequently, we obtain $\varphi(t_1, x_1, p) \leq \varphi(t_1, x_1, p_0)$. This is true for all $p \in \mathbb{R}^n$. As a result, $p_0 \in \ell(t_1, x_1)$ for any $(t_1, x_1) \in \mathcal{C}$, $t_1 \in [0, t_0]$ and the first assertion has been proved.

Next, let $p \notin \ell(t_0, x_0)$. Then $\eta(t_0, p) < 0$. If (i) holds, i.e. $\eta'(t, p) \geq 0$ then $\eta(t_1, p) \leq \eta(t_0, p) < 0$.

Otherwise, if (ii) holds, i.e. $\eta'(t,p) \le 0$, we have

$$\eta(t,p) \le \eta(0,p) = \langle y, p - p_0 \rangle - (\sigma^*(p) - \sigma^*(p_0)),\ t \in [0, t_0).$$

Since $p \ne p_0$, then $\eta(0,p) < 0$. Actually, if it is false, i.e. $\langle y, p - p_0 \rangle = (\sigma^*(p) - \sigma^*(p_0))$, then applying Lemma 2.7, we see that $[p, p_0]$ is contained in $\mathcal{D} = \{z \in \mathrm{dom}\sigma^* \mid \partial\sigma^*(z) \ne \emptyset\}$ and σ^* is not strictly convex on the straight line segment $[p, p_0]$. This is a contradiction, since $\sigma(x)$ is of $C^1(\mathbb{R}^n)$, then it is essentially strictly convex on $\mathcal{D}$. In particular, σ^* is stricly convex on $[p, p_0]$, see ([9], Thm. 26.3). This implies $\eta(t_1, p) < 0$.

Therefore, in any case, if $p \notin \ell(t_0, x_0)$ then $\eta(t_1, p) < 0$. Thus $p \notin \ell(t_1, x_1)$. The proof is then complete. □

We have seen that, if the characteristic curve $\mathcal{C}$ is of type (I) at (t_0, x_0) then it is of the type (I) at any point $(t, x) \in \mathcal{C},\ 0 \le t \le t_0$. Nevertheless, for the characteristic curve of type (II), we have the following result which is somewhat different.

Theorem 2.9. *Assume (A1) and (A2). In addition, suppose that H, σ are of class C^2. Let $\mathcal{C} : x = x(t) = x_0 + \int_{t_0}^t H_p(\tau, \sigma_y(y_0))d\tau$ be a characteristic curve of type (II) at some $(t_0, x_0) \in \Omega$. Then there exists $\theta \in (0, t_0)$ such that $\mathcal{C}$ is of type (I) at $(\theta, x(\theta))$ and $\mathcal{C}$ is of type (II) for all point $(t, x) \in \mathcal{C},\ t \in (\theta, t_0]$.*

Proof. Let $\mathcal{C}:\ x = x_0 + \int_{t_0}^t H_p(\tau, \sigma_y(y_0))d\tau$ be the characteristic curve of type (II) at (t_0, x_0) emanating from $(0, y_0)$. Then $\sigma_y(y_0) \in \sigma_y(\ell^*(t_0, x_0)) \setminus \ell(t_0, x_0)$.

By the Cauchy method of characteristics, the function defined by Hopf-type formula $u(t,x)$ coincides with the local C^2 solution of Problem (2.1) - (2.2), see [2, 11]. Then there exists $t_1 \in (0, t_0)$ such that $u(t,x)$ is differentiable at any point $(t, x(t)) \in \mathcal{C},\ u_x(t,x) = \sigma_y(y_0)$ and $\ell(t,x) = \{\sigma_y(y_0)\},\ 0 \le t \le t_1$. Let

$$\theta = \sup\{t_1 \in [0, t_0) \mid \ell(s, x(s)) = \{\sigma_y(y_0)\},\ 0 \le s \le t_1\}.$$

Since the multivalued mapping $(t,x) \mapsto \ell(t,x)$ is upper semicontinuous, we get that $\sigma_y(y_0) \in \ell(\theta, x(\theta))$. It is obvious that, $\theta < t_0$ since $\sigma_y(y_0) \notin \ell(t_0, x_0)$ and $\mathcal{C}$ is of type (I) at $(\theta, x(\theta))$. On the other hand, for $t \in (\theta, t_0],\ \mathcal{C}$ is of type (II) at $(t, x(t))$ by the definition of θ and Theorem 2.8. □

3 Strip of differentiability of Hopf-type formula

In this section we will study the strips of the form $V = (0, t_*) \times \mathbb{R}^n \subset \Omega$ so that the Hopf-type formula $u(t,x)$ is continuously differentiable on them.

Theorem 3.1. *Assume (A1) and (A2). Let $u(t,x)$ be the Hopf-type formula of Problem (2.1) - (2.2) defined by (2.3). Suppose that there exists $t_0 \in (0, T)$ such that the mapping: $\mathbb{R}^n \ni y \mapsto x(t_0, y) = y + \int_0^{t_0} H_p(\tau, \sigma_y(y))d\tau$ is injective. Then $u(t,x)$ is continuously differentiable in the open strip $(0, t_0) \times \mathbb{R}^n$.*

Proof. Let $(t_1, x_1) \in (0, t_0) \times \mathbb{R}^n$ and let $\mathcal{C}$:

$$x = x_1 + \int_{t_1}^{t} H_p(\tau, p_1) d\tau,$$

where $p_1 = \sigma_y(y_1) \in \ell(t_1, x_1)$ be the characteristic curve going through (t_1, x_1) defined as in Proposition 2.4.

Let (t_0, x_0) be the intersection point of $\mathcal{C}$ and plane $\Delta^{t_0} = \{(t_0, x) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n\}$. Since the mapping $y \mapsto x(t_0, y)$ is injective and $\ell(t_0, x_0) \neq \emptyset$, thus $\ell^*(t_0, x_0)$ is a singleton. Hence there is a unique characteristic curve passing (t_0, x_0). This characteristic curve is exactly $\mathcal{C}$. Therefore, we can rewrite $\mathcal{C}$ as follows:

$$x = x_0 + \int_{t_0}^{t} H_p(\tau, p_0) d\tau$$

where $p_0 \in \ell(t_0, x_0)$.

Since $\ell(t_0, x_0) \subset \sigma_y(\ell^*(t_0, x_0))$ and $\ell^*(t_0, x_0)$ is a singleton, so is $\ell(t_0, x_0)$. Consequently, by Theorem 2.8, for all $(t, x) \in \mathcal{C}$, $0 < t < t_0$, the curve $\mathcal{C}$ is of type (I) at (t, x) and $\ell(t, x) = \{p_0\}$ particularly, it holds at (t_1, x_1) and then, $p_0 = p_1$. Applying Theorem 2.1 we see that $u(t, x)$ is of class C^1 in $(0, t_0) \times \mathbb{R}^n$. $\square$

Note that at some point $(t_0, x_0) \in \Omega$ where $u(t, x)$ is differentiable there may be more than one characteristic curve goes through, that is $\ell^*(t_0, x_0)$ may not be a singleton. Next, we have:

Theorem 3.2. *Assume (A1) and (A2). Moreover, let σ be Lipschitz on $\mathbb{R}^n$. Take $t_0 \in (0, T]$ and suppose that for every point of the plane $\Delta^{t_0} = \{(t_0, x) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n\}$, the set $\ell(t_0, x)$ is a singleton. Then the Hopf-type formula $u(t, x)$ of Problem (2.1) - (2.2) defined by (2.3) is continuously differentiable in the open strip $(0, t_0) \times \mathbb{R}^n$.*

Proof. By assumption, the function $\sigma(x)$ is convex and Lipschitz on $\mathbb{R}^n$, then $D = \text{dom } \sigma^* = \{q \in \mathbb{R}^n \mid \sigma^*(q) < +\infty\}$ is a bounded (and convex) subset in $\mathbb{R}^n$. We thus have $\ell(t, x) \subset D$ for all $(t, x) \in \Omega$.

Let $(t_1, x_1) \in (0, t_0) \times \mathbb{R}^n$. We will check that $\ell(t_1, x_1)$ is a singleton.

For each $y \in \mathbb{R}^n$, we put

$$\Lambda(y) = x_1 - \int_{t_0}^{t_1} H_p(\tau, p(y)) d\tau,$$

where $p(y) \in \ell(t_0, y) \in D$. Since the multi-valued function $y \mapsto \ell(t_0, y)$ is u.s.c, see [10], and by the hypothesis, $\ell(t_0, y) = \{p(y)\}$ is a singleton for all $y \in \mathbb{R}^n$, we deduce that the single-valued function $y \mapsto p(y)$ is continuous. Therefore the function $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$, defined by $y \mapsto \Lambda(y)$ is also continuous on $\mathbb{R}^n$.

Since $p(y)$ is in the bounded set D and $H_p(t,p)$ is continuous, there exists $M > 0$ such that

$$|\Lambda(y) - x_1| \leq \int_{t_1}^{t_0} |H_p(\tau, p(y)|d\tau \leq M.$$

Therefore Λ is a continuous function from the closed ball $B'(x_1, M)$ into itself. By Brouwer theorem, Λ has a fixed point $x_0 \in B'(x_1, M)$, i.e., $\Lambda(x_0) = x_0$, hence

$$x_1 = x_0 + \int_{t_0}^{t_1} H_p(\tau, p(x_0))d\tau.$$

In other words, there exists a characteristic curve $\mathcal{C}$ of the type (I) at (t_0, x_0) described as in Theorem 2.8 passing (t_1, x_1). Since $\ell(t_0, x_0)$ is a singleton, so is $\ell(t_1, x_1)$. Applying Theorem 2.1, we see that $u(t,x)$ is continuously differentiable in $(0, t_0) \times \mathbb{R}^n$. □

We note that, the solution $u(t,x)$ is differentiable at (t_0, x_0) if and only if, $\ell(t_0, x_0)$ is a singleton. Thus we have the following corollary.

Corollary 3.3. *Assume (A1) and (A2). Moreover, let σ be Lipschitz on $\mathbb{R}^n$. Suppose that the Hopf-type formula $u(t,x)$ of Problem (2.1) - (2.2) defined by (2.3) is differentiable at every point of the plane $\Delta^{t_0} = \{(t_0, x) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n\},\ 0 < t_0 \leq T$. Then $u(t,x)$ is continuously differentiable in the strip $(0, t_0) \times \mathbb{R}^n$.*

Definition 3.4. We call a point $(t_0, x_0) \in \Omega$ *regular* for $u(t,x)$ if the function is differentiable at this point. If $u(t,x)$ is not differentiable at $(t_1, x_1) \in \Omega$ then this point is said to be a *singular* point or singularity of the function.

We study a simple propagation of singularities of viscosity solution $u(t,x)$ of the Cauchy problem (2.1) - (2.2) defined by the Hopf-type formula. Under minimum assumption we show that, if (t_0, x_0) is a singular point of $u(t,x)$, then there exists another singular one (t,x) for $t > t_0$ and x is near to x_0. It is worth noticing that, a comprehensive study of singularities of semiconcave/semiconvex functions is presented in [2].

Theorem 3.5. *Assume (A1) and (A2). Let $(t_0, x_0) \in \Omega$ be a singular point of the function $u(t,x)$ defined by the Hopf-type formula (2.3). Then for each $\epsilon > 0$ there exists $\delta > 0$ such that for any $t_* > t_0,\ |t_* - t_0| \leq \delta$, there exists $x_* \in B'(x_0, \epsilon)$ such that (t_*, x_*) is also a singular point.*

Proof. We use an idea of the proof of Lemma 6.5.1 in [2] with an appropriate adjustment. Let $(t_0, x_0) \in \Omega$ and let $\epsilon > 0$. Under assumption (A1), for all $(t,x) \in E = [t_0, T] \times B'(x_0, \epsilon)$, there exist positive numbers r_{tx} and N_{tx} such that for all (t', x') satisfying $|t' - t| + |x' - x| < r_{tx}$ then $\ell(t', x') \subset B'(0, N_{tx})$. Hence, we can cover the compact set E by a finite number balls centered at (t_i, x_i) with radii $r_{(tx)_i},\ i = 1, \dots, k$. We take the positive number $M = \max\{N_{(tx)_i},\ i = 1, \dots, k\}$, then for all $(t, x) \in E$ we get $\ell(t, x) \subset B'(0, M)$. Now we choose $\delta \in (0, T - t_0]$ satisfying

$$\delta \sup_{|t-t_0| \le T-t_0, |p| \le M} |H_p(t, p)| \le \epsilon$$

and fix a $t_* > t_0$ so that $t_* - t_0 \le \delta$.

By contradiction, if every point (t_*, y) where $y \in B'(x_0, \epsilon)$ is regular, then $\ell(t_*, y) = \{p(y)\}$ is a singleton. Since the multi-valued function $y \mapsto \ell(t_*, y)$ is u.s.c, then $y \mapsto p(y)$ is continuous on $B'(x_0, \epsilon)$. Thus, as in the proof of Theorem 3.2, we see that the function $\mathbb{R}^n \ni y \mapsto \Lambda(y) = x_0 - \int_{t_*}^{t_0} H_p(\tau, p(y)) d\tau$ is also continuous.

Note that, if $y \in B'(x_0, \epsilon)$ then

$$|\Lambda(y) - x| \le \int_{t_0}^{t_*} |H_p(\tau, p(y)| d\tau \le \delta \sup_{|t-t_0| \le T-t_0, |p| \le M} |H_p(t, p)| \le \epsilon.$$

Therefore Λ is a continuous function from the closed ball $B'(x_0, \epsilon)$ into itself. By Brouwer theorem, Λ has a fixed point $x_* \in B'(x_0, \epsilon)$, i.e., $\Lambda(x_*) = x_*$, hence,

$$x_0 = x_* + \int_{t_*}^{t_0} H_p(\tau, p(x_*)) d\tau.$$

In other words, there exists a characteristic curve $\mathcal{C}$ of the type (I) at (t_*, x_*) described as in Theorem 2.8 passing (t_0, x_0). Since $\ell(t_*, x_*)$ is a singleton, so is $\ell(t_0, x_0)$. This contradicts to the hypothesis. □

Remark 3.6. If $(t_0, x_0) \in \Omega$ is a singular point for $u(t, x)$ and $\epsilon > 0$, by the previous theorem, there exists $\delta > 0$ such that for any $t \in [t_0, t_0 + \delta]$ we can pick out $x = x(t) \in B'(x_0, \epsilon)$ so that (t, x) is singular. Put $\delta_1 = \delta$, $t_1 = t_0 + \delta_1$ and $x_1 = x(t_1)$. By induction, we can find $(\delta_k)_k$ and $x_k = x(t_k),\ t_k = t_{k-1} + \delta_k$ so that (t_k, x_k) is singular. Since δ_k is dependent on (t_k, x_k) there are two possibilities:

$$\sum_{k=1}^{\infty} \delta_k < T \quad \text{or} \quad \sum_{k=1}^{\infty} \delta_k \ge T.$$

In the first case, the singularities of $u(t, x)$ constructed by this way may not propagate to the boundary $t = T$, otherwise the singularities of $u(t, x)$ exist at some points (T, x_*). Nevertheless, if we assume $\sigma(x)$ is Lipschitz on $\mathbb{R}^n$ as an additional condition, then the number $\delta > 0$ in the proof of Theorem 3.5 can be chosen independently of $(t_i, x_i),\ i = 1, 2, \dots$

We have the following:

Theorem 3.7. *Assume (A1) and (A2). Moreover, let $\sigma(x)$ be a Lipschitz function on $\mathbb{R}^n$ and let (t_0, x_0) be a singular point for the Hopf-type formula $u(t,x)$ defined by (2.3). Then for each $\epsilon > 0$ there exists $\delta > 0$ such that for any $t_1 \in [t_0, t_0 + \delta]$ we can find $x_1 \in B'(x_0, \epsilon)$ such that (t_1, x_1) is also a singular point for $u(t,x)$..*

Proof. Since $\sigma(x)$ is convex and Lipschitz, then $D = \text{dom}\sigma^*$ is bounded. Hence, $D \subset B'(0, M)$ for some positive number M. Choose a fixed number $\delta > 0$ such that

$$\delta \sup_{0 \le t \le T, |p| \le M} |H_p(t,p)| \le \epsilon.$$

We argue similarly to the proof of Theorem 3.5. Let (t_0, x_0) be a singular point for $u(t,x)$. If there is $t_* \in (t_0, t_0 + \delta]$ such that (t_*, y) is regular for all $y \in B'(x_0, \epsilon)$ then the mapping

$$y \mapsto \Lambda(y) = x_0 - \int_{t_*}^{t} H_p(\tau, p(y))d\tau$$

is continuous from $B'(x_0, \epsilon)$ into itself. Thus, the mapping has a fixed point $x_* \in B'(x_0, \epsilon)$. This implies that there is a characteristics $\mathcal{C}$ of type (I) at (t_*, x_*) passing (t_0, x_0) and so (t_0, x_0) is regular. This is a contradiction. □

Corollary 3.8. *Assume (A1) and (A2) and let $\sigma(x)$ be a Lipschitz function on $\mathbb{R}^n$. If the Hopf-type formula $u(t,x)$ defined by (2.3) has a singular point $(t_0, x_0) \in \Omega$, then for any $\epsilon > 0$ and $t > t_0$, we can find another singular point (t, x) such that $|x - x_0| \le m\epsilon$, for some $m \in \mathbb{N}$. Therefore the singular points of $u(t,x)$ propagate with respect to t as t tends to T.*

Proof. Arguing as in Remark 3.6, we see that for $\epsilon > 0$ and $t_0 < t \le T$, there is $m \in \mathbb{N}$ such that $m\delta < t \le (m+1)\delta$, where $\delta > 0$ is defined as in Theorm 3.7. Let $t_i = i\delta$, $i = 0, \dots, m$. After m steps, we can take $x_m \in B'(x_{m-1}, \epsilon)$ such that (t, x_m) is singular and then

$$|x_m - x_0| \le |x_m - x_{m-1}| + \cdots + |x_1 - x_0| \le m\epsilon.$$

The proof is thus complete. □

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