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Hilbert Coefficients and the Depth of Associated Graded Rings with Respect to Parameter Ideals

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Abstract
In this paper, we investigate the non-positivity for the Hilbert coefficients of parameter ideals. Moreover, we establish a relationship between the vanishing of Hilbert coefficients and the depth of associated graded rings with respect to parameter ideals in the case of small regularity.

Keywords Hilbert coefficients · The depth of associated graded rings · Parameter ideals · Castelnuovo–Mumford regularity · Postulation number

Mathematics Subject Classification (2010) Primary 13D45 · 13D07 · Secondary 14B15

1 Introduction

Let (A, m) be a Noetherian local ring, $I \subset A$ an m-primary ideal and M a finitely generated A-module of dimension d. Denote by $G_I(A) = \oplus_{n\geq 0} I^n/I^{n+1}$ the associated graded ring of A with respect to I. Let $\ell(\cdot)$ denote the length of an A-module. The Hilbert–Samuel function of M with respect to I is the function $H_M : \mathbb{Z} \longrightarrow \mathbb{N}_0$ given by

$$H_M(n) = \begin{cases} \ell(M/I^nM) & \text{if } n \geq 0, \\ 0 & \text{if } n < 0. \end{cases}$$

Dedicated to Prof. Le Tuan Hoa on the occasion of his 60th birthday.

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Samuel showed that there exists a unique polynomial $P_M(x) \in \mathbb{Q}[x]$ (called the *Hilbert–Samuel polynomial*) of degree d such that $H_M(n) = P_M(n)$ for $n \gg 0$. We can always write $P_M(n)$ of the form

$$P_M(n) = \sum_{i=0}^{d} (-1)^i \binom{n+d-i-1}{d-i} e_i(I, M).$$

Then, the integers $e_i(I, M)'s$, $i = 0, \dots, d$, are called *Hilbert coefficients* of M with respect to I.

The aim of this paper is to study the non-positivity of the Hilbert coefficients and establish a relationship between the vanishing of Hilbert coefficients and the depth of associated graded rings.

The Hilbert coefficients give us structural information of rings and modules; so, they have been attracted attention of many mathematicians. In 2008, Vasconcelos [14] named $e_1(I, M)$ Chern number. Concerning Chern number, it is well known that $e_1(\mathfrak{q}, M) \le 0$ for every parameter ideal $\mathfrak{q}$ of M (see Mandal et al. [7]), while other Hilbert coefficients of parameter ideal would be positive. However, if $\operatorname{depth}(A) \ge \dim(A) - 1$, McCune [9] showed that $e_2(\mathfrak{q}, A) \le 0$. With the hypothesis $\operatorname{depth}(A) \ge \dim(A) - 1$, Saikia and Salony [11] proved that $e_3(\mathfrak{q}, A) \le 0$. In [9], McCune also proved that if $\mathfrak{q}$ is a parameter ideal such that $\operatorname{depth}(G_{\mathfrak{q}}(A)) \ge \dim(A) - 1$ then $e_i(\mathfrak{q}, A) \le 0$ for $i = 1, \dots, d$.

The first main result of this paper is an improvement of the McCune's result with a weaker assumption that $\operatorname{depth}(G_{\mathfrak{q}}(A)) \ge d - 2$.

Theorem 1 *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring with* $\dim(A) = d \ge 2$ *and* $\operatorname{depth} A \ge d - 1$. *Let* $\mathfrak{q}$ *be a parameter ideal of* A *such that* $\operatorname{depth} G_{\mathfrak{q}}(A) \ge d - 2$. *Then*

$$e_i(\mathfrak{q}) \le 0 \quad \textit{for all } \ i = 1, \dots d.$$

Next, we discuss on a relationship between the vanishing of Hilbert coefficients and the depth of the associated graded ring with respect to parameter ideals. In case A is unmixed, Ghezzi et al. [4] proved that $e_1(\mathfrak{q}) = 0$ if and only if A is Cohen–Macaulay. Lori Mccune [9] showed that $e_2(\mathfrak{q}) = 0$ if and only if $\operatorname{depth}(G_{\mathfrak{q}}(A)) \ge d - 1$. If $\mathfrak{q}$ is a parameter ideal generated by a d-sequence of an unmixed Noetherian local ring A, we get the following theorem.

Theorem 2 *Let* $(A, \mathfrak{m})$ *be a Noetherian unmixed local ring of dimension* $d \ge 2$ *and* $\mathfrak{q}$ *a parameter ideal of* A *generated by a* d*-sequence* $x_1, \dots, x_d$. *For each* $1 \le i \le d$, *we have*

$$e_i(\mathfrak{q}) = 0 \quad \textit{if and only if} \quad \operatorname{depth} G_{\mathfrak{q}}(A) \ge d - i + 1.$$

It is well known that if $\mathfrak{q}$ is a parameter ideal of A generated by a d-sequence, then $\operatorname{reg}(G_{\mathfrak{q}}(A)) = 0$. More generally, if $\mathfrak{q}$ is a parameter ideal of A such that $\operatorname{reg}(G_{\mathfrak{q}}(A)) \le 1$, we obtain the following result.

Theorem 3 *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring of dimension* $d \ge 3$ *and* $\operatorname{depth}(A) \ge k$, *for* $2 \le k \le d - 1$. *Let* $\mathfrak{q}$ *be a parameter ideal of* A *such that* $\operatorname{reg}(G_{\mathfrak{q}}(A)) \le 1$. *Then*

(i) $\operatorname{depth}(G_{\mathfrak{q}}(A)) \ge k$;
(ii) $e_{d-k+2}(\mathfrak{q}) = e_{d-k+3}(\mathfrak{q}) = \cdots = e_d(\mathfrak{q}) = 0$.

The paper is divided into three sections. In Section 2, we prepare some facts related to Hilbert coefficients. In Section 3, we prove the non-positivity for Hilbert coefficients of parameter ideals. In Section 4, we discuss the relationship between the vanishing of Hilbert coefficients and the depth of associated graded rings.

2 Preliminary

Let $R = \oplus_{n\geq 0} R_n$ be a finitely generated standard graded algebra over a Noetherian commutative ring R_0. Let R_+ be the ideal of R generated by the elements of positive degrees of R. Let E be a finitely generated graded R-module with $\dim(E) = d$. Denote by $H^i_{R_+}(E)$ the ith local cohomological module of E with support ideal R_+. Define

$$a_i(E) := \begin{cases} \max\{n \mid H^i_{R_+}(E)_n \neq 0\} & \text{if } H^i_{R_+}(E) \neq 0, \\ -\infty & \text{if } H^i_{R_+}(E) = 0. \end{cases}$$

The Castelnuovo–Mumford regularity of E is the number

$$\operatorname{reg}(E) := \max\{a_i(E) + i \mid i \geq 0\}.$$

If the basic ring R_0 of R is artinian, $h_E(n) := \ell(E_n)$ denote the Hilbert function of E. The unique polynomial $p_E(X)$ for which $h_E(n) = p_E(n)$ for $n \gg 0$ is called the Hilbert polynomial of E. It is written in the form

$$p_E(n) = \sum_{i=0}^{d-1} (-1)^i \binom{n+d-i-1}{d-i-1} e_i(E),$$

where $e_i(E)$ for $i = 0, 1, \ldots, d-1$ are integers, called the Hilbert coefficients of E. The postulation number $p(E)$ of E is defined to be the integer number

$$p(E) = \max\{n \mid h_E(n) \neq p_E(n)\}.$$

The relationship between Hilbert function and Hilbert polynomial is given by the following formula (see [8, Lemma 1.3] or [1, Theorem 17.1.7]):

$$h_E(n) - p_E(n) = \sum_{i=0}^{d} (-1)^i \ell(H^i_{R_+}(E)_n).$$

From this, we have the following property.

Lemma 1

$$p(E) \leq \max\{a_0(E), \ldots, a_d(E)\} \leq \operatorname{reg}(E).$$

Now, let $(A, \mathfrak{m})$ be a local Noetherian ring and I an $\mathfrak{m}$-primary ideal of A. Let M be a finitely generated A-module of dimension d. A numerical function

$$\begin{aligned} H_M : \mathbb{Z} &\longrightarrow \mathbb{N}_0 \\ n &\longmapsto H_M(n) = \begin{cases} \ell(M/I^n M) & \text{if } n \geq 0; \\ 0 & \text{if } n < 0. \end{cases} \end{aligned}$$

is said to be a *Hilbert–Samuel function* of M with respect to the ideal I. Samuel showed that there exists a polynomial $P_M \in \mathbb{Q}[x]$ of degree d such that $H_M(n) = P_M(n)$ for $n \gg 0$.

The polynomial P_M is called the Hilbert–Samuel polynomial of M with respect to the ideal I and it is written in the form

$$P_M(n) = \sum_{i=0}^{d} (-1)^i \binom{n+d-i-1}{d-i} e_i(I, M),$$

where $e_i(I, M)$ for $i = 0, \dots, d$ are integers, called *Hilbert coefficients of M with respect to I*. In particular, $e_0(I, M)$ and $e_1(I, M)$ are called the *multiplicity* and *Chern coefficient*, respectively. Denote

$$n_M(I) = \max\{n \mid H_M(n) \neq P_M(n)\}.$$

If $M = A$, we write $e_i(I)$ for $e_i(I, A)$ and $n(I)$ for $n_A(I)$.

Let $G_I(M) = \oplus_{n \geq 0} I^n M / I^{n+1} M$ denote the associated graded module of M with respect to I. Then,

$$e_i(G_I(M)) = e_i(I, M) \quad \text{for } i = 0, \dots, d-1.$$

Lemma 2 [2, Lemma 3.5]

$$n(I) = p(G_I(A)).$$

Suppose that $L = H^0_{\mathfrak{m}}(M)$ and $\overline{M} = M/L$. A relationship between $e_i(I, M)$ and $e_i(I, \overline{M})$ is given by the following lemma.

Lemma 3 [3, Lemma 3.4] *If $d = \dim(M) \geq 1$, then*

(i) $e_i(I, M) = e_i(I, \overline{M})$ *for $i = 0, \dots, d-1$;*
(ii) $e_d(I, M) = e_d(I, \overline{M}) + (-1)^d \ell(L)$.

If $d = 1$ and $I = \mathfrak{q}$ is a parameter ideal of M, then $\overline{M}$ is Cohen–Macaulay. This implies that $e_1(\mathfrak{q}, \overline{M}) = 0$. We get the following corollary.

Corollary 1 *If $\dim(M) = 1$ and $\mathfrak{q}$ is a parameter ideal of M, then*

$$e_1(\mathfrak{q}, M) = -\ell(L).$$

An element $x \in I \backslash \mathfrak{m}I$ is said to be *superficial* for I with respect to M if there exists a number $c \in \mathbb{N}$ such that $(I^n M : x) \cap I^c M = I^{n-1} M$ for $n > c$. If $A/\mathfrak{m}$ is infinite, then a superficial element for I always exists. Elements $x_1, \dots, x_r \in I \backslash \mathfrak{m}I$ is said to be a superficial sequence for I with respect to M if x_i is superficial for $I/(x_1, \dots, x_{i-1})$ with respect to $M/(x_1, \dots, x_{i-1}M)$, $i = 1, \dots, r$.

Suppose that x is a superficial element for I with respect to M and $N := M/xM$. The following lemma gives a relationship between $e_i(I, M)$ and $e_i(I, N)$.

Lemma 4 [10, 22.6] *Let M be a finitely generated A-module of dimension $d \geq 2$ and I an $\mathfrak{m}$-primary ideal of A. Let $x \in I \backslash \mathfrak{m}I$ be a superficial element for I with respect to M. Then*

(i) $e_i(I, M) = e_i(I, N)$ *for $i = 0, \dots, d-2$;*
(ii) $e_{d-1}(I, M) = e_{d-1}(I, N) + (-1)^d \ell(0 :_M x)$.

3 Non-positivity of Hilbert Coefficients with Respect to Parameter Ideals

Let $(A, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated A-module of dimension $d \geq 1$. Let $\mathfrak{q}$ be a parameter ideal of M. We begin with the non-positivity of the first Hilbert coefficient $e_1(\mathfrak{q}, M)$.

Proposition 1 [7, Theorem 3.5] *Let $(A, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated A-module of dimension $d \geq 1$. If $\mathfrak{q}$ is a parameter ideal of M, then $e_1(\mathfrak{q}, M) \leq 0$.*

The above proposition gives us the non-positivity of the Hilbert coefficient $e_1(\mathfrak{q}, M)$ of any parameter ideal. However, other Hilbert coefficients of parameter ideal would be positive. In [9], Lori McCune gave the following example to show that the second coefficient $e_2(\mathfrak{q})$ of a parameter ideal $\mathfrak{q}$ would be positive.

Example 1 Let $A = k[x, y, z, u, v, w]/I$, where $I = (x + y, z - u, w) \cap (z, u - v, y) \cap (x, u, w)$ and $\mathfrak{q} = (u - y, z + w, x - v)$. Then, A is an unmixed ring of dimension three and depth one and $\mathfrak{q}$ is a parameter ideal with

$$P_{\mathfrak{q}}(n) = 3\binom{n+2}{3} + 2\binom{n+1}{2} + n.$$

In particular, $e_2(\mathfrak{q}) = 1 > 0$.

Definition 1 Let $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ be a function. The i-difference function, $\Delta^i f$, is defined by

$$\Delta^1 f(n) = f(n+1) - f(n);$$

and

$$\Delta^i f = \Delta(\Delta^{i-1} f) \quad \text{if } i \geq 2.$$

For convenience, we write $f = \Delta^0 f$ and $\Delta f = \Delta^1 f$.

Remark 1 If $f(n) = 0$ for $n \gg 0$ and $\Delta f(n) \geq 0$ (respectively $\Delta f(n) \leq 0$) for all $n \geq k$, then $f(n) \leq 0$ (respectively $f(n) \geq 0$) for all $n \geq k$.

In the case of $\dim A = 1$, McCune [9, Proposition 2.2 (2)] provided the following property.

Lemma 5 *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension one and $\mathfrak{q}$ a parameter ideal of A. Then*

$$P_{\mathfrak{q}}(n) - H_{\mathfrak{q}}(n) \geq 0 \quad \textit{and} \quad \Delta(P_{\mathfrak{q}} - H_{\mathfrak{q}})(n) \leq 0 \qquad \textit{for all } n \geq -1.$$

The following lemma is a generalization of above lemma and that is a key point to prove the main result of this section.

Lemma 6 *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$ and* depth $A \geq d - 1$. *Let $\mathfrak{q}$ be a parameter ideal of A such that* depth $G_{\mathfrak{q}}(A) \geq d - 2$. *Then,*

(i) $(-1)^{d+1}[P_{\mathfrak{q}}(n) - H_{\mathfrak{q}}(n)] \geq 0$ *for all $n \geq -d$;*
(ii) $(-1)^d \Delta(P_{\mathfrak{q}} - H_{\mathfrak{q}})(n) \geq 0$ *for all $n \geq -d$.*

Proof We will prove by induction on d.

In case $d = 1$, the lemma holds from Lemma 5.

In case $d \geq 2$, we have depth $A \geq d-1 \geq 1$. We can choose a regular element $x \in \mathfrak{q}\backslash\mathfrak{m}\mathfrak{q}$ of A such that x is superficial for $\mathfrak{q}$. Denote $\bar{A} = A/(x)$ and $\bar{\mathfrak{q}} = \mathfrak{q}/(x)$. Then, $\bar{\mathfrak{q}}$ is also a parameter ideal of $\bar{A}$ and $\dim(\bar{A}) \geq 1$. From the following exact sequence

$$0 \longrightarrow (\mathfrak{q}^{n+1} : x)/\mathfrak{q}^n \longrightarrow A/\mathfrak{q}^n \xrightarrow{x} A/\mathfrak{q}^{n+1} \longrightarrow A/(\mathfrak{q}^{n+1}, x) \longrightarrow 0,$$

we get

$$\ell(A/(\mathfrak{q}^{n+1}, x)) = \ell(A/\mathfrak{q}^{n+1}) - \ell(A/\mathfrak{q}^n) + \ell((\mathfrak{q}^{n+1} : x)/\mathfrak{q}^n).$$

Hence,

$$H_{\bar{\mathfrak{q}}}(n+1) = H_{\mathfrak{q}}(n+1) - H_{\mathfrak{q}}(n) + \ell((\mathfrak{q}^{n+1} : x)/\mathfrak{q}^n). \tag{1}$$

Since x is regular, $\ell((\mathfrak{q}^{n+1} : x)/\mathfrak{q}^n) = \ell(0 : x) = 0$, for $n \gg 0$. From (1), we have

$$P_{\bar{\mathfrak{q}}}(n+1) = P_{\mathfrak{q}}(n+1) - P_{\mathfrak{q}}(n). \tag{2}$$

By subtracting (1) from (2), we obtain

$$\Delta(P_{\mathfrak{q}} - H_{\mathfrak{q}})(n) = P_{\bar{\mathfrak{q}}}(n+1) - H_{\bar{\mathfrak{q}}}(n+1) + \ell((\mathfrak{q}^{n+1} : x)/\mathfrak{q}^n) \tag{3}$$

for all $n \in \mathbb{Z}$.

If $d = 2$, then $\dim(\bar{A}) = 1$. By Lemma 5,

$$P_{\bar{\mathfrak{q}}}(n+1) - H_{\bar{\mathfrak{q}}}(n+1) \geq 0 \quad \text{for all } n \geq -2.$$

From (3), it follows that

$$\Delta(P_{\mathfrak{q}} - H_{\mathfrak{q}})(n) \geq 0 \quad \text{for all } n \geq -2.$$

By Remark 1, we have

$$P_{\mathfrak{q}}(n) - H_{\mathfrak{q}}(n) \leq 0 \quad \text{for all } n \geq -2.$$

So, the lemma holds for the case $d = 2$.

If $d \geq 3$, depth $G_{\mathfrak{q}}(A) \geq d-2 \geq 1$. Thus,

$$\ell((\mathfrak{q}^{n+1} : x)/\mathfrak{q}^n) = 0 \quad \text{for all } n \geq 0.$$

Then, (1) becomes

$$H_{\bar{\mathfrak{q}}}(n+1) = H_{\mathfrak{q}}(n+1) - H_{\mathfrak{q}}(n) \quad \text{for all } n \in \mathbb{Z}. \tag{4}$$

Subtracting (4) from (2) and multiplying both sides by $(-1)^d$, we get

$$(-1)^d \Delta(P_{\mathfrak{q}} - H_{\mathfrak{q}})(n) = (-1)^d (P_{\bar{\mathfrak{q}}}(n+1) - H_{\bar{\mathfrak{q}}}(n+1)), \tag{5}$$

for all $n \in \mathbb{Z}$. Since $\dim \bar{A} = d-1$ and $\bar{\mathfrak{q}}$ is a parameter ideal of $\bar{A}$, depth $G_{\bar{\mathfrak{q}}}(\bar{A}) \geq d-3$. By induction on d, we may assume that

$$(-1)^{d-1} \Delta(P_{\bar{\mathfrak{q}}} - H_{\bar{\mathfrak{q}}})(n) \geq 0 \quad \text{for all } n \geq -(d-1).$$

From Remark 1, we obtain

$$(-1)^{d-1}(P_{\bar{\mathfrak{q}}}(n) - H_{\bar{\mathfrak{q}}}(n)) \leq 0 \quad \text{for all } n \geq -(d-1).$$

Hence,

$$(-1)^d [P_{\bar{\mathfrak{q}}}(n+1) - H_{\bar{\mathfrak{q}}}(n+1)] \geq 0 \quad \text{for all } n \geq -d.$$

Thus, from (5), we have

$$(-1)^d \Delta(P_{\mathfrak{q}} - H_{\mathfrak{q}})(n) \geq 0 \quad \text{for all } n \geq -d.$$

By Remark 1,

$$(-1)^{d+1}[P_{\mathfrak{q}}(n) - H_{\mathfrak{q}}(n)] \geq 0 \quad \text{for all } n \geq -d. \qquad \square$$

Remark 2 Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$ and $\mathfrak{q}$ a parameter ideal of A. If $n(\mathfrak{q}) < i - d$ for some $i \in \{1, 2, \ldots, d\}$, then

$$e_i(\mathfrak{q}) = e_{i+1}(\mathfrak{q}) = \cdots = e_d(\mathfrak{q}) = 0.$$

Proof Since $n(\mathfrak{q}) < i - d \le 0$, $P_{\mathfrak{q}}(n) = 0$ for all $n = i-d, i-d+1, \ldots, 0$. Plugging the values $n = 0, -1, \ldots, i-d$ successively into $P_{\mathfrak{q}}(n)$, one can see that

$$e_d(\mathfrak{q}) = e_{d-1}(\mathfrak{q}) = \cdots = e_i(\mathfrak{q}) = 0. \qquad \square$$

The following theorem is a main result of this section.

Theorem 1 *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring with* $\dim(A) = d \ge 2$ *and* $\operatorname{depth} A \ge d - 1$. *Let* $\mathfrak{q}$ *be a parameter ideal of A such that* $\operatorname{depth} G_{\mathfrak{q}}(A) \ge d - 2$. *Then,*

$$e_i(\mathfrak{q}) \le 0 \quad \textit{for all} \;\; i = 1, \ldots d.$$

Proof First, we prove that $e_d(\mathfrak{q}) \le 0$. By Lemma 6 (i),

$$(-1)^{d+1}[P_{\mathfrak{q}}(n) - H_{\mathfrak{q}}(n)] \ge 0 \quad \text{for all} \;\; n \ge -d.$$

Then, the above inequality holds for $n = 0$. Therefore,

$$(-1)^{d+1}[(-1)^d e_d(\mathfrak{q}) - H_{\mathfrak{q}}(0)] \ge 0.$$

This implies that $e_d(\mathfrak{q}) \le 0$. So, the theorem is proved for $i = d$.

Now, we need to show $e_i(\mathfrak{q}) \le 0$ for $1 \le i < d$. From Proposition 1, $e_1(\mathfrak{q}) \le 0$. So, the theorem holds for $d = 2$.

If $d \ge 3$, then $\operatorname{depth} G_{\mathfrak{q}}(A) \ge d - 2 \ge 1$. We can choose a regular element $x \in \mathfrak{q} \backslash \mathfrak{m}\mathfrak{q}$ of A such that x is superficial for $\mathfrak{q}$. Set $\bar{A} = A/(x)$ and $\bar{\mathfrak{q}} = \mathfrak{q}/(x)$. We have $\bar{\mathfrak{q}}$ is a parameter ideal of $\bar{A}$. Since $\dim(\bar{A}) = d - 1$, by induction on d, we may assume that $e_i(\bar{\mathfrak{q}}) \le 0$ for all $i \in \{1, \ldots, d-1\}$. From Lemma 4, we have $e_i(\mathfrak{q}) = e_i(\bar{\mathfrak{q}}) \le 0$ for all $i \in \{1, \ldots, d-1\}$. $\square$

Notice that McCune ([9, Corollary 4.5]) proved that $e_i(\mathfrak{q}) \le 0$ under stronger assumption, that is $\operatorname{depth}(G_{\mathfrak{q}}(A)) \ge d - 1$. In [11], Saikia and Salony used a different method to prove this result. Theorem 4 also implies a result of McCune ([9, Theorem 3.5]).

Corollary 2 *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring with* $\dim(A) = d \ge 2$. *Let* $\mathfrak{q}$ *be a parameter ideal of A. If* $\operatorname{depth} A \ge d - 1$, *then* $e_2(\mathfrak{q}) \le 0$.

Proof If $d = 2$, it is obvious from Theorem 4. If $d > 2$, without loss of generality, we may assume that $x_1, \ldots, x_{d-2}$ is a superficial sequence for $\mathfrak{q}$. Set $\bar{A} := A/(x_1, \ldots, x_{d-2})$ and $\bar{\mathfrak{q}} := \mathfrak{q}\bar{A}$. Then, $\dim(\bar{A}) = 2$ and $\operatorname{depth}(\bar{A}) \ge 1$. Applying Lemma 4 and Theorem 4, we obtain

$$e_2(\mathfrak{q}) = e_2(\bar{\mathfrak{q}}) \le 0. \qquad \square$$

From Lemma 6, we get the following corollary.

Corollary 3 *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring with* $\dim(A) = d \ge 2$ *and* $\operatorname{depth} A \ge d - 1$. *Let* $\mathfrak{q}$ *be a parameter ideal of A such that* $\operatorname{depth} G_{\mathfrak{q}}(A) \ge d - 2$. *If* $e_i(\mathfrak{q}) = 0$ *for some* $i \in \{1, \ldots, d-1\}$, *then* $e_{i+1}(\mathfrak{q}) = \cdots = e_d(\mathfrak{q}) = 0$.

Proof Suppose that $e_1(\mathfrak{q}) = 0$. By [7, Proposition 3.4], A is Cohen–Macaulay. It follows that $e_2(\mathfrak{q}) = \cdots = e_d(\mathfrak{q}) = 0$.

Suppose that $d \geq 3$ and $\mathfrak{q} = (x_1, \ldots, x_d)$. It is sufficient to prove for $i = d-1$. Without loss of generality, assume that $x = x_1$ is superficial for $\mathfrak{q}$. Set $\bar{A} = A/(x)$ and $\bar{\mathfrak{q}} = \mathfrak{q}\bar{A}$. By Lemma 4, $e_{d-1}(\bar{\mathfrak{q}}) = e_{d-1}(\mathfrak{q}) = 0$. Hence, $P_{\bar{\mathfrak{q}}}(0) = 0$. By applying Lemma 6 for the ring $\bar{A}$, we have

$$(-1)^{d-1}\Delta(P_{\bar{\mathfrak{q}}} - H_{\bar{\mathfrak{q}}})(n) \geq 0 \quad \text{for all } n \geq 1-d.$$

This implies

$$(-1)^{d-1}[P_{\bar{\mathfrak{q}}}(n+1) - H_{\bar{\mathfrak{q}}}(n+1)] \geq (-1)^{d-1}[P_{\bar{\mathfrak{q}}}(n) - H_{\bar{\mathfrak{q}}}(n)] \quad \text{for all } n \geq 1-d.$$

In particular, we have

$$0 = (-1)^{d-1}[P_{\bar{\mathfrak{q}}}(0) - H_{\bar{\mathfrak{q}}}(0)] \leq (-1)^{d-1}[P_{\bar{\mathfrak{q}}}(n) - H_{\bar{\mathfrak{q}}}(n)] \leq 0 \quad \text{for all } n \geq 0.$$

It follows that

$$H_{\bar{\mathfrak{q}}}(n) = P_{\bar{\mathfrak{q}}}(n) \quad \text{for all } n \geq 0.$$

This gives $n(\bar{\mathfrak{q}}) \leq -1$. Since depth $G_{\mathfrak{q}}(A) \geq d-2 \geq 1$, $n(\bar{\mathfrak{q}}) = n(\mathfrak{q}) + 1$. Therefore, $n(\mathfrak{q}) = n(\bar{\mathfrak{q}}) - 1 \leq -2$. By Remark 2, we get $e_d(\mathfrak{q}) = 0$. □

4 Hilbert Coefficients and the Depth of Associated Graded Ring

In this section, we study a relationship between the vanishing of Hilbert coefficients $e_i(\mathfrak{q})$ and the depth of $G_{\mathfrak{q}}(A)$. Recall that a sequence $x_1, \ldots, x_s$ of A is said to be *d-sequence* if it satisfies one of the following two equivalent conditions:

(a) $(x_1, \ldots, x_{i-1}) : x_i x_k = (x_1, \ldots, x_{i-1}) : x_k$ for $1 \leq i \leq k \leq s$;
(b) $[(x_1, \ldots, x_{i-1}) : x_i] \cap \mathfrak{q} = (x_1, \ldots, x_{i-1})$ for $1 \leq i \leq s$, and $\mathfrak{q} = (x_1, \ldots, x_s)$.

Let $\mathfrak{q}$ be a parameter ideal of A generated by a d-sequence $x_1, \ldots, x_d$. Without loss of generality, assume that the residue field $A/\mathfrak{m}$ is infinite and $x_1, \ldots, x_d$ is a superficial sequence for $\mathfrak{q}$. By [12, Theoerem 1.1], $H^0_{\mathfrak{m}}(A/(x_1, \ldots, x_{i-1})) = (0_{A/(x_1,\ldots,x_{i-1})} : x_i)$. Applying [12, Theorem 4.1], we obtain

$$(-1)^d e_d(\mathfrak{q}) = \ell(H^0(A)) \tag{6}$$

and

$$(-1)^{d-i} e_{d-i}(\mathfrak{q}) = \ell(H^0(A/(x_1, \ldots, x_i))) - \ell(H^0(A/(x_1, \ldots, x_{i-1}))) \geq 0 \tag{7}$$

for $i = 1, \ldots, d-1$. As a consequence of [12, Theorem 4.1], the following proposition gives the sign of Hilbert coefficients.

Proposition 2 *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 1$. If $\mathfrak{q}$ is a parameter ideal of A generated by a d-sequence $x_1, \ldots, x_d$, then*

$$(-1)^i e_i(\mathfrak{q}) \geq 0 \quad \textit{for all } i = 1, \ldots, d.$$

From (6) and (7), we get the following corollary.

Corollary 4 *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 1$ and $\mathfrak{q}$ a parameter ideal of A generated by a d-sequence $x_1, \ldots, x_d$. For each $1 \leq i \leq d$,*

$$e_j(\mathfrak{q}) = 0 \quad \textit{for all } j \geq i \qquad \textit{if and only if} \qquad \text{depth } A \geq d - i + 1.$$

Lemma 7 *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 1$ with infinite residue field. Let $\mathfrak{q}$ be a parameter ideal of A generated by a superficial sequence $x_1, \ldots, x_d$ for $\mathfrak{q}$. The following conditions are equivalent:*

(i) $x_1, \ldots, x_d$ *is a d-sequence;*
(ii) $\operatorname{reg}(G_{\mathfrak{q}}(A)) = 0$.

Proof See [13, Corollary 5.2]. □

A Noetherian local ring $(A, \mathfrak{m})$ of dimension d is called *unmixed* if $\dim(\widehat{A}/\mathfrak{p}) = d$, for every $\mathfrak{p} \in \operatorname{Ass}(\widehat{A})$, where $\widehat{A}$ denotes the $\mathfrak{m}$-adic completion of A. If A is an unmixed ring, then we obtain a relationship between the vanishing of Hilbert coefficients and the depth of A.

Lemma 8 *Let $(A, \mathfrak{m})$ be a Noetherian unmixed local ring of dimension $d \geq 2$ and $\mathfrak{q}$ a parameter ideal of A generated by a d-sequence. Then for each $1 \leq i \leq d$, we have*

$$e_i(\mathfrak{q}) = 0 \quad \textit{if and only if} \quad \operatorname{depth} A \geq d - i + 1.$$

Proof By Lemma 4, the lemma holds for $i = d$. So, we need only to prove for the case $i < d$.

For $i \leq d - 1$, without loss of generality we may assume that the residue field of A is infinite. We can choose an element $x \in \mathfrak{q}/\mathfrak{m}\mathfrak{q}$ such that x is superficial for $\mathfrak{q}$. Since A is unmixed, we may assume that x is a regular element of A. Set $\bar{A} := A/(x)$ and $\bar{\mathfrak{q}} := \mathfrak{q}\bar{A}$. From Lemma 4, we have

$$e_{d-1}(\bar{\mathfrak{q}}, \bar{A}) = e_{d-1}(\mathfrak{q}, A).$$

Then, $e_{d-1}(\bar{\mathfrak{q}}, \bar{A}) = 0$ if and only if depth $\bar{A} \geq 1$, from Lemma 4. This is equivalent to depth $A =$ depth $\bar{A} + 1 \geq 2$. So, the lemma holds for $i = d - 1$.

If $d = 2$, the proof of the lemma is completed. If $d > 2$, by induction, we may assume that the lemma holds for $i = d - k$, $k = 1, \ldots, d - 2$. We need to prove that it holds for $i = d - k - 1$. By [4, Proposition 2.2], we may choose a superficial $a \in \mathfrak{q}/\mathfrak{m}\mathfrak{q}$ such that

$$\operatorname{Ass}(A/(a)) \subseteq \operatorname{Assh}(A/(a)) \cup \mathfrak{m},$$

where $\operatorname{Assh}(A) = \{\mathfrak{p} \in \operatorname{Ass}(A) \mid \dim(A/\mathfrak{p}) = \dim(A)\}$. Rewrite $S = A/(a)$ and $Q = \mathfrak{q}S$. Since $\operatorname{reg}(G_Q(S)) \leq \operatorname{reg}(G_{\mathfrak{q}}(A)) = 0$, $\operatorname{reg}(G_Q(S)) = 0$. By Lemma 7, Q is a parameter ideal generated by a d-sequence of S (note that S is not necessary unmixed). Denote $U = U_S(0)$ the unmixed component of (0) in A and $\overline{S} = S/U$, $\overline{Q} = QS$. By arguing as in the proof of [4, Theorem 2.1], we have $\overline{S}$ is an unmixed ring of dimension $d - 1$ and $\overline{Q}$ is a parameter ideal of $\overline{S}$ generated by a d-sequence.

To prove the lemma holds for $i = d - k - 1$, first suppose that $e_{d-k-1}(\mathfrak{q}, A) = 0$. Then, we have

$$e_{d-k-1}(\overline{Q}, \overline{S}) = e_{d-k-1}(Q, S) = e_{d-k-1}(\mathfrak{q}, A) = 0.$$

It follows, by inductive hypothesis, that depth $\overline{S} \geq (d - 1) - (d - k - 1) + 1 = k + 1$. Therefore, $H^i_{\mathfrak{m}}(\overline{S}) = 0 \; \forall i = 0, 1, \cdots, k$. From a short exact sequence

$$0 \longrightarrow U \longrightarrow S \longrightarrow \overline{S} \longrightarrow 0,$$

we get a long exact sequence of local cohomology

$$0 \longrightarrow H^0_{\mathfrak{m}}(U) \longrightarrow H^0_{\mathfrak{m}}(S) \longrightarrow H^0_{\mathfrak{m}}(\overline{S}) \longrightarrow H^1_{\mathfrak{m}}(U) \longrightarrow H^1_{\mathfrak{m}}(S) \longrightarrow H^1_{\mathfrak{m}}(\overline{S}) \longrightarrow \cdots$$
$$\longrightarrow H^k_{\mathfrak{m}}(U) \longrightarrow H^k_{\mathfrak{m}}(S) \longrightarrow H^k_{\mathfrak{m}}(\overline{S}) \longrightarrow \cdots$$

Since $\operatorname{Ass}(S) \subseteq \operatorname{Assh}(S) \cup \mathfrak{m}$, $H^0_{\mathfrak{m}}(S) = U$. Hence $H^i_{\mathfrak{m}}(U) = 0$ for all $i \geq 1$. From the above exact sequence, we have $H^i_{\mathfrak{m}}(S) = 0$ for all $i = 1, \ldots, k$. Now, consider the short exact sequence

$$0 \longrightarrow A \xrightarrow{a} A \longrightarrow S \longrightarrow 0,$$

we get

$$0 \longrightarrow H^0_{\mathfrak{m}}(A) \xrightarrow{a} H^0_{\mathfrak{m}}(A) \longrightarrow H^0_{\mathfrak{m}}(S) \longrightarrow H^1_{\mathfrak{m}}(A) \xrightarrow{a} H^1_{\mathfrak{m}}(A) \longrightarrow 0.$$

As A is unmixed, $H^1_{\mathfrak{m}}(A)$ is finitely generated. On the other hand, from above exact sequence, we have $H^1_{\mathfrak{m}}(A) = aH^1_{\mathfrak{m}}(A)$. By applying Nakayama Lemma, we obtain $H^1_{\mathfrak{m}}(A) = 0$. This implies that $H^i_{\mathfrak{m}}(S) = 0$ for all $i = 0, 1, \ldots, k$. Hence, depth $S \geq k + 1$ and depth $A = \operatorname{depth} S + 1 \geq k + 2 = d - (d - k - 1) + 1$.

Conversely, suppose that $\operatorname{depth}(A) \geq d - (d - k - 1) + 1 = k + 2$. Assume that $\mathfrak{q} = (x_1, \ldots, x_d)$ and $x_1, \ldots, x_{k+1}$ is a regular sequence of A. From $\operatorname{depth}(A) \geq k + 2$, we have $\operatorname{depth}(A/(x_1, \ldots, x_i)) \geq 1$ for $i = 1, \ldots, k + 1$. It follows that

$$H^0_{\mathfrak{m}}(A/(x_1, \ldots, x_i)) = 0 \quad \text{for } i = 0, \ldots, k + 1.$$

From (6) and (7), we get $e_{d-k-1}(\mathfrak{q}) = 0$. □

Now, we continue to study a relationship between the vanishing of Hilbert coefficients and the depth of the associated graded ring $G_{\mathfrak{q}}(A)$.

Lemma 9 *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 1$ and $\mathfrak{q}$ a parameter ideal of A generated by a d-sequence. Then,*

$$e_d(\mathfrak{q}) = 0 \quad \textit{if and only if} \quad \operatorname{depth}(G_{\mathfrak{q}}(A)) \geq 1.$$

Proof Suppose that depth $G_{\mathfrak{q}}(A) \geq 1$. Then, depth $A \geq 1$. It follows that $L = H^0_{\mathfrak{m}}(A) = 0$. From (6), we have $e_d(\mathfrak{q}) = 0$.

Conversely, assume that $e_d(\mathfrak{q}) = 0$. Since $\mathfrak{q}$ is an ideal generated by a d-sequence, $\operatorname{reg}(G_{\mathfrak{q}}(A)) = 0$. By the definition of Castelnuovo–Mumford regularity,

$$\operatorname{reg}(G_{\mathfrak{q}}(A)) = \max\{a_i(G_{\mathfrak{q}}(A)) + i \mid i = 0, \ldots, d\}.$$

Hence, $a_i(G_{\mathfrak{q}}(A)) + i \leq 0$ for all $i \geq 0$. On the other hand, from Lemma 4, we get depth $A > 0$. Thus, $a_0(G_{\mathfrak{q}}(A)) < a_1(G_{\mathfrak{q}}(A)) < 0$, by [5, Theorem 5.2]. This implies that $H^0_{G_+}(G_{\mathfrak{q}}(A)) = 0$ and hence $\operatorname{depth}(G_{\mathfrak{q}}(A)) \geq 1$. □

Lemma 10 [6, Lemma 2.2] *Let $(A, \mathfrak{m})$ be a Noetherian local ring and I an $\mathfrak{m}$-primary ideal of A. Let $x_1, \ldots, x_k$ be a superficial sequence for I. Denote $\bar{I} = I/(x_1, \ldots, x_k)$ and $\bar{A} = A/(x_1, \ldots, x_k)$. If* $\operatorname{depth}(G_{\bar{I}}(\bar{A})) \geq 1$, *then* $\operatorname{depth}(G_I(A)) \geq k + 1$.

The following theorem is a main result of this section.

Theorem 2 *Let $(A, \mathfrak{m})$ be a Noetherian unmixed local ring of dimension $d \geq 2$ and $\mathfrak{q}$ a parameter ideal of A generated by a d-sequence $x_1, \ldots, x_d$. Then for each $1 \leq i \leq d$, we have*

$$e_i(\mathfrak{q}) = 0 \quad \textit{if and only if} \quad \operatorname{depth} G_{\mathfrak{q}}(A) \geq d - i + 1.$$

Proof Suppose that depth $G_{\mathfrak{q}}(A) \geq d - i + 1$. This implies that depth $A \geq d - i + 1$. By Lemma 8, $e_i(\mathfrak{q}) = 0$.

Conversely, suppose that $e_i(\mathfrak{q}) = 0$. By Lemma 9, the theorem holds for $i = d$. By induction on i, we may assume that the lemma holds for $i = d-k$, for some $k \in \{1, \dots, d-2\}$, and need to prove it holds for $i = d-k-1$. From $e_{d-k-1}(\mathfrak{q}) = 0$ and by Lemma 8, we have depth $A \geq k+2$. Without loss of generality, assume that $x_1, \dots, x_{k+1}$ is a superficial sequence for $\mathfrak{q}$. Set $\bar{A} = A/(x_1, \dots, x_{k+1})$ and $\bar{\mathfrak{q}} = \mathfrak{q}\bar{A}$. Then,

$$0 \leq \operatorname{reg}(G_{\bar{\mathfrak{q}}}(\bar{A})) \leq \operatorname{reg}(G_{\mathfrak{q}}(A)) = 0.$$

Hence, $\operatorname{reg}(G_{\bar{\mathfrak{q}}}(\bar{A})) = 0$. By Lemma 7, $\bar{\mathfrak{q}}$ is the parameter ideal generated by a d-sequence. Notice that $\dim(\bar{A}) = d-k-1$, $\operatorname{depth}(\bar{A}) \geq 1$ and $e_{d-k-1}(\bar{\mathfrak{q}}) = e_{d-k-1}(\mathfrak{q}) = 0$. Applying Lemma 9, we obtain depth $G_{\bar{\mathfrak{q}}}(\bar{A}) \geq 1$. From Lemma 10, we get depth $G_{\mathfrak{q}}(A) \geq 1 + (k+1) = k+2$. □

If $\mathfrak{q}$ is a parameter ideal generated by a d-sequence and $\operatorname{depth}(A) \geq d-1$, by applying Theorem 5, we obtain the following corollary.

Corollary 5 *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 2$ and* $\operatorname{depth}(A) \geq d-1$. *Let $\mathfrak{q}$ be a parameter ideal of A generated by a d-sequence $x_1, \dots, x_d$. Then*

(i) $\operatorname{depth}(G_{\mathfrak{q}}(A)) \geq d-1$;
(ii) $e_2(\mathfrak{q}) = e_3(\mathfrak{q}) = \cdots = e_d(\mathfrak{q}) = 0$.

It is well known that if $\mathfrak{q}$ is an ideal generated by a d-sequence, then $\operatorname{reg}(G_{\mathfrak{q}}(A)) = 0$. More generally, we consider parameter ideals $\mathfrak{q}$ such that $\operatorname{reg}(G_{\mathfrak{q}}(A)) \leq 1$.

Theorem 3 *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 3$ and* $\operatorname{depth}(A) \geq k$ *for* $2 \leq k \leq d-1$. *Let $\mathfrak{q}$ be a parameter ideal of A such that* $\operatorname{reg}(G_{\mathfrak{q}}(A)) \leq 1$. *Then,*

(i) $\operatorname{depth}(G_{\mathfrak{q}}(A)) \geq k$;
(ii) $e_{d-k+2}(\mathfrak{q}) = e_{d-k+3}(\mathfrak{q}) = \cdots = e_d(\mathfrak{q}) = 0$.

Proof (i) Suppose that $\mathfrak{q} = (x_1, \dots, x_d)$. Without loss of generality, we assume that $x_1, \dots, x_{k-1}$ is a superficial sequence for $\mathfrak{q}$. Set $\bar{A} := A/(x_1, \dots, x_{k-1})$ and $\bar{\mathfrak{q}} := \mathfrak{q}\bar{A}$. Then, $\dim(\bar{A}) = d-k+1$ and $\operatorname{depth}(\bar{A}) \geq 1$. By [5, Theorem 5.2], $a_0(G_{\bar{\mathfrak{q}}}(\bar{A})) < a_1(G_{\bar{\mathfrak{q}}}(\bar{A}))$. Therefore,

$$\operatorname{reg}(G_{\bar{\mathfrak{q}}}(\bar{A})) = \max\{a_1(G_{\bar{\mathfrak{q}}}(\bar{A})) + 1, \dots, a_d(G_{\bar{\mathfrak{q}}}(\bar{A})) + d\} \leq \operatorname{reg}(G_{\mathfrak{q}}(A)) \leq 1.$$

Hence, $a_0(G_{\bar{\mathfrak{q}}}(\bar{A})) < a_1(G_{\bar{\mathfrak{q}}}(\bar{A})) \leq 0$. Thus, $H^0_{G_+}(G_{\bar{\mathfrak{q}}}(\bar{A})) = 0$. It follows that $\operatorname{depth}(G_{\bar{\mathfrak{q}}}(\bar{A})) \geq 1$. By Lemma 10, we get $\operatorname{depth}(G_{\mathfrak{q}}(A)) \geq 1 + (k-1) = k$.

(ii) Since $\operatorname{depth}(G_{\mathfrak{q}}(A)) \geq k$,

$$\operatorname{reg}(G_{\mathfrak{q}}(A)) = \max\{a_k(G_{\mathfrak{q}}(A)) + k, \dots, a_d(G_{\mathfrak{q}}(A)) + d\} \leq 1.$$

Hence, $a_i(G_{\mathfrak{q}}(A)) \leq 1-k$ for all $i \geq 0$. By Lemma 1 and Lemma 2,

$$n(\mathfrak{q}) \leq 1-k < 2-k.$$

From hypothesis $2 \leq k \leq d-1$ and Remark 2, we get

$$e_{d-k+2}(\mathfrak{q}) = e_{d-k+3}(\mathfrak{q}) = \cdots = e_d(\mathfrak{q}) = 0.$$

□

Corollary 6 *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring of dimension* $d \geq 3$ *and* $\operatorname{depth}(A) \geq d-1$. *Let* $\mathfrak{q}$ *be a parameter ideal of* A *such that* $\operatorname{reg}(G_{\mathfrak{q}}(A)) \leq 1$. *Then,*

(i) $\operatorname{depth}(G_{\mathfrak{q}}(A)) \geq d-1$;
(ii) $e_3(\mathfrak{q}) = e_4(\mathfrak{q}) = \cdots = e_d(\mathfrak{q}) = 0$.

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