Annales Univ. Sci. Budapest., Sect. Comp. 46 (2017) 327–340

SEGRE'S UPPER BOUND FOR THE REGULARITY INDEX OF $2n+2$ NON-DEGENERATE DOUBLE POINTS IN $\mathbb{P}^n$

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Communicated by Bui Minh Phong

(Received May 20, 2017; accepted August 6, 2017)

Abstract. We prove the Segre's upper bound for the regularity index of $2n+2$ non-degenerate double points that do not exist n+1 points lying on a $(n-2)$-plane in $\mathbb{P}^n$.

1. Introduction

Let $P_1, ..., P_s$ be a set of distinct points in a projective space with n-dimension $\mathbb{P}^n := \mathbb{P}^n_k$, with k as an algebraically closed field. Let $\wp_1, ..., \wp_s$ be the homogeneous prime ideals of the polynomial ring $R := k[x_0, ..., x_n]$ corresponding to the points $P_1, ..., P_s$. Let $m_1, ..., m_s$ be positive integers and $I = \wp_1^{m_1} \cap \cdots \cap \wp_1^{m_1}$. Denote $Z = m_1P_1 + \cdots + m_sP_s$ the zero-scheme defined by I, and we call Z a set of s fat points in $\mathbb{P}^n$.

The homogeneous coordinate ring of Z is

$$A = R/(\wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s}).$$

The ring $A = \oplus_{t \geq 0} A_t$ is a one-dimension Cohen-Macaulay k-graded algebra whose multiplicity is $e(A) = \sum\limits_{i=1}^{s} \binom{m_i+n-1}{n}$. The Hilbert function $H_A(t) =$

Key words and phrases: Fat points, regularity index, zero-scheme.
2010 Mathematics Subject Classification: Primary 14C20, Secondary 13D40.

$= \dim_k A_t$ increases strictly until it reaches the multiplicity $e(A)$, at which it stabilizes. The regularity index of Z is defined to be the least integer t such that $H_A(t) = e(A)$, and we denote it by $\operatorname{reg}(Z)$ (or $\operatorname{reg}(A)$).

In 1961, Segre (see [10]) showed the upper bound for regularity index of generic fat points $Z = m_1P_1 + \cdots + m_sP_s$ in $\mathbb{P}^2$:

$$\operatorname{reg}(Z) \le \max\Big\{m_1 + m_2 - 1, \Big[\frac{m_1 + \cdots + m_s}{2}\Big]\Big\}$$

with $m_1 \ge \cdots \ge m_s$.

For arbitrary fat points $Z = m_1P_1 + \cdots + m_sP_s$ in $\mathbb{P}^2$, in 1969 Fulton (see [9]) gave the following upper bound:

$$\operatorname{reg}(Z) \le m_1 + \cdots + m_s - 1.$$

This bound was later extended to arbitrary fat points in $\mathbb{P}^n$ by Davis and Geramita (see [6]). They also showed that this bound is attained if and only if points $P_1, ..., P_s$ lie on a line in $\mathbb{P}^n$.

A set of fat points $Z = m_1P_1 + \cdots + m_sP_s$ in $\mathbb{P}^n$ is said to be in general position if no $j + 2$ of the points $P_1, \ldots, P_s$ are on any j-plane for $j < n$. A set of fat points $Z = m_1P_1 + \cdots + m_sP_s$ of $\mathbb{P}^n$ is said to be non-degenerate if all points $P_1, \ldots, P_s$ do not lie on a hyperplane of $\mathbb{P}^n$. In 1991, Catalisano (see [3], [4]) extended Segre's result to fat points in general position in $\mathbb{P}^2$, and later Catalisano, Trung and Valla (see [5]) extended the result to fat points in general position in $\mathbb{P}^n$, they proved:

$$\operatorname{reg}(Z) \le \max\Big\{m_1 + m_2 - 1, \Big[\frac{m_1 + \cdots + m_s + n - 2}{n}\Big]\Big\}.$$

In 1996, N.V. Trung gave the following conjecture: Let $Z = m_1P_1 + \cdots +$ $+m_sP_s$ be arbitrary fat points in $\mathbb{P}^n$. Then

$$\operatorname{reg}(Z) \le \max\Big\{T_j \mid j = 1, ..., n\Big\},$$

where

$$T_j = \max\Big\{\Big[\frac{\sum_{l=1}^{q} m_{i_l} + j - 2}{j}\Big] \mid P_{i_1}, ..., P_{i_q} \text{ lie on a } j\text{-plane}\Big\}.$$

This upper bound nowadays is called the Segre's upper bound.

The Segre's upper bound is proved right in projective spaces with $n = 2$, $n = 3$ (see [12], [13]), for the case of double points $Z = 2P_1 + \cdots + 2P_s$ in $\mathbb{P}^n$ with $n = 4$ (see [14]) by Thien; also for case $n = 2, n = 3$, independently by Fatabbi and Lorenzini (see [7], [8]).

In 2012, Benedetti, Fatabbi and Lorenzini proved the Segre's bound for any set of $n+2$ non-degenerate fat points $Z = m_1P_1 + \cdots + m_{n+2}P_{n+2}$ of $\mathbb{P}^n$ (see [1]), and independently Thien also proved the Segre's bound for a set of $s+2$ fat points which is not on a $(s-1)$-space in $\mathbb{P}^n, s \le n$ (see [15]).

Recently, Ballico, Dumitrescu and Postinghen proved the Segre's upper bound for the case $n+3$ non-degenerate fat points $Z = m_1P_1 + \cdots + m_{n+3}P_{n+3}$ in $\mathbb{P}^n$ (see [2]) and Sinh proved the Segre's upper bound for the regularity index of $2n+1$ double points $Z = 2P_1 + \cdots + 2P_{2n+1}$ that do not exist $n+1$ points lying on a $(n-2)$-plane in $\mathbb{P}^n$ (see [11]). Up to now, there have not been any other result of Trung's conjecture published yet.

In this article, we prove the Segre's upper bound in the case $2n+2$ non-degenerate double points $Z = 2P_1 + \cdots + 2P_{2n+2}$ that do not exist $n+1$ points lying on a $(n-2)$-plane in $\mathbb{P}^n$.

2. Preliminaries

We will use the following lemmas which have been proved. The first lemma allows us to compute the regularity index by induction.

Lemma 2.1. [5, Lemma 1]. *Let $P_1, ..., P_r, P$ be distinct points in $\mathbb{P}^n$, and let $\wp$ be the defining ideal of P. If $m_1, ..., m_r$ and a are positive integers, $J = \wp_1^{m_1} \cap \cdots \cap \wp_r^{m_r}$, and $I = J \cap \wp^a$, then*

$$\operatorname{reg}(R/I) = \max\Big\{a-1, \operatorname{reg}(R/J), \operatorname{reg}(R/(J+\wp^a))\Big\}.$$

To compute $\operatorname{reg}(R/(J+\wp^a))$, we need the following lemma.

Lemma 2.2. [5, Lemma 3]. *Let $P_1, ..., P_r$ be distinct points in $\mathbb{P}^n$ and $a, m_1, ..., m_r$ positive integers. Put $J = \wp_1^{m_1} \cap \cdots \cap \wp_r^{m_r}$ and $\wp = (x_1, ..., x_n)$. Then*

$$\operatorname{reg}(R/(J+\wp^a)) \le b$$

if and only if $x_0^{b-i}M \in J + \wp^{i+1}$ for every monomial M of degree i in $x_1, ..., x_n$, $i = 0, ..., a-1$.

To find such a number b, we will find t hyperplanes $L_1, ..., L_t$ avoiding P such that $L_1 \cdots L_t M \in J$. For $j = 1, ..., t$, since we can write $L_j = x_0 + G_j$ for some linear form $G_j \in \wp$, we get $x_0^t M \in J + \wp^{i+1}$. Therefore, if we put

$$\delta = \max\Big\{t + i | M \text{ is a monomial of degree } i,\, 0 \le i \le a-1\Big\}$$

then

$$\operatorname{reg}(R/(J+\wp^a)) \leq \delta.$$

The hyperplanes $L_1, ..., L_t$ will be constructed by the help of the following lemma.

Lemma 2.3. [5, Lemma 4]. *Let $P_1, ..., P_r, P$ be distinct points in general position in $\mathbb{P}^n$, let $m_1 \geq \cdots \geq m_r$ be positive ingeters, and let $J = \wp_1^{m_1} \cap \cdots \cap \wp_r^{m_r}$. If t is an integer such that $nt \geq \sum_{i=1}^{r} m_i$ and $t \geq m_1$, we can find t hyperplanes, say $L_1, ..., L_t$ avoiding P such that for every $P_l, l = 1, ..., r$, there exist m_l hyperplanes of $\{L_1, ..., L_t\}$ passing through P_l.*

The two following lemmas are used to prove main results by induction.

Lemma 2.4. [11, Proposition 2.1]. *Let $X = \{P_1, ..., P_{2n+1}\}$ be a set of $2n+1$ distinct points that do not exist $n+1$ points of X lying on a $(n-2)$-plane in $\mathbb{P}^n$. Let $\wp_i$ be the homogeneous prime ideal corresponding $P_i, i = 1, ..., 2n+1$. Let*

$$Z = 2P_1 + \cdots + 2P_{2n+1}.$$

Put

$$T_j = \max\{[\frac{1}{j}(2q+j-2)] \mid P_{i_1}, ..., P_{i_q} \text{ lie on a } j\text{-plane}\},$$

$$T_Z = \max\{T_j \mid j = 1, ..., n\}.$$

Then, there exists a point $P_{i_0} \in X$ such that

$$\operatorname{reg}(R/(J+\wp_{i_0}^2)) \leq T_Z,$$

where

$$J = \bigcap_{k \neq i_0} \wp_k^2.$$

Lemma 2.5. [11, Proposition 2.2]. *Let $X = \{P_1, ..., P_{2n+1}\}$ be a set of $2n+1$ distinct points which do not exist $n+1$ points of X lying on a $(n-2)$-plane in $\mathbb{P}^n$. Let $Y = \{P_{i_1}, ..., P_{i_s}\}, 2 \leq s \leq 2n$, be a subset of X. Let $\wp_i$ be the homogeneous prime ideal corresponding $P_i, i = 1, ..., 2n+1$. Let*

$$Z = 2P_1 + \cdots + 2P_{2n+1}.$$

Put

$$T_j = \max\{[\frac{1}{j}(2q+j-2)] \mid P_{i_1}, ..., P_{i_q} \text{ lie on a } j\text{-plane}\},$$

$$T_Z = \max\{T_j \mid j = 1, ..., n\}.$$

Then, there exists a point $P_{i_0} \in Y$ *such that*

$$\operatorname{reg}(R/(J + \wp_{i_0}^2)) \le T_Z,$$

where

$$J = \bigcap_{P_k \in Y \setminus \{P_{i_0}\}} \wp_k^2.$$

3. Segre's upper bound for the regularity index of $2n+2$ non-degenerate double points in $\mathbb{P}^n$

From now on, we consider a hyperplane and its identical defining linear form. These following propositions are important for proving of Segre' upper bound.

Proposition 3.1. *Let* $X = \{P_1, ..., P_{2n+2}\}$ *be a non-degenerate set of* $2n+2$ *distinct points that do not exist* $n+1$ *points of* X *lying on a* $(n-2)$*-plane in* $\mathbb{P}^n$*. Let* $\wp_i$ *be the homogeneous prime ideal corresponding* $P_i, i = 1, ..., 2n+2$*, and*

$$Z = 2P_1 + \cdots + 2P_{2n+2}.$$

Put

$$T_j = \max\left\{ \left[\tfrac{1}{j}(2q + j - 2)\right] \mid P_{i_1}, ..., P_{i_q} \text{ lie on a } j\text{-plane} \right\},$$

$$T_Z = \max\{T_j \mid j = 1, ..., n\}.$$

Then, there exists a point $P_{i_0} \in X$ *such that*

$$\operatorname{reg}(R/(J + \wp_{i_0}^2)) \le T_Z,$$

where

$$J = \bigcap_{k \ne i_0} \wp_k^2.$$

Proof. We denote $|H|$ by the number points of X lying on a j-plane H. The proposition was proved in projective spaces with $n \le 4$ (see [7], [8], [12]–[14]). Thus, we will prove the case with $n \ge 5$.

We can see that there are $(n-1)$-planes $H_1, ..., H_d$ in $\mathbb{P}^n$ with d as the least integer such that the two following conditions satisfied:

(i) $X \subset \cup_{i=1}^{d} H_i$,

(ii) $\mid H_i \cap (X) \setminus \bigcup_{j=1}^{i-1} H_j \mid = \max\{|H \cap (X \setminus \bigcup_{j=1}^{i-1} H_j)| \mid H$ is an $(n-1)$-plane$\}$.

Since X non-degenerate and $n+1$ points do not lie on a $(n-2)$-plane, $2 \leq d \leq 3$. We consider the following cases:

Case 1. $d = 3$. Since a hyperplane always passes through at least n points of X and $d = 3$, we have the two following cases:

(i) $|H_1| = n,\ |H_2| = n,\ |H_3| = 2$.

(ii) $|H_1 = n+1| = |H_2 \backslash H_1| = n,\ |H_3| = 1$.

Case 1.1. $|H_1| = n,\ |H_2| = n,\ |H_3| = 2$. Since $|H_1| = n$, there do not exist $n+1$ points of X lying on a hyperplane. Therefore, X is general position. By Lemma 2.3 and Lemma 2.2 we have

$$\operatorname{reg}(R/(J + \wp_{i_0}^2)) \leq T_Z.$$

Case 1.2. $|H_1| = n+1, |H_2| = n, |H_3| = 1$. We may assume that $P_1 \in H_3$. Choose $P_1 = P_{i_0} = (1, 0, ..., 0)$, then $\wp_{i_0} = (x_1, ..., x_n)$. Clearly, H_1, H_2 avoiding P_{i_0}. We have $H_1 H_1 H_2 H_2 \in J$ for every monomial $M = x_1^{c_1} \cdots x_n^{c_n}, c_1 + \cdots + {} + c_n = i,\ i = 0, 1$. By Lemma 2.2 we have

$$\operatorname{reg}(R/(J + \wp_{i_0}^2)) \leq 4 + i \leq 5 \leq T_Z.$$

Case 2. $d = 2$. We have $X \subset H_1 \cup H_2$. Therefore, $|H_1| \geq n+1$ and $H_1 \geq |H_2|$. We call q the number points of X lying on $H_2 \backslash H_1$, we have $1 \leq q \leq n+1$, without loss of generality, we assume $P_1, ..., P_q \in H_2 \backslash H_1$. Put $Y = \{P_1, ..., P_q\}$. Since $n+1$ points of X do not lie on a $(n-2)$-plane, Y does not lie on a $(q-3)$-plane. We consider the following cases:

Case 2.1. Y lies on a $(q-1)$-plane and Y does not lie on a $(q-2)$-plane. Choose $P_q = P_{i_0} = (1, 0, ..., 0)$, $P_1 = (0, \underbrace{1}_{2}, ..., 0), ..., P_{q-1} = (0, ..., \underbrace{1}_{q}, ..., 0)$, then $\wp_{i_0} = (x_1, ..., x_n)$. Since we always have a $(q-2)$-plane, say K, passing through $P_1, ..., P_{q-1}$ and avoiding P_{i_0}; therefore, we always have a hyperplane, say L, containing K and avoiding P_{i_0}. We have $H_1 H_1 L L \in J$. Thus $H_1 H_1 L L M \in J$ for every monomial $M = x_1^{c_1} \cdots x_n^{c_n}, c_1 + \cdots + c_n = i, i = 0, 1$. By Lemma 2.2 we have

$$\operatorname{reg}(R/(J + \wp_{i_0}^2)) \leq 4 + i \leq 5 \leq T_Z.$$

Case 2.2. Y lies on a $(q-2)$-plane $\alpha, q \geq 3$. We consider the following cases of Y :

Case 2.2.1. There are $q-1$ points of Y lying on a $(q-3)$-plane. Assume that $P_1, ..., P_{q-1}$ lying on a $(q-3)$-plane, say K and $P_q \notin K$. Choose $P_q = P_{i_0} = {} = (1, 0, ..., 0)$, then $\wp_{i_0} = (x_1, ..., x_n)$. Since $q \leq n+1$, we have $q - 3 \leq n - 2$ and $P_{i_0} \notin K$, we always have a hyperplane L containing K and avoiding

P_{i_0}. We have $H_1H_1LL \in J$, thus $H_1H_1LLM \in J$ for every monomial $M = x_1^{c_1}\cdots x_n^{c_n}, c_1+\cdots+c_n = i, i = 0,1$. By Lemma 2.2 we have

$$\operatorname{reg}(R/(J+\wp_{i_0}^2)) \leq 4+i \leq 5 \leq T_Z.$$

Case 2.2.2. There are not $q-1$ points of Y lying on a $(q-3)$-plane. We consider the three following cases of q :

Case 2.2.2.1. $q \geq 5$. Since any $(q-3)$-planes only pass through $q-2$ points of Y. Choose $P_q = P_{i_0} = (1,0,...,0), P_1 = (0,\underbrace{1}_{2},0...,0), ..., P_{q-2} =$ $= (0,...0,\underbrace{1}_{q-1},0,...,0)$. Put $m_l = 2-i+c_l, l = 1,...,q-2, m_{q-1} = 2$ and

$$t = \max\Big\{2, [(\sum_{i=1}^{q-1} m_l + (q-2)-1)/(q-2)]\Big\}.$$

We have

$$\begin{aligned} t+i = {} & \max\{2,[(\textstyle\sum_{i=1}^{q-1} m_l+q-3)/(q-2)]\}+i \leq \\ & \leq \max\{2+i,[(\textstyle\sum_{i=1}^{q-1} m_l+(q-2)i+q-3)/(q-2)]\} \leq \\ & \leq \max\{2+i,[(3q-4)/(q-2)] \leq 3. \end{aligned}$$

Therefore,

$$t \leq 3-i.$$

By Lemma 2.2, we can find t $(q-3)$-planes, say $G_1,...,G_t$ avoiding P_{i_0} such that for every point $P_l, l = 1,...,q-1$, there are m_l $(q-3)$-planes of $G_1,...,G_t$ passing through P_l. With $j = 1,...,t$ we find a hyperplane L_j containing G_j and avoiding P_{i_0}. Therefore

$$L_1\cdots L_t \in \wp_1^{m_1}\cap\cdots\cap\wp_{q-2}^{m_{q-2}}\cap\wp_{q-1}^2.$$

Moreover, since $H_1H_1 \in \wp_{q+1}^2\cap\cdots\cap\wp_{2n+2}^2$ and $M \in \wp_1^{i-c_1}\cap\cdots\cap\wp_{q-2}^{i-c_{q-2}}$, then

$$H_1H_1L_1\cdots L_tM \in J.$$

By Lemma 2.2 we have

$$\operatorname{reg}(R/(J+\wp_{i_0}^2)) \leq 2+(3-i)+i \leq T_Z.$$

Case 2.2.2.2. $q = 4$. We have $P_1, P_2, P_3, P_4 \notin H_1$. Choose $P_1 = P_{i_0} =$ $= (1,0,...,0)$, $P_3 = (0,\underbrace{1}_{2},0,...,0)$, $P_4 = (0,0,\underbrace{1}_{3},0,...,0)$, ..., $P_{n+1} =$ $= (0,...,0,\underbrace{1}_{n},0)$, $P_{n+2} = (0,...,0,\underbrace{1}_{n+1})$, therefore $\wp_{i_0} = (x_1,...,x_n)$. We call l_1 a line passing through P_2, P_3; l_2 a line passing through P_3, P_4; l_3 a line passing through P_2, P_4. We consider the two following cases of i:

a) $i = 0$. With $j = 1, 2, 3$, since $P_{i_0} \notin l_j$, then we always have a hyperplane L_j containing l_j and avoiding P_{i_0}. We have $H_1H_1L_1L_2L_3 \in J$, thus $H_1H_1L_1L_2L_3M \in J$. By Lemma 2.2 we have

$$\operatorname{reg}(R/(J + \wp_{i_0}^2)) \leq 5 \leq T_Z.$$

b) $i = 1$. Since $c_1 + \cdots + c_n = 1$, then there exists $j \in \{1, ..., n\}$ such that $c_j = 1, c_k = 0, k \in \{1, ..., n\} \backslash \{j\}$.

$\circ$ If $j \in \{1, 2\}$, assume that $c_1 = 1$ then

$$M \in \wp_4 \cap \wp_5 \cap \cdots \cap \wp_{n+2}.$$

We have a $(n-2)$-plane, say K_1 passing through $P_{n+3}, ..., P_{2n-1}$ and l_1, a $(n-2)$-plane, say K_2 passing through P_{2n}, P_{2n+1} and l_1, a $(n-2)$-plane, say K_3 passing through P_4, P_{2n+2} avoiding P_{i_0}. With $i = 1, 2, 3$, we always have hyperplanes L_i containing K_i and avoiding P_{i_0}. We have

$$H_1L_1L_2L_3 \in \wp_2^2 \cap \wp_3^2 \cap \wp_4 \cap \wp_5 \cap \cdots \cap \wp_{n+2} \cap \wp_{n+3}^2 \cap \cdots \cap \wp_{2n+2}^2.$$

Therefore

$$H_1L_1L_2L_3M \in J.$$

By Lemma 2.2 we have

$$\operatorname{reg}(R/(J + \wp_{i_0}^2)) \leq 4 + i \leq T_Z.$$

$\circ$ If $j \in \{3, ..., n\}$, assume that $c_3 = 1$ then

$$M \in \wp_3 \cap \wp_4 \cap \wp_6 \cap \cdots \cap \wp_{n+2}.$$

We call l_1 a line passing through P_2, P_3 and l_2 a line passing through P_2, P_4. With $i = 1, 2$, since $P_{i_0} \notin l_i$, then we always have hyperplanes L_i containing l_i and avoiding P_{i_0}. We have

$$L_1L_2 \in \wp_2^2 \cap \wp_3 \cap \wp_4$$

Since $H_1H_1 \in \wp_5^2 \cap \cdots \cap \wp_{2n+2}^2$ then

$$H_1H_1L_1L_2M \in J.$$

By Lemma 2.2 we have

$$\operatorname{reg}(R/(J + \wp_{i_0}^2)) \leq 4 + i \leq T_Z.$$

Case 2.2.2.3. $q = 3$. We have $P_1, P_2, P_3 \notin H_1$. We call l a line passing through P_1, P_2, P_3 and $W = \{P_4, ..., P_{2n+2}\}$ are the points of X lying on $H_1 \cap X$, then there are $(n-2)$-planes $Q_1, ..., Q_r$ in $\mathbb{P}^n$ such that the two following conditions satisfied:

(i) $W \subset \cup_{i=1}^{r} Q_i$,

(ii) $| Q_i \cap (W \setminus \bigcup_{j=1}^{i-1} Q_j) | = \max\{|Q \cap (W \setminus \bigcup_{j=1}^{i-1} Q_j)| \mid Q$ is a $(n-2)$-plane$\}$.

Since $n+1$ of X do not lie on a $(n-2)$-plane, then we consider the two following cases of Q_1:

a) $|Q_1| = n$. We have $r = 2$ and $|Q_2| = n-1$. Put $U = \{P_4, ..., P_{n+2}\}$ to be $n-1$ points lying on Q_2 v $T = \{P_1, ..., P_{n+2}\}$. We consider the two following cases of T:

a.1) T does not lie on a $(n-1)$-plane. Since P_1, P_2, P_3 lie on a line l, then we always have a hyperplane containing l and passing through $n-2$ points of U. Assume that L to be a hyperplane containing l and passing through points $P_4, ..., P_{n+1}$. Clearly, the hyperplane L avoiding P_{n+2} (if not, then T lies on a $(n-1)$-plane). Choose $P_{n+2} = P_{i_0} = (1, 0, ..., 0)$, then $\wp_{i_0} = (x_1, ..., x_n)$. Since $P_{i_0} \notin Q_1$, therefore we always have a hyperplane L_1 containing Q_1 and avoiding P_{i_0}. We have $LLL_1L_1 \in J$ then $LLL_1L_1M \in J$ for every monomial $M = x_1^{c_1} \cdots x_n^{c_n}, c_1 + \cdots + c_n = i, i = 0, 1$. By Lemma 2.2 we have

$$\operatorname{reg}(R/(J + \wp_{i_0}^2)) \leq 4 + i \leq 5 \leq T_Z.$$

a.2) T lies on a $(n-1)$-plane, say H. Assume that $|Q_1 \cap H \cap X| = s$. When hyperplane H passing through $n+2+s$ points of X. Consider $n-s$ points lying on $Q_1 \setminus H$, say $P_{i_1}, ..., P_{i_{n-s}} \in Q_1 \setminus H$.

a.2.1) Case $P_{i_1}, ..., P_{i_{n-s}}$ lie on a $(n-s-1)$-plane and they do not lie on a $(n-s-2)$-plane. Choose $P_{i_1} = P_{i_0} = (1, 0, ..., 0)$, then $\wp_{i_0} = (x_1, ..., x_n)$. Since we always have a $(n-s-2)$-plane, say β passing through $P_{i_2}, ..., P_{i_{n-s-1}}$. Moreover, since $n-s-2 \leq n-2$ then we always have a hyperplane L containing β and avoiding P_{i_0}. We have $HHLL \in J$ then $HHLLM \in J$ for every monomial $M = x_1^{c_1} \cdots x_n^{c_n}, c_1 + \cdots + c_n = i, i = 0, 1$. By Lemma 2.2 we have

$$\operatorname{reg}(R/(J + \wp_{i_0}^2)) \leq 4 + i \leq 5 \leq T_Z.$$

a.2.2) Case $P_{i_1}, ..., P_{i_{n-s}}$ lie on a $(n-s-2)$-plane. Since P_1, P_2, P_3 lie on a line, then $P_1, P_2, P_3, P_{i_1}, ..., P_{i_{n-s}}$ lie on a $(n-s)$-plane. So, $n-1 \leq n-s \leq n$ or $0 \leq s \leq 1$.

• If $\{P_{i_1}, ..., P_{i_{n-s}}\}$ has $n-s-1$ points lying on a $(n-s-3)$-plane, say γ. Assume that $P_{i_1} \notin \gamma$, then choose $P_{i_1} = P_{i_0} = (1, 0, ..., 0)$, then $\wp_{i_0} = (x_1, ..., x_n)$. Since $P_{i_0} \notin \gamma$ therefore we always have a hyperplane L containing γ and avoiding P_{i_0}. We have $LLHH \in J$ then $LLHHM \in J$ for every monomial $M = x_1^{c_1} \cdots x_n^{c_n}, c_1 + \cdots + c_n = i, i = 0, 1$. By Lemma 2.2 we have

$$\operatorname{reg}(R/(J + \wp_{i_0}^2)) \leq 4 + i \leq 5 \leq T_Z.$$

• If $\{P_{i_1}, ..., P_{i_{n-s}}\}$ without $n-s-1$ points lying on a $(n-s-3)$-plane, then any $(n-s-3)$-plane only pass through $n-s-2$ points of $\{P_{i_1}, ..., P_{i_{n-s}}\}$. Choose $P_{i_1} = P_{i_0} = (1, 0, ..., 0)$, $P_{i_2} = (0, \underbrace{1}_{2}, 0, ..., 0), ..., P_{i_{n-s-1}} = (0, ..., 0, \underbrace{1}_{n-s-1}, 0,$ $..., 0)$ then $\wp_{i_0} = (x_1, ..., x_n)$. Put $m_l = 2 - i + c_l, l = 2, ..., n-s-1, m_{n-s} = 2$ and

$$t = \max\Big\{2, [(\sum_{i=1}^{n-s-1} m_l + (n-s-2) - 1)/(n-s-2)]\Big\}.$$

We have

$$\begin{aligned} t + i &= \max\{2, [(\textstyle\sum_{i=1}^{n-s-1} m_l + n - s - 3)/(n-s-2)]\} + i \leq \\ &\leq \max\{2+i, [(\textstyle\sum_{i=1}^{n-s-1} m_l + (n-s-2)i + n-s-3)/(n-s-2)]\} \leq \\ &\leq \max\{2+i, [(3(n-s-2)+2)/(n-s-2)]. \end{aligned}$$

✓ $s = 0$ or $n \geq 6$, we have

$$t \leq 3 - i.$$

By Lemma 2.3 we can find t $(q-3)$-planes, say $G_1, ..., G_t$ avoiding P_{i_0} such that for every point $P_l, l = 1, ..., q-1$, there are m_l $(q-3)$-planes of $G_1, ..., G_t$ passing through. With $j = 1, ..., t$ we find a hyperplane L_j containing G_j and avoiding P_{i_0}. Therefore

$$L_1 \cdots L_t \in \wp_{i_2}^{m_2} \cap \cdots \cap \wp_{i_{n-s-1}}^{m_{n-s-1}} \cap \wp_{i_{n-s}}^2.$$

So, $HHL_1 \cdots L_t M \in J$ for every monomial $M = x_1^{c_1} \cdots x_n^{c_n}, c_1 + \cdots + c_n = i, i = 0, 1$. By Lemma 2.2 we have

$$\operatorname{reg}(R/(J + \wp_{i_0}^2)) \leq 4 + i \leq 5 \leq T_Z.$$

✓ $s = 1$ and $n = 5$. Then hyperplane H pass through eight points of X and there are four points $P_{i_1}, P_{i_2}, P_{i_3}, P_{i_4}$ lying on a 2-plane, say $\gamma_1 \backslash H$. According to Case 2.2.2.2 we have proved it.

b) If $|Q_1| = n-1$, then $W = \{P_4, ..., P_{2n+2}\}$ lie on the general position in H_1. We call H a hyperplane containing l and passing through $n-3+u$ points of $W \cap H_1$. We have $u \geq 1$.

• If $u = 1$, then consider $n+1$ points of $H_1 \backslash H$. Without loss of generality, assume that $P_{n+2}, ..., P_{2n+2} \in H_1 \backslash H$. Put $V = \{P_{n+2}, ..., P_{2n+2}\}$. Since there do not exist n points of V lying on a $(n-2)$-plane. Choose $P_{n+2} = P_{i_0} =$

$= (1, 0, ..., 0)$, $P_{n+3} = (0, \underbrace{1}_{2}, 0, ..., 0), ..., P_{2n+1} = (0, , ..., 0, \underbrace{1}_{n}, 0)$ then $\wp_{i_0} =$ $= (x_1, ..., x_n)$. Put $m_l = 2 - i + c_l,\ l = n+3, ..., 2n+1,\ m_{2n+2} = 2$ and

$$t = \max\Big\{2, [(\sum_{i=n+3}^{2n+2} m_l + (n-1) - 1)/(n-1)]\Big\}.$$

We have

$$\begin{aligned} t + i = \max\{2, [(\sum_{i=n+3}^{2n+2} m_l + n - 2)/(n-1)]\} + i \leq \\ \leq \max\{2 + i, [(\sum_{i=n+3}^{2n+2} m_l + (n-1)i + n - 2)/(n-1)]\} \leq \\ \leq \max\{2 + i, [(3n-1)/(n-1)]\} \leq 3. \end{aligned}$$

Therefore

$$t \leq 3 - i.$$

By Lemma 2.3 we can find t $(n-2)$-planes $G_1, ..., G_t$ avoiding P_{i_0} such that for every $P_l, l = n+3, ..., 2n+2$, there are m_l $(n-2)$-planes of $G_1, ..., G_t$ passing through. With $j = 1, ..., t$ we find a hyperplane L_j containing G_j and avoiding P_{i_0}. Therefore

$$L_1 \cdots L_t \in \wp_{n+3}^{m_{n+3}} \cap \cdots \cap \wp_{2n+1}^{m_{2n+1}} \cap \wp_{2n+2}^2.$$

Moreover, since $HH \in \wp_1^2 \cap \cdots \cap \wp_{n+1}^2$ and $M \in \wp_{n+3}^{i-c_1} \cap \cdots \cap \wp_{2n+1}^{i-c_{n-1}}$ then

$$H_1 H_1 L_1 \cdots L_t M \in J.$$

By Lemma 2.2 we have

$$\operatorname{reg}(R/(J + \wp_{i_0}^2)) \leq 2 + (3 - i) + i \leq T_Z.$$

• If $u \geq 2$, then there are $n+2-u$ points, assume that $P_{i_1}, ..., P_{n+2-u} \in H_1 \backslash H$. Since $u \geq 2$ then $n+2-u \leq n$. Moreover, since $P_{i_1}, ..., P_{n+2-u}$ lie on the general position in H_1, then we have a $(n-u)$-plane, say π, passing through $n+1-u$ points $P_{i_2}, ..., P_{n+2-u}$ and avoiding P_{i_1}. Choose $P_{i_1} = P_{i_0} = (1, 0, ..., 0)$, then $\wp_{i_0} = (x_1, ..., x_n)$. Since $P_{i_0} \notin \pi$, we always have a hyperplane, say L, containing π and avoiding P_{i_0}. We have $HHLL \in J$, therefore $HHLLM \in J$ for every monomial $M = x_1^{c_1} \cdots x_n^{c_n}, c_1 + \cdots + c_n = i, i = 0, 1$. By Lemma 2.2 we have

$$\operatorname{reg}(R/(J + \wp_{i_0}^2)) \leq 4 + i \leq 5 \leq T_Z.$$

The proof of proposition 3.1 is completed. ■

From Lemma 2.4, Lemma 2.5 and Proposition 3.1, we get the following remark.

Remark 3.1. Let $X = \{P_1, ..., P_{2n+2}\}$ be a non-degenerate set of $2n+2$ distinct points that do not exist $n+1$ points of X lying on a $(n-2)$-plane in $\mathbb{P}^n$. Let $Y = \{P_{i_1}, ..., P_{i_s}\}, 2 \leq s \leq 2n+1$, be a subset of X. Let $\wp_i$ be the homogeneous prime ideal corresponding $P_i, i = 1, ..., 2n+1$, and

$$Z = 2P_1 + \cdots + 2P_{2n+2}.$$

Put

$$T_j = \max\Big\{\Big[\frac{1}{j}(2q+j-2)\Big] \mid P_{i_1}, ..., P_{i_q} \text{ lie on a } j\text{-plane}\Big\},$$

$$T_Z = \max\{T_j \mid j = 1, ..., n\}.$$

Then, there exists a point $P_{i_0} \in Y$ such that

$$\operatorname{reg}(R/(J + \wp_{i_0}^2)) \leq T_Z,$$

where

$$J = \bigcap_{P_k \in Y \setminus \{P_{i_0}\}} \wp_k^2.$$

The theorem below is the main result of this paper.

Theorem 3.2. *Let $X = \{P_1, ..., P_{2n+2}\}$ be a non-degenerate set of $2n+2$ distinct points that do not exist $n+1$ points of X lying on a $(n-2)$-plane in $\mathbb{P}^n$. Let*

$$Z = 2P_1 + \cdots + 2P_{2n+2}.$$

Then

$$\operatorname{reg}(Z) \leq \max\Big\{T_j \mid j = 1, ..., n\Big\} = T_Z,$$

where

$$T_j = \Big\{\Big[\frac{2q+j-2}{j}\Big] \mid P_{i_1}, ..., P_{i_q} \text{ lie on a } j\text{-plane}\Big\}.$$

Proof. Firstly, we have the following claim:

Let $X = \{P_1, ..., P_{2n+2}\}$ in $\mathbb{P}^n$, $Y = \{P_{i_1}, ..., P_{i_s}\}$ be a subset of X, $1 \leq s \leq$ $\leq 2n+1$. Then

$$\operatorname{reg}(R/J_s) \leq T_Z,$$

where

$$J_s = \bigcap_{P_i \in Y} \wp_i^2.$$

We will prove this claim by induction on number points of Y.
If $s = 1$. Let $\wp_1$ be the defining homogeneous prime ideal of P_1. Put $J_1 =$ $= \wp_1^2,\ A = R/J_1$. Then,

$$\operatorname{reg}(R/J_1) = 1 \leq T_Z.$$

Assume that the claim is right for all subsets Y of X, whose number points are smaller or equal $s-1$. Let $Y = \{P_{i_1}, ..., P_{i_s}\}$. By Remark 3.1, there exists a point $P_{i_0} \in Y$ such that

$$\operatorname{reg}(R/(J_{s-1} + \wp_{i_0}^2)) \leq T_Z, \tag{1}$$

where $J_{s-1} = \bigcap\limits_{P_i \in Y \setminus \{P_{i_0}\}} \wp_i^2$. Note that, J_{s-1} is the intersection of ideals containing $s-1$ double points of Y. By conjecture of induction, we have

$$\operatorname{reg}(R/J_{s-1}) \leq T_Z. \tag{2}$$

By Lemma 2.1 we have

$$\operatorname{reg}(R/J_s) = \Big\{1, \operatorname{reg}(R/(J_{s-1}), \operatorname{reg}(R/(J_{s-1} + \wp_{i_0}^2))\Big\}. \tag{3}$$

From (1), (2) and (3) we have

$$\operatorname{reg}(R/J_s) \leq T_Z.$$

The proof of the above claim is completed.

Now, we prove Theorem 3.2. Let $X = \{P_1, ..., P_{2n+2}\}$ in $\mathbb{P}^n$, by Proposition 3.1, there exists a point $P_{i_0} \in X$ such that

$$\operatorname{reg}(R/(J + \wp_{i_0}^2)) \leq T_Z. \tag{4}$$

where $J = \bigcap\limits_{P_i \in X \setminus \{P_{i_0}\}} \wp_i^2$. Note that, J is the intersection of ideals containing $2n+1$ double points of X. Therefore, by the above claim with $s = 2n+1$, we have

$$\operatorname{reg}(R/J) \leq T_Z. \tag{5}$$

By Lemma 2.1 we have

$$\operatorname{reg} R/I = \Big\{1, \operatorname{reg}(R/J), \operatorname{reg}(R/(J + \wp_{i_0}^2))\Big\} \tag{6}$$

where $I = J \cap \wp_{i_0}^2$.

From (4), (5) and (6) we have

$$\operatorname{reg}(Z) \leq T_Z.$$

The proof of Theorem 3.2 is completed. ■

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