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A WEIGHTED VOLUME ESTIMATE AND ITS APPLICATION TO BERNSTEIN TYPE THEOREMS IN GAUSS SPACE

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Abstract. A weighted area estimate for entire graphs with bounded weighted mean curvature in a Gauss space is given, with a short proof. Bernstein type theorems for self-shrinkers (Wang, 2011) as well as for graphic λ-hypersurfaces (Cheng and Wei, 2014) are immediate consequences.

1. Introduction. A *manifold with density* is a Riemannian manifold with a positive function e^{-f} used to weight both the volume and the perimeter area. The *weighted volume* of a region E is $\mathrm{Vol}_f(E) = \int_E e^{-f}\,dV$ and the *weighted area* of a hypersurface Σ is $\mathrm{Area}_f(\Sigma) = \int_\Sigma e^{-f}\,dA_\Sigma$, where dV and dA are the $(n+1)$-dimensional Riemannian volume and the n-dimensional Riemannian perimeter area elements, respectively.

The *weighted mean curvature* of a hypersurface Σ in such a manifold is defined as follows:

$$H_f(\Sigma) = H(\Sigma) + \langle \nabla f, \mathbf{n}\rangle,$$

where $\mathbf{n}$ is the unit normal vector field and $H = -\operatorname{div}\mathbf{n}$ is the Euclidean mean curvature of the hypersurface. If $H_f(\Sigma) = \lambda$, a constant, then Σ is called a λ-*hypersurface*, and if $H_f(\Sigma) = 0$, then Σ is said to be f-*minimal.*

The *Gauss space* $\mathbb{G}^{n+1}$, the Euclidean space $\mathbb{R}^{n+1}$ with the Gaussian probability density $e^{-f} = (2\pi)^{-(n+1)/2}e^{-|x|^2/2}$, is a typical example of a manifold with density and is of much interest to probabilists. For more details about manifolds with density, we refer the reader to [M1]–[M3], [MW].

In the Gauss space, f-minimal hypersurfaces are self-shrinkers and hyperplanes are λ-hypersurfaces. The weighted mean curvature of the hyperplane $\sum_{i=1}^{n+1} a_i x_i + a_0 = 0$ is $-a_0/(\sum_{i=1}^{n+1} a_i^2)^{1/2}$. It is well-known that hyperplanes solve the weighted isoperimetric problem, i.e., they minimize the weighted area for given weighted volume (see [B], [ST]). It should be mentioned that

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both the weighted volume of $\mathbb{G}^{n+1}$ and the weighted area of a hyperplane are finite.

In this paper, we use a short proof to establish a weighted area estimate for entire graphs with bounded weighted mean curvature. The Bernstein type theorems for graphic self-shrinkers [W] as well as for graphic λ-hypersurfaces [CW] are immediate consequences. These results were also proved by Ecker and Huisken [EH] and Guang [G], respectively, under the polynomial volume growth conditions.

2. Weighted area estimate and Bernstein type theorems. In $\mathbb{G}^{n+1}$, let Σ be the graph of a smooth function $u(\mathbf{x}) = x_{n+1}$ for $\mathbf{x} \in \mathbb{R}^n$, and $\mathbf{n}$ be its upward unit normal field. Extending $\mathbf{n}$ by translations along the x_{n+1}-axis we obtain a smooth vector field on $\mathbb{R}^{n+1}$, also denoted by $\mathbf{n}$. Along any vertical line, since $\operatorname{div}(\mathbf{n})$ remains unchanged while $\langle \nabla f, \mathbf{n}\rangle$ is increasing, $H_f = -\operatorname{div}(\mathbf{n}) + \langle \nabla f, \mathbf{n}\rangle$ is increasing.

Consider the differential n-form

$$w(X_1, \ldots, X_n) = \det(X_1, \ldots, X_n, \mathbf{n}),$$

where X_i, $i = 1, \ldots, n$, are smooth vector fields. Then $|w(X_1, \ldots, X_n)| \le 1$ for any unit normal vector fields X_i, $i = 1, \ldots, n$, and equality holds if and only if $X_1, \ldots, X_n$ are tangent to Σ.

We use the following notations:

- $E_1 = \{(\mathbf{x}, x_{n+1}) : x_{n+1} \le u(\mathbf{x})\}$, $E_2 = \{(\mathbf{x}, x_{n+1}) : x_{n+1} \le a\}$, with $a \in \mathbb{R}$, such that $\operatorname{Vol}_f(E_1) = \operatorname{Vol}_f(E_2)$, and let P the hyperplane $x_{n+1} = a$;
- $F = (E_1 - E_2) \cup (E_2 - E_1)$, the region bounded by P and Σ;
- $F^+ = E_2 - E_1$ and $F^- = E_1 - E_2$, the parts of F above and below Σ, respectively;
- B_R, S_R the $(n+1)$-ball and n-hypershere in $\mathbb{R}^{n+1}$ with center O and radius R, respectively;
- $\Sigma_R = \Sigma \cap B_R$, $P_R = P \cap B_R$, $F_R = F \cap B_R$, $F_R^+ = F^+ \cap B_R$ and $F_R^- = F^- \cap B_R$.

THEOREM 2.1. *If $H_f(\Sigma)$ is bounded, then*

$$\operatorname{Area}_f(\Sigma) \le \operatorname{Area}_f(P) + \tfrac{1}{2}(M - m)\operatorname{Vol}_f(F), \tag{2.1}$$

where $M = \sup H_f(\Sigma)$ and $m = \inf H_f(\Sigma)$.

Proof. Let R be so large that B_R intersects both F^+ and F^-. By Stokes' theorem with suitably chosen orientations of boundary parts (see the figure),

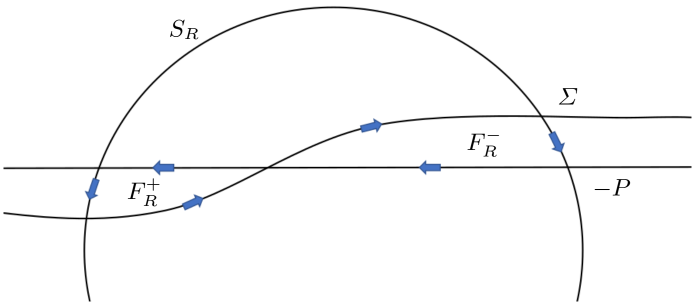
$$\operatorname{Area}_f(\Sigma_R) - \operatorname{Area}_f(P_R) + \int_{F\cap S_R} e^{-f} w$$

$$\le \int_{\Sigma_R} e^{-f} w - \int_{P_R} e^{-f} w + \int_{F\cap S_R} e^{-f} w = \int_{F_R} d(e^{-f} w) = \int_{F_R} d(\operatorname{div}(e^{-f}\mathbf{n}))\, dV$$

$$= \int_{F_R} (e^{-f} \operatorname{div}(\mathbf{n}) - e^{-f} \langle \nabla f, \mathbf{n} \rangle) \, dV = - \int_{F_R} e^{-f} H_f \, dV$$
$$= - \int_{F_R^+} e^{-f} H_f \, dV + \int_{F_R^-} e^{-f} H_f \, dV \le -m \operatorname{Vol}_f(F_R^+) + M \operatorname{Vol}_f(F_R^-).$$

Thus,

$$\operatorname{Area}_f(\Sigma_R) - \operatorname{Area}_f(P_R) + \int_{F_R} e^{-f} w \le -m \operatorname{Vol}_f(F_R^+) + M \operatorname{Vol}_f(F_R^-). \tag{2.2}$$



It is not hard to check that

$$\lim_{R\to\infty} \int_{S_R \cap F} e^{-f} w = \lim_{R\to\infty} e^{-R} \int_{S_R \cap F} w = 0,$$

and by the assumption that $\operatorname{Vol}_f(E_1) = \operatorname{Vol}_f(E_2)$,

$$\lim_{R\to\infty} \operatorname{Vol}_f(F_R^+) = \operatorname{Vol}_f(F^+) = \lim_{R\to\infty} \operatorname{Vol}_f(F_R^-) = \operatorname{Vol}_f(F^-) = \frac{1}{2} \operatorname{Vol}_f(F).$$

Taking the limit of both sides of (2.2) as R tends to infinity, we get (2.1). ∎

Corollary 2.2 (Bernstein type theorem for λ-hypersurfaces [CW]). *If Σ is an entire graphic λ-hypersurface, then it must be a hyperplane.*

Proof. Because $M - m = 0$ and P is weighted area minimizing. ∎

Corollary 2.3 (Bernstein type theorem for self-shrinkers [W]). *If Σ is an entire graphic self-shrinker, then it must be a hyperplane passing through the origin.*

Proof. Because among all hyperplanes, only the ones passing through the origin have zero weighted mean curvature. ∎

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