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On Automorphism-Invariant Rings with Chain Conditions

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Abstract
It is shown that if R is a right automorphism-invariant ring and satisfies ACC on right annihilators, then R is a semiprimary ring. By this useful fact, we study finiteness conditions which ensure an automorphism-invariant ring is quasi-Frobenius (QF). Thus, we prove, among other results, that: (1) R is QF if and only if R is right automorphism-invariant, right min-CS and satisfies ACC on right annihilators; (2) R is QF if and only if R is left Noetherian, right automorphism-invariant and every complement right ideal of R is a right annihilator; (3) If R is right CPA, right automorphism-invariant and every complement right ideal of R is a right annihilator, then R is QF.

Keywords Automorphism-invariant ring · NCS ring · QF ring

Mathematics Subject Classification (2010) 16D50 · 16D60 · 16D80

1 Introduction

A ring R is said to be a *QF-ring* if R is right or left Artinian and right or left self-injective. QF-rings form an important class of associative rings known for its application to representation theory of finite groups. A ring R is called right *mininjective* if, for any minimal right ideal I of R, every R-homomorphism from I to R extends to an R-homomorphism from R to R. In [19, Lemma 2.3], it is shown that if R is a right minsymmetric ring with ACC on

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right annihilators in which $\mathrm{Soc}(R_R) \leq^e R_R$, then R is semiprimary (a ring R is called right *minsymmetric* if for any minimal right ideal kR of R, Rk is a minimal left ideal of R). By this useful lemma, it is also proved that if R is a left and right mininjective ring with ACC on right annihilators in which $\mathrm{Soc}(R_R) \leq^e R_R$, then R is QF (see [19, Theorem 2.5]).

In [15], Lee and Zhou introduced the notion of an automorphism-invariant (sub)-module. They defined a submodule N of M to be an *automorphism-invariant submodule* if $\sigma(N) \leq N$ for every automorphism σ of M. A module is called *automorphism-invariant* if it is an automorphism-invariant submodule of its injective hull. Some other properties of automorphism-invariant modules have been studied in [9, 14, 18].

In the present paper, we prove that if R is a right automorphism-invariant ring and satisfies ACC on right annihilators, then R is a semiprimary ring (see Theorem 1). By this key result, we have R is QF if and only if R is right automorphism-invariant, every right ideal of R is a right annihilator and satisfies ACC on right annihilators (see Theorem 2). As an application, we prove in Theorem 2 that if R is a right automorphism-invariant, right CS ring with ACC on essential right ideals, then R is a QF ring. It is proved, among other results, if R is a right automorphism-invariant, right CS ring with ACC on essential left ideals, then R is a QF ring.

According to [10], a ring R is called right CPA if every cyclic right R-module is a direct sum of a projective module and an Artinian module. We use [10, Theorem 2.1] to show that if R is a right CPA and right C2 ring, then R is right Artinian. As an application, we prove, in Corollary 2, that a ring R is QF if and only if R is right CPA, right automorphism-invariant and every complement right ideal of R is a right annihilator.

We next study some properties of right automorphism-invariant rings satisfying ACC on essential left ideals. It shows that these rings satisfy $J(R)$ a nilpotent ideal of R, $r(J(R)) \leq^e R_R$ and $J(R) = lr(J(R))$. Then, we show that R is QF if and only if R is left Noetherian, right automorphism-invariant and every complement right ideal of R is a right annihilator (Theorem 3).

Throughout this article, unless otherwise stated, all rings have unity and all modules are unital. A submodule K of an R-module M is said to be a *complement* to a submodule N of M if K is maximal with respect to the property that $K \cap N = 0$. A submodule N of an R-module M is called essential in M, denoted by $N \leq^e M$, if for any nonzero submodule L of M, $L \cap M \neq 0$. A submodule N of M is called *closed* in M if it has no proper essential extension in M. A nonzero module M is called *uniform* if any two nonzero submodules of M intersect nontrivially. Dually, M is called *hollow*, if every proper submodule of M is small in M. For a nonempty subset X of a ring R, the left annihilator of X in R is $l(X) = \{r \in R : rx = 0 \text{ for all } x \in X\}$. For any $a \in R$, we write $l(a)$ for $l(\{a\})$. Right annihilators $r(X)$ are defined similarly. We write $J(R)$, $Z(R_R)$, $\mathrm{Soc}(R_R)$, $\mathrm{Soc}({}_RR)$ for the Jacobson radical of R, the right singular ideal of R, the right socle of R, and the left socle of R, respectively. We also write $N \leq^e M$ and $N \leq^{\oplus} M$ to indicate that N is an essential submodule of M and a direct summand of M, respectively. For an integer $n \geq 2$, we use $\mathbb{Z}_n$ to denote the ring of integers modulo n. We also use $\mathbb{N}$ to denote the set of natural numbers. For other concepts of rings and modules not defined here, we refer to the texts [3, 5, 16, 20].

2 Automorphism-Invariant Rings

Let M be a module. A submodule N of M is said to be an *automorphism-invariant submodule* if $\sigma(N) \leq N$ for every automorphism σ of M. A module is called an *automorphism-invariant* *module* if it is an automorphism-invariant submodule of its injective hull [15]. A ring R is called right *automorphism-invariant* if R_R is an automorphism-invariant module.

It is clear that a right self-injective ring is right automorphism-invariant. The following example shows that the converse is not true in general.

Example 1 ([7, Example 9]) The ring

$$R = \left\{ (x_n)_n \in \prod_{n=1}^{\infty} \mathbb{Z}_2 : \text{ all except finitely many } x_n \text{ are equal to some a} \in \mathbb{Z}_2 \right\}$$

is a commutative automorphism-invariant ring which is not self-injective.

Lemma 1 *Assume that R is right automorphism-invariant. If $r(x) = r(y)$ for all $x, y \in R$, then $Rx = Ry$.*

Proof This is clear. □

We recall that a ring R is called *semiprimary* if the Jacobson radical $J(R)$ of R is nilpotent and the ring $R/J(R)$ is a semisimple Artinian ring.

Theorem 1 *If R is a right automorphism-invariant ring and satisfies ACC on right annihilators, then R is a semiprimary ring.*

Proof Consider the chain

$$Rx_1 \geq Rx_2 \geq \cdots$$

of cyclic left ideals of R. Then we have $r(x_1) \leq r(x_2) \leq \cdots$. By hypothesis, there exists $n \in \mathbb{N}$ such that $r(x_n) = r(x_{n+k})$ for all $k \in \mathbb{N}$. By Lemma 1, $Rx_n = Rx_{n+k}$ for all $k \in \mathbb{N}$. Thus R is right perfect.

Now we consider the ascending chain

$$r(J(R)) \leq r(J(R)^2) \leq \cdots .$$

By assumption, there is $n \in \mathbb{N}$ such that $r(J(R)^n) = r(J(R)^{n+k})$ for all $k \in \mathbb{N}$. Let $B = J(R)^n$. Then, $r(B) = r(B^2)$ and $B^2 \neq 0$. Now, we shall show that $J(R)$ is nilpotent. Assume $J(R)$ is not nilpotent. Let

$$\mathcal{S} = \{r(b) |\ b \in B \text{ and } Bb \neq 0\}.$$

It is easy to see that $\mathcal{S}$ is a non-empty set. Then $\mathcal{S}$ has a maximal element, say $r(b_0)$ where $b_0 \in B$. Now $BBb_0 = 0$ implies that $b_0R \leq r(B^2) = r(B)$ and hence $Bb_0 = 0$, a contradiction. Therefore there exists an element of B, say x, such that $Bxb_0 \neq 0$. However, since $r(b_0) \leq r(xb_0)$, the maximality of $r(b_0)$ implies that $r(b_0) = r(xb_0)$. By Lemma 1, we obtain that $Rb_0 = Rxb_0$, i.e., $b_0 = sxb_0$ for some $s \in R$ or $b_0(1 - sx) = 0$. Since $sx \in B \leq J(R)$, we have $b_0 = 0$, a contradiction. □

For an R-module M, we have the following definitions [16].

(CS) Every submodule of M is essential in a direct summand of M.

(C2) Every submodule of M that is isomorphic to a direct summand of M is itself a direct summand of M.

(C3) If A and B are two direct summands of M with $A \cap B = 0$, then the sum $A + B$ is a direct summand of M.

We remark that:

(∗) An automorphism-invariant module need not be CS;
(∗∗) Any automorphism-invariant module satisfies (C2)-condition and so (C3). Hence an automorphism-invariant CS module is continuous;
(∗ ∗ ∗) A continuous module need not necessarily be automorphism-invariant.

According to Huynh [11], a module M is called *NCS* if no nonzero complement submodule is small. A ring R is *right NCS* if R_R is NCS.

Clearly every CS module is NCS, but the converse is not true:

- The $\mathbb{Z}$-module $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ is NCS but not CS.
- Let K be a division ring and V be a left K-vector space of infinite dimension. Let $S = \mathrm{End}_K(V)$ and $R = \begin{pmatrix} S & S \\ S & S \end{pmatrix}$. Then R is a right NCS ring but not a right CS ring.

For a hollow module M, it can be easily checked that M is NCS if and only if M is uniform and M is CS.

In [11], Huynh showed that:

Proposition 1 *Let R be a semiperfect ring. If R is right NCS, then R is a right CS ring.*

Recall that a module M is called *pseudo-injective* if, for any submodule A of M, every monomorphism $A \to M$ can be extended to some element of $\mathrm{End}(M)$ [13].

Lemma 2 *(*[7, Theorem 16]*) A module M is automorphism-invariant if and only if it is pseudo-injective.*

A ring R is called right *min-CS* if every minimal right ideal is essential in a direct summand of R_R [17].

A ring R is called right *mininjective* if $lr(a) = Ra$, where aR is a simple right ideal of R.

An idempotent element e of R is called local idempotent if $\mathrm{End}(eR)$ is a local ring.

Theorem 2 *The following statements are equivalent for a ring R:*

1. *R is QF.*
2. *R is right automorphism-invariant, every complement right ideal of R is a right annihilator and satisfies ACC on right annihilators.*
3. *R is right automorphism-invariant, right NCS and satisfies ACC on right annihilators.*
4. *R is right automorphism-invariant, right min-CS and satisfies ACC on right annihilators.*
5. *R is right automorphism-invariant and satisfies ACC on right annihilators with eR is uniform for any local idempotent $e \in R$.*

Proof (1) ⇒ (2) and (1) ⇒ (5) are obvious.

(2) ⇒ (3) By Theorem 1, the ring R is semiprimary. Since R is right pseudo-injective by Lemma 2, R is right mininjective and so $\mathrm{Soc}(R_R) \leq \mathrm{Soc}({}_R R)$. It follows that R is left Kasch by [17, Lemma 1.48]. Thus R is right continuous [21, Theorem 10].

(3) ⇒ (4) By Theorem 1, the ring R is semiprimary. Since a semiprimary ring is right and left perfect, R is right CS by Proposition 1. Hence R is right min-CS.

(4) $\Rightarrow$ (1) By Theorem 1, the ring R is semiprimary. Assume that $\mathrm{Soc}(R_R) = \oplus_{i\in I} S_i$, where each S_i is simple for any $i \in I$. Since R is right min-CS, there exist idempotent elements e_i of R such that S_i essential in $e_i R$. Note that $\{e_i R\}_{i\in I}$ is an independent family since $\{S_i\}_{i\in I}$ is an independent family. Hence $\oplus_{i\in I} e_i R$ is essential in R_R. By $(**)$, we obtain that $\oplus_{i\in I} e_i R$ is a local direct summand of R_R. Since R satisfies ACC on right annihilators, we have $\oplus_{i\in I} e_i R$ is a closed submodule of R_R by [5, Lemma 8.1(1)]. By $(**)$, we obtain that $R_R = \oplus_{i\in I} e_i R$ and each $e_i R$ is uniform. Hence R is right self-injective by [1, Lemma 3.5]. Thus R is QF.

(5) $\Rightarrow$ (1) As we pointed out in the proof of (2) $\Rightarrow$ (3), the ring R is semiperfect. Hence $R = \oplus_{i=1}^{n} e_i R$ where e_i are local idempotent elements. By the hypothesis, $e_i R$ is uniform for all $i = 1, 2, \ldots, n$. It follows that R is right self-injective by [1, Lemma 3.5]. □

Recall that a right CS ring with ACC on essential right ideals is a right Noetherian ring [6, Corollary 18.7]. We have the following result:

Corollary 1 *The following statements are equivalent for a ring R:*

1. *R is QF.*
2. *R is a right automorphism-invariant, right CS ring with ACC on essential right ideals (or left ideals).*

A ring R is called *right CPA* if every cyclic right R-module is a direct sum of a projective module and an Artinian module [10].

Proposition 2 *If R is a right CPA and right C2 ring, then R is right Artinian.*

Proof By [10, Theorem 2.1], R has a direct decomposition

$$R_R = A \oplus U^{(1)} \oplus \cdots \oplus U^{(n)},$$

where A is an ideal of R such that A_R is Artinian and each $U^{(i)}$ is a uniform right R-module with $\mathrm{Soc}(U_R^{(i)}) = 0$. We will prove that $U^{(i)} = 0$ for every i. Assume $U^{(i)} \neq 0$ for some i. Take $0 \neq x \in U^{(i)}$. Since R is right CPA, $xR = P_R \oplus B_R$ where P_R is projective and B_R is Artinian. However $\mathrm{Soc}(xR_R) = 0$ which implies that $B = 0$, i.e., xR is projective. It follows that $r(x)$ is a direct summand of R_R. Thus xR is a direct summand of R_R by condition C2. So

$$R = xR \oplus I,$$

where $I \leq R_R$. Therefore,

$$U^{(i)} = (xR \oplus I) \cap U^{(i)} = xR \oplus \left(I \cap U^{(i)}\right).$$

Since $xR_R \neq 0$ and $U^{(i)}$ is uniform, we obtain $I \cap U^{(i)} = 0$. So $U^{(i)} = xR$ for each $0 \neq x \in U^{(i)}$, which implies that $U^{(i)}$ is simple, a contradiction since $\mathrm{Soc}(U_R^{(i)}) = 0$. Hence $U^{(i)} = 0$, $i = 1, 2, \ldots, n$, and so $R = A$. Therefore R is a right Artinian ring. □

Corollary 2 *If R is right CPA, right automorphism-invariant and every complement right ideal of R is a right annihilator, then R is QF.*

Proof It follows immediately from Theorem 2 and Proposition 2. □

The series of higher left socles $\{S^l_\alpha\}$ of a ring R are defined inductively as

$$S^l_1 = \mathrm{Soc}({}_R R)$$

and

$$S^l_{\alpha+1}/S^l_\alpha = \mathrm{Soc}({}_R(R/S^l_\alpha))$$

for each ordinal $\alpha \geq 1$.

Lemma 3 *If R is a right automorphism-invariant ring, then $J(R) = Z(R_R)$ and $R/J(R)$ is a von Neumann regular ring.*

Proof By [8, Proposition 1]. □

The following lemma is inspired by Lemma 9 in [4].

Proposition 3 *If R is a right automorphism-invariant ring and satisfies ACC on essential left ideals, then*

(1) $r(J(R)) \leq^e R_R$,
(2) $J(R)$ *is nilpotent,*
(3) $J(R) = lr(J(R))$.

Proof (1) Since R has ACC on essential left ideals, the ring $R/\mathrm{Soc}({}_R R)$ is left Noetherian (see [2, 6] or [12]). There exists $k > 0$ such that $S^l_k = S^l_{k+1} = \cdots$ and R/S^l_k is a right Noetherian ring. Now we show that $S^l_k \leq^e R_R$. Assume that $xR \cap S^l_k = 0$ for some $0 \neq x \in R$. Let $\bar{R} = R/S^l_k$ and $l_{\bar{R}}(\bar{a})$ be a maximal element of the set $\{l_{\bar{R}}(\bar{y}) |\ 0 \neq y \in xR\}$. Since $S^l_k = S^l_{k+1}$, we get $\mathrm{Soc}({}_{\bar{R}}\bar{R}) = 0$, and so $\bar{R}\bar{a}$ is not simple as a left $\bar{R}$-module. Thus there exists $t \in R$ such that $0 \neq \bar{R}\bar{t}\bar{a} < \bar{R}\bar{a}$.

If $\bar{a}\bar{t}\bar{a} = \bar{0}$, then $ata \in aR \cap S^l_k = 0$, and so $ata = 0$. If $r(a) = r(ta)$, then $Ra = Rta$ by Lemma 1, a contradiction. Thus $r(a) < r(ta)$. Then there exists $b \in R$ such that $ab \neq 0$ and $tab = 0$. It follows that $0 \neq ab \in xR$ and $l_{\bar{R}}(\bar{a}) < l_{\bar{R}}(\overline{ab})$, a contradiction.

If $\bar{a}\bar{t}\bar{a} \neq \bar{0}$, then $0 \neq \bar{R}\bar{a}\bar{t}\bar{a} < \bar{R}\bar{a}$. We have R is right automorphism-invariant, and so if $r(ata) = r(b)$, $b \in R$ then $b \in Rata$. It follows that $r(a) < r(ata)$. Let $c \in r(ata) \setminus r(a)$. Then $0 \neq ac \in xR$, $\bar{a}\bar{t} \in l_{\bar{R}}(\overline{ac}) \setminus l_{\bar{R}}(\bar{a})$, a contradiction.

Thus $S^l_k \leq^e R_R$ and hence $r(J(R)) \leq^e R_R$ (since $S^l_k \leq r(J(R))$).

(2) See [4, Lemma 9(ii)].

(3) By Lemma 3, $Z(R_R) = J(R)$. On the other hand, for any $x \in lr(J(R))$, then $r(J(R)) \leq r(x)$. We have that $r(J(R)) \leq^e R_R$ and obtain that $r(x) \leq^e R_R$. This gives $x \in Z(R_R) = J(R)$. So $lr(J(R)) \leq J(R)$. We deduce that $lr(J(R)) = J(R)$. □

Using Proposition 3, we obtain another characterization of QF-rings as follows.

Theorem 3 *The following statements are equivalent for a ring R:*

1. *R is QF.*
2. *R is left Noetherian, right automorphism-invariant and every complement right ideal of R is a right annihilator.*

Proof (1) ⇒ (2), (3), (4) This is obvious.

(2) ⇒ (1) As R is left Noetherian, $R/J(R)$ is also a left Noetherian ring. Thus $R/J(R)$ is a semisimple Artinian ring, since $R/J(R)$ is a von Neumann regular ring by Lemma 3. By Proposition 3, $J(R)$ is nilpotent and so R is semiprimary. Thus R is a left Artinian ring which implies that R satisfies ACC on right annihilators. By assumption, every complement right ideal of R is a right annihilator. Thus R is QF by Theorem 2. □

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References

1. Alahmadi, A., Er, N., Jain, S.K.: Modules which are invariant under monomorphisms of their injective hulls. J. Austr. Math. Soc. **79**, 349–360 (2005)
2. Armendariz, E.P.: Rings with DCC on essential left ideals. Commun. Algebra **8**, 299–308 (1980)
3. Anderson, F.W., Fuller, K.R.: Rings and Categories of Modules. Springer, New York (1974)
4. Chen, J., Shen, L., Zhou, Y.: Characterizations of QF rings. Commun. Algebra **35**, 281–288 (2007)
5. Dung, N.V., Huynh, D.V., Smith, P.F., Wisbauer, R.: Extending Modules. Pitman Research Notes in Mathematics Series, vol. 313. Longman Scientific & Tecnical, Essex (1996)
6. Dung, N.V., Huynh, D.V., Wisbauer, R.: Quasi-injective modules with ACC or DCC on essential submodules. Arch. Math. **53**, 252–255 (1989)
7. Er, N., Singh, S., Srivastava, A.K.: Rings and modules which are stable under automorphisms of their injective hulls. J. Algebra **379**, 223–229 (2013)
8. Guil Asensio, P.A., Srivastava, A.K.: Automorphism-invariant modules satisfy the exchange property. J. Algebra **388**, 101–106 (2013)
9. Guil Asensio, P.A., Quynh, T.C., Srivastava, A.K.: Additive unit structure of endomorphism rings and invariance of modules. Bull. Math. Sci. **7**, 229–246 (2017)
10. Huynh, D.V., Dân, P.: On rings with restricted minimum condition. Arch. Math. **51**, 313–326 (1988)
11. Huynh, D.V.: Rings in which no nonzero complement is small. Preprint
12. Huynh, D.V.: Rings with ACC on essential right ideals. Math. Jpn. **35**, 707–712 (1990)
13. Jain, S.K., Singh, S.: Quasi-injective and pseudo-injective modules. Can. Math. Bull. **18**, 359–366 (1975)
14. Koşan, M.T., Quynh, T.C., Srivastava, A.K.: Rings with each right ideal automorphism-invariant. J. Pure Appl. Algebra **220**, 1525–1537 (2016)
15. Lee, T.-K., Zhou, Y.: Modules which are invariant under automorphisms of their injective hulls. J. Algebra Appl. **12**, 1250159 (2013)
16. Mohammed, S.H., Müller, B.J.: Continuous and Discrete Modules. London Mathematical Society Lecture Notes Series, vol. 147. Cambridge University Press, Cambridge (1990)
17. Nicholson, W.K., Yousif, M.F.: Quasi-Frobenius Rings. Cambridge Tracts in Mathematics, vol. 158. Cambridge University Press, Cambridge (2003)
18. Quynh, T.C., Koşan, M.T.: On automorphism-invariant modules. J. Algebra Appl. **14**, 1550074 (2015)
19. Shen, L., Chen, J.: New characterizations of quasi-Frobenius rings. Commun. Algebra **34**, 2157–2165 (2006)
20. Wisbauer, R.: Foundations of Module and Ring Theory. Gordon and Breach Science Publishers, Philadelphia (1991)
21. Yousif, M., Zhou, Y.: Pseudo-Frobenius rings: characterizations and questions. Commun. Algebra **31**, 4473–4484 (2003)