Stochastic Pretopology as a Tool for Topological Analysis of Complex Systems

Quang Vu Bui[1,2(✉)], Soufian Ben Amor[3], and Marc Bui[1,4]

[1] CHArt Laboratory EA 4004, EPHE, PSL Research University, Paris, France
quang-vu.bui@etu.ephe.fr
[2] Hue University of Sciences, Hue, Vietnam
[3] LI-PARAD Laboratory, University of Versailles-Saint- Quentin-en-Yvelines, Versailles, France
[4] University Paris 8, Paris, France

Abstract. We are proposing in this paper a more general network modeling framework for complex system representation by introducing *Stochastic Pretopology*, a result of the combination of Pretopology theory and Random Sets. After giving the definition and some examples for building stochastic pretopology in many situations, we show how this approach generalizes graph, random graph, multi-relational networks and we present an application by giving *Pretopology Cascade Model* as a general model for information diffusion process that can take place in more complex networks such as multi-relational networks or stochastic graphs.

Keywords: Random set · Pretopology · Random graph
Complex network · Social network · Complex system
Multi-relational network · Information diffusion

1 Introduction

Complex system is a system composed of many interacting parts, such that the collective behavior of its parts together is more than the "sum" of their individual behaviors [10]. The topology of complex systems (who interact with whom) is often specified in terms of networks that are usually modeled by graphs, composed by vertices or nodes and edges or links. Graph theory has been widely used the conceptual framework of network models, such as random graphs, small world networks, scale-free networks [5,11].

However, having more complicated non-regular topologies, complex systems need a more general framework for their representation [10]. To overcome this issue, we propose using Stochastic Pretopology built from the mixing between Pretopology theory and Random Sets theory. Pretopology [2] is a mathematical tool for modeling the concept of proximity which allows us to follow structural transformation processes as they evolve while random sets theory [9,12] provides the good ways for handling what happens in a stochastic framework at the sets' level point ot views.

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In this paper, after recalling basics of pretopology and the definition of graphs in the *Berge sense* [3], we first show pretopology as an extension of graph theory which leads us to the definition of pretopology networks as a general framework for network representation. Connected to random sets theory, we then give the definition of Stochastic Pretopology and propose different ways for building such a pretopology that is useful for modeling the topological structure of complex systems in different spaces such as metric space, valued or binary relation spaces. These models can be convenient to handle phenomena in which collective behavior of a group of elements can be different from the summation of element behaviors composing the group. After presenting *Independent Cascade model* [6] and *Independent Threshold model* [7] under stochastic pretopology language, we will propose pretopological information diffusion model as a general diffusion model that can take place in more complex networks such as multi-relational networks or stochastic graphs. Stochastic graphs presented in this paper are defined by extending the definition of graph in the *Berge sense* [3] $G = (V, \Gamma)$. In this approach, by considering Γ function as a finite random set defined from a degree distribution, we give a general graph-based network model in which *Erdős-Rényi model* and *scale-free networks* are special cases.

The rest of this paper is organized as follows: Sect. 2 briefly recalls basic concepts of pretopology theory prepared for building stochastic pretopology in Sect. 3; we then conclude by presenting an application of stochastic pretopology in information diffusion.

2 Pretopology as a Group Modeling in Complex Networks

For modeling the dynamic processes on complex networks, topology theory is not suitable since the idempotent property of its closure function makes it impossible for changing unless changing the topological structure. So, we propose in this section a new insight on networks modeling with pretopology theory. Pretopology [2] is considered as an extension of topology obtained by the relaxing of its axiomatic. The pretopology is a tool for modeling the concept of proximity that allows monitoring step by step the evolution of a set. It establishes a powerful tools for the structure analysis, classification, and multi-criteria clustering [4]. It also applies for group modeling in social networks. Based on set theory, by considering a group of elements as a set, pretopology formalism allows us to consider a group as a whole independent entity.

2.1 Pseudo-Closure Function

Definition 1. *We call pseudo-closure defined on a set* V*, any function* $a(.)$ *from* $\mathcal{P}(V)$ *into* $\mathcal{P}(V)$ *such as:*

(P1): $a(\emptyset) = \emptyset$;
(P2): $A \subset a(A) \quad \forall A, A \subset V$

(V, a) *is then called pretopological space.*

Pseudo-closure allows, for each of its applications, to add elements to a set departure according to defined characteristics. The starting set gets bigger but never reduces. There are different ways to build a pseudo-closure function.

(a) *V* is equipped with a metric. When space V is equipped with a metric d, we can build a *pseudo-closure* function **a(.)** on V with a closed ball of center x and radius r ($B(x,r) = \{y \in V | d(x,y) \leq r\}$):

$$\forall A \in \mathcal{P}(V), \quad a(A) = \{y \in V | B(x,r) \cap A \neq \emptyset\} \tag{1}$$

The *pseudo-closure* $a(A)$ is a set of all elements $y \in V$ such that y is within a distance of at most radius r from at least one element of A.

(b) The elements of *V* are linked by a valued relation. In order to model certain problems such as model in weighted graph, we often need the space V are bound by a valued relation. For instance, we can define an real value ν on relations as a function from $V \times V \rightarrow \mathbb{R}$ as: $(x,y) \rightarrow \nu(x,y)$. The *pseudo-closure* **a(.)** can build such as:

$$\forall A \in \mathcal{P}(V), \quad a(A) = \{y \in V - A | \sum_{x \in A} \nu(x,y) \geq s\} \cup A; s \in \mathbb{R} \tag{2}$$

(c) The elements of *V* are linked by n reflexive binary relations. Suppose we have a family $(R_i)_{i=1,\ldots,n}$ of binary reflexive relations on a finite set V. For each relation R_i, we can define pretopological structure by considering the following subset: $\forall i = 1, 2, \ldots, n, \forall x \in V, V_i(x)$ defined by:

$$V_i(x) = \{y \in V | x \, R_i \, y\}$$

We can define the *pseudo-closure* **a(.)** by:

$$\forall A \in \mathcal{P}(V), \quad a(A) = \{x \in V | \forall i = 1, 2, \ldots, n, V_i(x) \cap A \neq \emptyset\} \tag{3}$$

(d) The elements of *V* are equipped with a neighborhood function. Let us consider a multivalued function $\Gamma : V \rightarrow \mathcal{P}(V)$ as a neighborhood function. $\Gamma(x)$ is a set of neighborhoods of element x. We define a *pseudo-closure* **a(.)** as follows:

$$\forall A \in \mathcal{P}(V), \quad a(A) = A \cup (\bigcup_{x \in A} \Gamma(x)) \tag{4}$$

2.2 Pretopology as an Extension of Graph Theory

(a) Graphs in the Berge sense. By using the knowledge from multivalued function, Claude Berge [3] defined a graph such as:

Definition 2. *A graph, which is denoted by $G = (V, \Gamma)$, is a pair consisting of a set V of vertices or nodes and a multivalued function Γ mapping V into $\mathcal{P}(V)$.*

The pair (x,y), with $y \in \Gamma(x)$ is called an arc or edge of the graph. We therefore can also denote a graph by a pair $G = (V, E)$, which V is a set of nodes and E is a set of edges. Conversely, if we denote a graph as $G = (V, E)$, we can define the Γ function as: $\Gamma(x) = \{y \in V | (x,y) \in E\}$. $\Gamma(x)$ is a set of neighbors of node x.

(b) Pretopology as an extension of Graph theory. In this part, we show reflexive graph (V, Γ) which is a special case of pretopology. More specifically, as it is known, a finite reflexive graph (V, Γ) complies the property: $\forall A \subset V, a(A) = \cup_{x \in A} a(\{x\})$ where pseudo-closure function defined as $a(A) = \cup_{x \in A} \Gamma(x)$. For this reason, graph may be represented by a $\mathcal{V}_D$-type pretopological space. Conversely, we can build a pretopology space (V, a) presented a graph such as: $a(A) = \{x \in V | \Gamma(x) \cap A \neq \emptyset\}$ where $\Gamma(x) = \{y \in V | x\, R\, y\}$ built from a binary relation R on V. Therefore, a graph (V, Γ) is a pretopological space (V, a) in which the pseudo-closure function built from a binary relation or built from a neighborhood function in Eq. (4).

By using a graph, a network is represented with only one binary relation. In the real world, however, a network is a structure made of nodes that are tied by one or more specific types of binary or value relations. As we show in the previous, by using pretopology theory, we can generalize the definition of complex network such as:

Definition 3 *(Pretopology network). A pretopology network, which is denoted by $G^{(Pretopo)} = (V, a)$, is a pair consisting of a set V of vertices and a pseudo-closure function **a(.)** mapping $\mathcal{P}(V)$ into $\mathcal{P}(V)$.*

3 Stochastic Pretopology (SP)

Complex systems usually involve structural phenomena, under stochastic or uncontrolled factors. In order to follow these phenomena step by step, we need concepts which allow modelling dynamics of their structure and take into account the factors' effects. As we showed in the previous section, we propose to use pretopology for modelling the dynamics of phenomena; the non idempotents of its pseudo-closure function makes it suitable for such a modelling. Then, we introduce stochastic aspects to handle the effects of factors influencing the phenomena. For that, we propose using a theory of random sets by considering that, given a subset A of the space, its pseudo-closure $a(A)$ is considered as a random set. So, we have to consider the pseudo-closure not only as a set transform but also as a random correspondence.

Stochastic pretopology was first basically introduced in Chap. 4 of [2] by using a special case of random set (the simple random set) to give three ways to define stochastic pretopology. We have also given some applications of stochastic pretopology such as: modeling pollution phenomena [8] or studying complex networks via a stochastic pseudo-closure function defined from a family of random relations [1]. Since we will deal with complex networks in which set of nodes V is a finite set, we propose in this paper another approach for building stochastic pretopology by using finite random set theory [12].

From now on, V denotes a finite set. $(\Omega, \mathcal{A}, \mathbb{P})$ will be a *probability space*, where: Ω is a set, representing the *sample space* of the experiment; $\mathcal{A}$ is a σ-algebra on Ω, representing *events* and $\mathbb{P} : \Omega \to [0, 1]$ is a *probability measure*.

3.1 Finite Random Set

Definition 4. *A finite random set (FRS) with values in $\mathcal{P}(V)$ is a map $X : \Omega \to \mathcal{P}(V)$ such as*

$$X^{-1}(\{A\}) = \{\omega \in \Omega : X(\omega) = A\} \in \mathcal{A} \text{ for any } A \in \mathcal{P}(V) \tag{5}$$

The condition (5) is often called *measurability condition*. So, in other words, a FRS is a measurable map from the given probability space $(\Omega, \mathcal{A}, P)$ to $\mathcal{P}(V)$, equipped with a σ-algebra on $\mathcal{P}(V)$. We often choose σ-algebra on $\mathcal{P}(V)$ is the discrete σ-algebra $\mathcal{E} = \mathcal{P}(\mathcal{P}(V))$. Clearly, a *finite random set* X is a *random element* when we refer to the *measurable space* $(\mathcal{P}(V), \mathcal{E})$. This is because $X^{-1}(\mathcal{E}) \subseteq \mathcal{A}$ since $\forall \mathbb{A} \in \mathcal{E}; X^{-1}(\mathbb{A}) = \cup_{A \in \mathbb{A}} X^{-1}(A)$.

3.2 Definition of Stochastic Pretopology

Definition 5. *We define **stochastic pseudo-closure** defined on $\Omega \times V$, any function **a(.,.)** from $\Omega \times \mathcal{P}(V)$ into $\mathcal{P}(V)$ such as:*

(P1): $a(\omega, \emptyset) = \emptyset \quad \forall \omega \in \Omega$;
(P2): $A \subset a(\omega, A) \quad \forall \omega \in \Omega, \forall A, A \subset V$;
(P3): $a(\omega, A)$ *is a finite random set* $\forall A, A \subset V$

$(\Omega \times V, a(.,.))$ *is then called Stochastic Pretopological space.*

By connecting the finite random set theory [9,12], we can build stochastic pseudo-closure function with different ways.

3.3 SP Defined from Random Variables in Metric Space:

By considering a *random ball* $B(x, \xi)$ with ξ is a non-negative random variable, we can build a *stochastic pseudo-closure* **a(.)** in metric space such as:

$$\forall A \in \mathcal{P}(V), \quad a(A) = \{x \in V | B(x, \xi) \cap A \neq \emptyset\} \tag{6}$$

3.4 SP Defined from Random Variables in Valued Space:

We present two ways to build stochastic pseudo-closure by extending the definition of pseudo-closure function on valued space presented in Eq. (2). Firstly, by considering threshold s is a *random variable* η, we can define a *stochastic pseudo-closure* **a(.)** such as:

$$\forall A \in \mathcal{P}(V), \quad a(A) = \{y \in V - A | \sum_{x \in A} \nu(x, y) \geq \eta\} \cup A \tag{7}$$

where threshold η is *random variable*.

Secondly, by considering the weight function $\nu(x, y)$ between two elements x, y as a *random variable*, we can define a *stochastic pseudo-closure* **a(.)** such as:

$$\forall A \in \mathcal{P}(V), \quad a(A) = \{y \in V - A | \sum_{x \in A} \nu_{\Omega}(x, y) \geq s\} \cup A \tag{8}$$

where $\nu_{\Omega}(x, y)$ is a *random variable*.

3.5 SP Defined from a Random Relation Built from a Family of Binary Relations

Suppose we have a family $(R_i)_{i=1,\dots,m}$ of *binary reflexive relations* on a finite set V. We call $L = \{R_1, R_2, \dots, R_m\}$ is a set of relations. Let us define a *random relation* $R : \Omega \to L$ as a random variable:

$$P(R(\omega) = R_i) = p_i; \quad p_i \geq 0; \sum_{i=1}^{m} p_i = 1.$$

For each $x \in V$, we can build a random set of neighbors of x with random relation R:

$$\Gamma_{R(\omega)}(x) = \{y \in V | x\, R(\omega)\, y\}$$

We can define a *stochastic pseudo-closure* **a(.,.)** such as:

$$\forall A \in \mathcal{P}(V), \quad a(\omega, A) = \{x \in V | \Gamma_{R(\omega)}(x) \cap A \neq \emptyset\} \tag{9}$$

3.6 SP Defined from a Family of Random Relations

We can extend the previous work by considering many *random relations.* Suppose we have a family $(R_i)_{i=1,\dots,n}$ of random binary reflexive relations on a set V. For each $x \in V$, we can build a random set of neighbors of x with random relation $R_i, i = 1, 2, \dots, n$:

$$\Gamma_{R_i(\omega)}(x) = \{y \in V | x\, R_i(\omega)\, y\}$$

We can define a *stochastic pseudo-closure* $a(.,.)$ such as:

$$\forall A \in \mathcal{P}(V), \quad a(\omega, A) = \{x \in V | \forall i = 1, 2, \dots, n, \Gamma_{R_i(\omega)}(x) \cap A \neq \emptyset\} \tag{10}$$

3.7 SP Defined from a Random Neighborhood Function

Let us consider a random neighborhood function as a random set $\Gamma : \Omega \times V \to \mathcal{P}(V)$. $\Gamma(\omega, x)$ is a random set of neighborhoods of element x. We define a *stochastic pseudo-closure* **a(.,.)** as follows:

$$\forall A \in \mathcal{P}(V), \quad a(\omega, A) = A \cup (\bigcup_{x \in A} \Gamma(\omega, x)) \tag{11}$$

We have shown in this section how we construct stochastic pseudo-closure functions for various contexts. That is to say how proximity with randomness can be delivered to model complex neighborhoods formation in complex networks. In the two next sections, we will show how stochastic pretopology can be applied for modeling dynamic processes on complex networks by representing classical information diffusion models under stochastic pretopology language and then proposing Pretopology Cascade Model as a general information diffusion model in which complex random neighborhoods set can be captured by using stochastic pseudo-closure functions.

4 Stochastic Pretopology as a General Information Diffusion Model on Single Relational Networks

Information diffusion has been widely studied in networks, aiming to model the spread of information among objects when they are connected with each other. In a single relational network, many diffusion models have been proposed such as *tipping model*, *threshold models*, *cascade models*, ... [5,11]. We assume a network $G = (V, \Gamma, W)$, where $W : V \times V \to \mathbb{R}$ is weight function. $W(x, y)$ is the weight of edge between two nodes x, y in threshold model or the probability of node y infected from node x in cascade model.

The diffusion process occurs in discrete time steps t. If a node adopts a new behaviour or idea, it becomes active, otherwise it is inactive. An inactive node has the ability to become active. The set of active nodes at time t is considered as A_t. We present in the following two scenarios in which stochastic pretopology as extensions of both *Independent Cascade* (IC) model [6] and *Independent Threshold* (IT) model [7].

4.1 Stochastic Pretopology as an Extension of IC Model

Independent Cascade model. Under the IC model, at each time step t where A_{t-1}^{new} is the set of newly activated nodes at time $t-1$, each $x \in A_{t-1}^{new}$ infects the inactive neighbors $y \in \Gamma(x)$ with a probability $W(x, y)$.

Representing IC model under stochastic pretopology language: We can represent IC model by giving a definition of stochastic pretopology based on two definitions in subsection 3.4,3.7: we firstly define a random set of actived nodes from each node $x \in A_{t-1}^{new}$ and then use a random neighbor function to define the random active nodes in the time t. Specifically, A_t^{new} is defined via two steps:

i. For each $x \in A_{t-1}^{new}$, set of actived nodes from x, $\Gamma^{(active)}(x)$, defined as:

$$\Gamma^{(active)}(x) = \{y \in \Gamma(x) | W(x, y) \geq \eta\}; \quad \eta \sim U(0, 1) \tag{12}$$

ii. The set of newly active nodes, A_t^{new}, defined as:

$$A_t^{new} = a(A_{t-1}^{new}) - A_{t-1}; A_t = A_{t-1} \cup a(A_{t-1}^{new}) \tag{13}$$

where:

$$a(A_{t-1}^{new}) = A_{t-1}^{new} \bigcup \Big(\bigcup_{x \in A_{t-1}^{new}} \Gamma^{(active)}(x) \Big) \tag{14}$$

4.2 Stochastic Pretopology as an Extension of IT Model

Independent Threshold model: Under the IT model, each node y selects a randomly threshold $\theta_y \sim U(0, 1)$. Then, at each time step t where A_{t-1} is the set of nodes activated at time $t-1$ or earlier, each inactive node y becomes active if $\sum_{x \in \Gamma^{-1}(y) \cap A_{t-1}} W(x, y) \geq \theta_y$ where $\Gamma^{-1}(y) = \{x \in V | y \in \Gamma(x)\}$.

Representing IT model under stochastic pretopology language: We can represent IC model by giving a definition of stochastic pretopology such as:

$$A_t = a(A_{t-1}) = \{y \in V - A_{t-1} | \sum_{x \in \Gamma^{-1}(y) \cap A_{t-1}} W(x,y) \geq \eta\} \cup A_{t-1}; \quad \eta \sim U(0,1)$$

5 Pretopology Cascade Models for Modeling Information Diffusion on Complex Networks

Most of information diffusion models are defined via node's neighbors. In general, at each time step t, the diffusion process can be described in two steps:

Step 1: define set of neighbors $N(A_{t-1})$ of set of active nodes A_{t-1}.
Step 2: each element $x \in N(A_{t-1}) - A_{t-1}$ will be influenced by all elements in A_{t-1} to be active or not active node by following a diffusion rule.

We consider the way to define set of neighbors $N(A_{t-1})$ in step 1. In classical diffusion model with complex network represented by a graph $G = (V, \Gamma)$, $N(A_{t-1})$ is often defined such as: $N(A_{t-1}) = \cup_{x \in A_{t-1}} \Gamma(x)$. By using the concepts of stochastic pretopology theory introducted in the Sect. 3, the information diffusion process can be generalized by defining a set of neighbors $N(A_{t-1})$ as a *stochastic pseudo-closure* function $N(A_{t-1}) = a_{\Omega}(A_{t-1})$. We therefore propose the *Pretopological Cascade Model* presented in the following as a general information diffusion model which can be captured more complex random neighborhoods set in diffusion processes.

Definition 6. *Pretopological Cascade model:*
Under the Pretopological Cascade model, at each time step t, the diffusion process takes place in two steps:

Step 1: define set of neighbors $N(A_{t-1})$ of A_{t-1} as a stochastic pseudo-closure function $N(A_{t-1}) = a_{\Omega}(A_{t-1})$.
Step 2: each element $x \in N(A_{t-1}) - A_{t-1}$ will be influenced by A_{t-1} to be active or not active node by following a "diffusion rule".

For defining $N(A_{t-1})$ in step 1, we can apply different ways to define stochastic pseudo-closure function presented in Sect. 3. "Diffusion rule" in step 2 can be chosen by various ways such as:

- Probability based rule: element x infects the inactive elements $y \in N(A_{t-1})$ with a probability $P_{x,y}$.
- Threshold rule: inactive elements $y \in N(A_{t-1})$ will be actived if sum of all influence of all incoming elements of y greater than a threshold θ_y.

We present in the following two examples of the pretopological cascade model: the first takes place in a stochastic graph by defining random neighbors sets based on nodes' degree distribution and the second takes place in multi-relational networks where random neighbors set is built from a family of relations.

5.1 Pretopological Cascade Model on Stochastic Graphs

Definition 7 *(Stochastic Graph). A stochastic graph, which is denoted by $G^{\Omega} = (V, \Gamma_{\Omega})$ is a pair consisting of a set V of vertices and a finite random set Γ_{Ω} mapping $\Omega \times V$ into $\mathcal{P}(V)$.*

The random neighbor function Γ_{Ω} in the definition 7 can be defined in a general way from finite random set theory [12]. Since the *nodes' degree distribution* is necessary for studying network structure, we propose here the way to defining the random neighbor function Γ_{Ω} via two steps:

1. Defining *probability law* of the cardinality of Γ_{Ω} (in fact, Γ_{Ω} is a degree distribution of network).

$$Prob(|\Gamma_{\Omega}| = k) = p_k \quad for \quad k = 1, 2, \ldots, \infty \tag{15}$$

2. Assigning *probability law* on V^k for $k = 1, 2, \ldots, \infty$

$$Prob(\Gamma_{\Omega}^{(1)} = x^{(1)}, \ldots, \Gamma_{\Omega}^{(k)} = x^{(k)} || \Gamma_{\Omega}| = k) \quad for \quad x^{(1)}, \ldots, x^{(k)} \in V \tag{16}$$

We can see some classical network models are specical cases of this kind of stochastic graph. For example, we have *Erdős-Rényi* model if $|\Gamma_{\Omega}| \sim U(0, 1)$ and *scale-free networks* model when $|\Gamma_{\Omega}|$ follows a *power-law distribution*. We also have other network models by using other probability distributions such as Poisson distribution, Geometry distribution, Binomial distribution, etc.

Pretopological cascade model on Stochastic Graph. Under the *Pretopological Cascade model* on stochastic graph, at each time step t, each $x \in A_{t-1}$ generates a random number of neighbors η following a degree distribution given by the Eq. (15) and then generates random neighbors set $\Gamma_{\Omega}(x)$ following a point distribution given by the Eq. (16); after that x infects the inactive neighbors $y \in \Gamma_{\Omega}(x)$ with a probability $P_{x,y}$.

5.2 Pretopological Cascade Model on Multi-relational Networks

Multi-relational network. A *multi-relational network* can be represented as a multi-graph, which allows multiple edges between node-pairs. A *multi-relational network*, which is denoted by $G^{(multi)} = (V, (\Gamma_1, \Gamma_2, \ldots, \Gamma_m))$, is a pair consisting of a set V of vertices and a set of multivalued functions $(\Gamma_i)_{i=1,2,\ldots,m}$ mapping V into 2^V. Γ_i is a neighbor function following the relation R_i.

Defining Random neighbors set on Multi-relational network. Let us define a random index η takes values on $\{1, 2, \ldots, m\}$ such as a random variable:

$$P(\eta = i) = p_i; i = 1, 2, \ldots, m; \quad p_i \geq 0; \sum_{i=1}^{m} p_i = 1 \tag{17}$$

We define a random neighbor function Γ_{η} based on random index η such as: $\Gamma_{\eta} = \Gamma_i$ if $\eta = i$. For each $x \in V$, we can build a random set of neighbors of x: $\Gamma_{\eta}(x) = \Gamma_i(x)$ if $\eta = i, i = 1, 2, \ldots, m$.

Pretopological cascade model on Multi-relational network. Under the *pretopological cascade model* on multi-relational networks, at each time step t, each $x \in A_{t-1}$ generates a random index η given by the Eq. (17) then generates random neighbors set $\Gamma_\eta(x)$; after that x infects the inactive neighbors $y \in \Gamma_\eta(x)$ with a probability $P_{x,y}$.

We can extend this model by choosing randomly a set $S_\eta \subset \{1, 2, \ldots, m\}$ and then using interset or union operator to generate a random set of neighbors of x. For example, we can define $\Gamma_\eta(x) = \cup_{k_i \in S_\eta} \Gamma_{k_i}(x)$ or $\Gamma_\eta(x) = \cap_{k_i \in S_\eta} \Gamma_{k_i}(x)$.

6 Conclusion

In this paper, we proposed *Stochastic Pretopology* as a general mathematical framework for complex systems analysis. The advantage of this approach is that we can not only deal with uncontrolled factors by using random sets but also work with a set as a whole entity, not as a combination of elements. We illustrate our approach by introducing various ways to define a stochastic pseudo-closure function in many situations. Furthermore, we presented an application by proposing *Pretopology Cascade Model*, a general information diffusion model which can apply on diffirent kinds of complex networks such as stochastic graphs, multi-relational networks. A point not discussed in this paper which can be seen as a perspective is practical aspects of the proposed model. In future works can be developed a software library for implementing stochastic pretopology algorithms and applying the proposed model for real-world complex systems.

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