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Abstract	In this paper, we study rings with the property that every cyclic module is almost-injective (CAI). It is shown that *R* is an Artinian serial ring with $J(R)^2 = 0$ if and only if *R* is a right CAI-ring with the finitely generated right socle (or I-finite) if and only if every semisimple right *R*-module is almost injective, R_R is almost injective and has finitely generated right socle. Especially, *R* is a two-sisded CAI-ring if and only if every (right and left) *R*-module is almost injective. From this, we have the decomposition of a CAI-ring via an SV-ring for which Loewy $(R) \leq 2$ and an Artinian serial ring whose squared Jacobson radical vanishes. We also characterize a Noetherian right almost V-ring via the ring for which every semisimple right *R*-module is almost injective.

Keywords (separated by '-')	Almost-injective module - Almost V-ring - V-ring - CAI-ring
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ORIGINAL PAPER

Rings Characterized via Some Classes of Almost-Injective Modules

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Abstract

In this paper, we study rings with the property that every cyclic module is almost-injective (CAI). It is shown that R is an Artinian serial ring with $J(R)^2 = 0$ if and only if R is a right CAI-ring with the finitely generated right socle (or I-finite) if and only if every semisimple right R-module is almost injective, R_R is almost injective and has finitely generated right socle. Especially, R is a two-sisded CAI-ring if and only if every (right and left) R-module is almost injective. From this, we have the decomposition of a CAI-ring via an SV-ring for which Loewy $(R) \leq 2$ and an Artinian serial ring whose squared Jacobson radical vanishes. We also characterize a Noetherian right almost V-ring via the ring for which every semisimple right R-module is almost injective.

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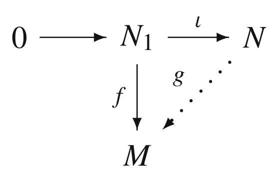
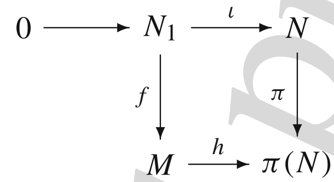
1 Introduction

Throughout this paper, all rings R are associative with unit and all modules are right unital. Let M and N be right R-modules. The module M is said to be *almost N-injective* (or *almost injective respect to N*) if, for every submodule N_1 of N and for every homomorphism $f : N_1 \to M$, either there is a homomorphism $g : N \to M$ such that $f = g \circ \iota$, i.e., the diagram (1) commutes, or there is a nonzero idempotent $\pi \in \mathrm{End}(N)$ and a homomorphism $h : M \to \pi(N)$ such that $h \circ f = \pi \circ \iota$, i.e., the diagram (2) commutes, where $\iota : N_1 \to N$ is the embedding of N_1 into N. The module M is said to be *almost injective* if it is almost injective with respect to every right R-module.

(1)   (2)

This concept was defined by Baba in many years ago, however, many related results were obtainned in recent years, for examples, see [1–8], ... Of course, injective $\Rightarrow$ almost injective, but the converse isn't true, in general. It is proved that a ring R is semisimple if and only if every right (left) R-module is injective and then a well-known result of Osofsky said that it is equivalent to every cyclic right (left) R-module is injective. In [4], the authors consider the structure of a ring R over which every module is almost injective. It is natural to ask how is the structure of a ring R for which every cyclic module is almost injective. We continue prove that the class of rings whose all cyclic right R-modules are almost injective contains the class of Artinian serian rings with squared Jacobson radical vanishes. So Theorem 1 and it's Corollaries from [4] are followed from our result, i.e., in cases of if Soc (R_R) is finitely generated (or R is semiperfect, or R_R is extending, or R is of finite reduced rank), then two above classes and the class of the rings whose all right R-modules are almost injective coincide. Especially, a ring R is two-sided CAI if and only if every (right and left) R-module is almost injective. From this result, we have the decomposition of a CAI-ring via an SV-ring for which Loewy $(R) \leq 2$ and an Artinian serial ring whose squared Jacobson radical vanishes.

Recall that R is a right *V-ring* if every simple right R-module is injective. In [3], the authors consider a generalization of a V-ring, that is almost V-ring, i.e., if every simple right R-module is almost injective. A module M is called *simple-extending* (*semisimple-extending*, resp.) if the complement of any simple (semisimple, resp.) submodule of M is a direct summand of M. Now we write the *class 1* stands for all rings R for which every simple module is almost injective, i.e., R is an almost V-ring, the *class 2* stands for all rings R for which every semisimple module is almost injective, the *class 3* stands for all rings R for which every module is simple-extending. In [3], the authors proved that the class 1 and class 3 coincides (see [3], Theorem 2.9). It is also proved that the intersection of the class 1 and the class of all right Noetherian rings is equal to the class 2 (see [5], Theorem 2.4). Our aim is to consider

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the weaker conditions of Noetherian, that are having finite Goldie dimesion or finitely generated right socle together the class 1 will be replaced by class 2 and we also obtain a characterization of a right Noetherian right almost V-ring. From this, we give back some characterizations of an Artinian serial ring with squared Jacobson radical vanishes via class 2.

For a submodule N of M, we use $N \leq M$ $(N < M)$ to mean that N is a submodule of M (respectively, proper submodule), and we write $N \leq^e M$ to indicate that N is an essential submodule of M. A module is called a *CS-module*, or *extending*, provided every complement submodule is a direct summand. A module is called *uniform* if the intersection of any two nonzero submodules is nonzero. A ring R is called *I-finite* if it contains no infinite orthogonal family of idempotents. Let M be an arbitrary module. Recall that $Z(M) = \{m \in M | mI = 0$ for some $I \leq^e R_R\}$ is called the *singular submodule* of M, and if $Z(M) = M$ ($Z(M) = 0$, resp.), then M is called *singular* (*nonsingular*. resp.) (see [9]). The *Goldie torsion* (or *second singular*) *submodule* of M denoted by $Z_2(M)$ satisfies $Z(M/Z(M)) = Z_2(M)/Z(M)$. The (*Goldie*) *reduced rank* of M is the uniform dimension of $M/Z_2(M)$. Every module of finite uniform dimension is of finite reduced rank. Let M, N be arbitrary modules. M is called *essentially N-injective* if for every embedding $\iota : A \to N$ and every homomorphism $f : A \to M$ such that $\mathrm{Ker} f \leq^e A$, there exists a homomorphism $g : N \to M$ such that $\iota \circ g = f$. The module M is said to be *essentially injective* if it is essentially N-injective with each $N \in$ Mod-R. Moreover, R is a right *SC-ring* if every singular R-module is continuous. M is called a *uniserial module*, if the set of submodules of M is linear ordered. A ring R is called *semiperfect* in case $R/J(R)$ is semisimple and idempotents lift modulo $J(R)$. It is equivalent to every its finitely generated right (left) R-module has a projective cover. A ring R is called a right *perfect ring* in case $R/J(R)$ is semisimple and $J(R)$ is right T-nilpotent. It is equivalent to every its right R-module has a projective cover.

By the *Loewy series* of a module M_R we mean the ascending chain

$$0 \leq \mathrm{Soc}_1(M) = \mathrm{Soc}(M) \leq \cdots \leq \mathrm{Soc}_\alpha(M) \leq \mathrm{Soc}_{\alpha+1}(M) \leq \ldots,$$

where

$$\mathrm{Soc}_\alpha(M)/\mathrm{Soc}_{\alpha-1}(M) = \mathrm{Soc}(M/\mathrm{Soc}_{\alpha-1}(M))$$

for every nonlimit ordinal α and

$$\mathrm{Soc}_\alpha(M) = \bigcup_{\beta<\alpha} \mathrm{Soc}_\beta(M)$$

for every limit ordinal α. Denote by $L(M)$ the submodule of the form $\mathrm{Soc}_\xi(M)$, where ξ stands for the least ordinal for which $\mathrm{Soc}_\xi(M) = \mathrm{Soc}_{\xi+1}(M)$. A module M is semiartinian if and only if $M = L(M)$. In this case, ξ is called the *Loewy length* of the module M and is denoted by Loewy (M). A ring R is said to be *right semiartinian* if

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the module R_R is semiartinian. In this case, every nonzero (principal) right R-module has a nonzero socle and a ring R is right perfect if and only if it is left semiartinian and I-finite. The class of right semiartinian right V-rings, which we call *right SV-rings*. A ring R is called right *nonsingular* if $Z(R_R) = 0$, right *serial* if R_R is a direct sum of uniserial modules. In this paper, we denote by $\mathrm{Rad}(M)$, $\mathrm{Soc}(M)$, $E(M)$, and $\mathrm{length}(M)$ the Jacobson radical, the socle, the injective hull and the composition length of M, respectively. The full subcategory of Mod-R whose objects are all R-modules subgenerated by M is denoted by $\sigma[M]$.

Left-sided for these above notations are defined similarly. All terms such as "artinian", "serial", ... when applied to a ring will apply all both sided. For any terms not defined here the reader is referred to [9–13].

2 On Rings with Cyclic Almost-Injective Modules

Firstly, we include the following known result related to finite decomposition of almost-injective modules for the sake of completeness.

Lemma 2.1 [8, Lemma 1.14] *Let $N, V_1, V_2, \ldots, V_n$ be a family of modules over a ring R. Then $M = \bigoplus_{i=1}^{n} V_i$ is almost N-injective if and only if every V_i is almost N-injective.*

The second author gave the following problem in [1]: describe the rings over which every cyclic right R-module is almost-injective. In this section, we will study on this problem and give some characterizations of rings for which every cyclic right R-module is almost-injective.

Definition 2.2 A ring R is called *right CAI*, if every cyclic right R-module is almost-injective. If R is a right and left CAI-ring, then R is called a CAI-ring.

Example 2.3 (1) Every semisimple ring is CAI.

(2) Let F be a field. Then, the ring $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ is a right CAI-ring.

Firstly, we give the following key lemma:

Lemma 2.4 *Let R be a right CAI-ring. If M is a right R-module, then M/A is a semisimple module for every essential submodule A of M.*

Proof Let A be an essential submodule of M. We show that M/A is a semisimple module. By [11, Corollary 7.14], it is necessary to prove that every cyclic right R-module in the category $\sigma[M/A]$ is M/A-injective. In fact, let N be a cyclic right R-module (in the category $\sigma[M/A]$) and $f : A'/A \to N$ be a homomorphism from an arbitrary submodule A'/A of M/A to N. We show that f is extended to M/A. Call $\pi_1 : A' \to A'/A$, $\pi_2 : M \to M/A$ the natural projections and $\iota_1 : A' \to M$, $\iota_2 : A'/A \to M/A$ the inclusions. We consider the homomorphism $f \circ \pi_1 : A' \to N$. We show that $f \circ \pi_1$ is extended to M. Otherwise, since N is almost-injective, there

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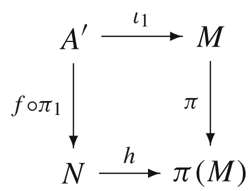
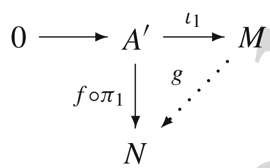
exist an idempotent π of $\mathrm{End}(M)$ and a homomorphism $h : N \to \pi(M)$ such that $\pi \circ \iota_1 = h \circ (f \circ \pi_1)$.



Then, we have

$$\pi(A) = (\pi \circ \iota_1)(A) = (h \circ f)(\pi_1(A)) = 0.$$

It means that $A \leq \mathrm{Ker}(\pi) = (1 - \pi)(M)$, and so $(1 - \pi)(M)$ is essential in M. This gives a contradiction. Thus, there is a homomorphism $g : M \to N$ such that $g \circ \iota_1 = f \circ \pi_1$.



We have

$$g(A) = (g \circ \iota_1)(A) = (f \circ \pi_1)(A) = 0$$

It shows that there is a homomorphism $g' : M/A \to N$ such that $g = g' \circ \pi_2$. From this gives

$$f \circ \pi_1 = g \circ \iota_1 = (g' \circ \pi_2) \circ \iota_1 = g' \circ (\pi_2 \circ \iota_1) = g' \circ (\iota_2 \circ \pi_1)$$

It follows that $f = g' \circ \iota_2$. Thus, N is M/A-injective. □

Corollary 2.5 *Every right CAI-ring is a right SC-ring.*

From Lemma 2.4 and [14], we have the following fact:

Fact 2.6 *If R is a right CAI-ring, then*

1. *$J(R) \leq Soc(R_R)$.*
2. *$J(R)^2 = 0$.*
3. *$R/Soc(R_R)$ is a right Noetherian ring.*

Theorem 2.7 *The following statements are equivalent for a ring R:*

1. *R is an Artinian serial ring with $J(R)^2 = 0$.*
2. *R is a right CAI-ring and $R/J(R)$ is I-finite.*
3. *R is a I-finite right CAI-ring.*
4. *R is a right CAI-ring with the finitely generated right socle.*

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Proof (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4) Suppose that R is a I-finite right CAI-ring. Then there exist primitive idempotents $e_1, e_2 \ldots, e_n$ such that $1 = e_1 + e_2 + \cdots + e_n$. Note that all $e_i R$ are indecomposable modules. Since R is a right CAI-ring, by [7, Lemma 3.1, Theorem 3.5], then $e_i R$ is uniform and $\text{End}(e_i R)$ is local for all $i \in \{1, 2, \ldots, n\}$. It follows that R is a semiperfect ring. We deduce, from Fact 2.6, that R is a semiprimary ring with $J(R)^2 = 0$. Moreover, inasmuch as $e_i R$ is uniform which implies that $\text{Soc}(e_i R)$ is simple for all $i \in \{1, 2, \ldots, n\}$. Thus, $\text{Soc}(R_R)$ is finitely generated.

(4) $\Rightarrow$ (1) Assume that R is a right CAI-ring with the finitely generated right socle. Then, R is a right Noetherian ring by Fact 2.6. We can write $R = e_1 R \oplus e_2 R \oplus \cdots \oplus e_n R$, where $e_1, e_2 \ldots, e_n$ are primitive idempotents such that $1 = e_1 + e_2 + \cdots + e_n$ and all right ideals $e_i R$ are uniform. By the proof of (3) $\Rightarrow$ (4), R is a semiprimary ring with $J(R)^2 = 0$. We deduce that R is a right Artinian ring. Note that $(R \oplus R)_R$ is an extending right R-module by [7, Remark 3.2]. It follows that $E(R_R)$ is a projective right R-module by [15, Theorem 3.3].

Next, we show that $e_i R$ is either simple or injective with the length of two. In fact, for any nonzero submodule U of $e_i R$, then $e_i R/U$ is a semisimple module by Lemma 2.4. Moreover, $e_i R/U$ is an indecomposable module. We deduce that $e_i R$ is either simple or length of two. On the other hand, we have that $E(e_i R)$ is a uniform projective module and obtain that $E(e_i R) \cong e_j R$ for some $j \in \{1, 2, \ldots, n\}$. Now, we assume that $e_k R$ is the module with length of two with $k \in \{1, 2, \ldots, n\}$. Then $E(e_k R)$ is indecomposable and projective. Therefore length $(E(e_k R)) \leq 2$, and so $E(e_k R) = e_k R$, i.e., $e_k R$ is injective. Thus, R is an Artinian serial ring with $J(R)^2 = 0$ by [11, 13.5]. □

Corollary 2.8 *The following statements are equivalent for a ring R.*

1. *R is an Artinian serial ring with $J(R)^2 = 0$.*
2. *R is a right CAI-ring with $\text{Soc}(R_R)/J(R)$ is finitely generated.*

Example 2.9 Consider the ring R consisting of all eventually constant sequences of elements from $\mathbb{F}_2$. Clearly, R is a CAI-ring and $\text{Soc}(R)$ is not finitely generated.

Lemma 2.10 *If R is a right CAI-ring, then*

1. *$R/\text{Soc}(R_R)$ is semisimple.*
2. *R is a right semi-Artinian ring.*

Proof (1) Assume that R is a right CAI-ring. One can check that $R/\text{Soc}(R_R)$ is also a right CAI-ring. From Fact 2.6 and Theorem 2.7 gives that $R/\text{Soc}(R_R)$ is a right Artinian ring. Note that $R/\text{Soc}(R_R)$ is a right V-ring by [3, Proposition 2.3]. We deduce that $R/\text{Soc}(R_R)$ is semisimple.

(2) is followed from (1). □

Proposition 2.11 *Let R be a right CAI-ring. Then the followings hold:*

1. *Every direct sum of uniform right R-modules is extending.*
2. *Every uniform right R-module has length at most 2.*
3. *$R_R = (\oplus_{i \in I} L_i) \oplus N$, where L_i is a local injective module of length two for every $i \in I$, $J(N) = 0$ and $\text{End}(N)$ is a right SV-ring.*

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Proof (1) From Lemma 2.10, R is a right semiartinian ring. By [11, 13.1], we need to prove that $H_1 \oplus H_2$ is an extending module for any uniform modules H_1 and H_2. In fact, let H_1 and H_2 are uniform right R-module. Since H_1 and H_2 are uniform with essential socles, $\mathrm{Soc}(H_1 \oplus H_2)$ is finitely generated and essential in $H_1 \oplus H_2$. Inasmuch as R is a right CAI-ring, we have every simple right R-module is almost-injective, and so $H_1 \oplus H_2$ is extending by [3, Theorem 2.9, Corollary 2.13.].

(2) is followed by (1) and [11, 13.1].

(3) By Zorn's Lemma, there is a maximal independent set of submodules $\{L_i\}_{i\in I}$ of R_R such that L_i is a local injective module of length two for every $i \in I$. Since by Fact 2.6(3), $R/\mathrm{Soc}(R_R)$ is a right Noetherian ring, then I is a finite set. Then, we have a decomposition $R_R = (\oplus_{i\in I} L_i) \oplus N$ for some right ideal N of R. Suppose that $J(N) \neq 0$. From Lemma 2.10(2) gives $J(N)$ containing a simple submodule S. Let N_0 be a complement of the submodule S in the module N. It follows that N/N_0 is a uniform nonsimple module whose socle is isomorphic to the module S. Thus, it follows from (1) and [3, Theorem 3.1] that N/N_0 is a projective module and length of N/N_0 is equal to two. Hence $N = N_0 \oplus L$, where L is a local injective module of length two, which contradicts the choice of the set $\{L_i\}_{i\in I}$. We deduce that $J(N) = 0$. One can check that the module N can be considered as a projective $R/J(R)$-module. By [3, Proposition 2.3] and Lemma 2.10, we have $R/J(R)$ is a right SV-ring. It follows from [16, Theorem 2.9] that $\mathrm{End}(N)$ is a right SV-ring.

□

For two-sided CAI-rings, we have:

Theorem 2.12 *The following statements are equivalent for a ring R:*

1. *Every R-module is almost injective.*
2. *Every finitely generated R-module is almost injective.*
3. *R is a CAI-ring.*
4. *R is a direct product of an SV-ring for which Loewy $(R) \leq 2$ and an Artinian serial ring whose squared Jacobson radical vanishes.*

Proof (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4) By Proposition 2.11, there exists an idempotent $e \in R$ such that $R_R = eR \oplus (1-e)R$, where $eR = \oplus_{i\in I} L_i$, L_i is a local injective module of length two for every $i \in I$, $J((1-e)R) = 0$ and $(1-e)R(1-e)$ is a right SV-ring. One can check that $\mathrm{Hom}(eR, (1-e)R) = 0$ and $J(R) = J(\oplus_{i\in I} L_i)$. Then $eR(1-e)$ is a submodule of ${}_RR$ and $eR(1-e) \leq J(R)$. It follows, from the left-sided analogue of Proposition 2.11(3), that there exists a set of orthogonal idempotents $\{f_1, \ldots, f_n\}$ such that $eR(1-e) = J(Rf_1 \oplus \cdots \oplus Rf_n)$ and Rf_i is a local injective module of length two for every $1 \leq i \leq n$. Consider the two-sided Peirce decomposition of the ring R corresponding to the decomposition $1 = e + (1-e)$

$$R = \begin{pmatrix} eRe & eR(1-e) \\ 0 & (1-e)R(1-e) \end{pmatrix}.$$

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Then for every $1 \le i \le n$ the following equalities hold

$$f_i = \begin{pmatrix} er_ie & em_i(1-e) \\ 0 & (1-e)s_i(1-e) \end{pmatrix},$$
$$(er_ie)^2 = er_ie, \ ((1-e)s_i(1-e))^2 = (1-e)s_i(1-e)$$

and

$$em_i(1-e) = er_iem_i(1-e) + em_i(1-e)s_i(1-e).$$

Let $S := (1-e)R(1-e)$ and $g_i := (1-e)s_i(1-e)$ for every $1 \le i \le n$. Fix an arbitrary index $1 \le i \le n$. We have that

$$J(R)f_i = \begin{pmatrix} eJ(R)e & eR(1-e) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} er_ie & em_i(1-e) \\ 0 & g_i \end{pmatrix} \le \begin{pmatrix} 0 & eR(1-e) \\ 0 & 0 \end{pmatrix}$$

and obtain $eJ(R)er_ie = 0$. On the other hand, for every $j \in J(R)$ and $m \in eR(1-e)$ we have

$$\begin{aligned} &\begin{pmatrix} eje & em(1-e) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} er_ie & em_i(1-e) \\ 0 & g_i \end{pmatrix} \\ &= \begin{pmatrix} 0 & ejem_i(1-e) + emg_i \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & eje(er_iem_i(1-e) + em_ig_i) + emg_i \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & e(jem_i + m)g_i \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

We deduce that $J(R)f_i \le \begin{pmatrix} 0 & eRg_i \\ 0 & 0 \end{pmatrix}$. Since $J(R)f_i \ne 0$, then $g_i \ne 0$. Inasmuch as the idempotent $f_i + J(R) \in R/J(R)$ is primitive and $J(R)^2 = 0$ we have $er_ie = 0$ and $eJ(R)eR(1-e) = 0$. Consequently,

$$\begin{pmatrix} 0 & eR(1-e) \\ 0 & 0 \end{pmatrix} = \bigoplus_{i=1}^{n} J(R)f_i = \bigoplus_{i=1}^{n} \begin{pmatrix} 0 & eR(1-e)g_i \\ 0 & 0 \end{pmatrix}.$$

It means that $eR(1-e) = \oplus_{i=1}^n eR(1-e)g_i$ and $eR(1-e)(1 - \sum_{i=1}^n g_i) = 0$. If, for some primitive idempotent g_0 of the ring S, the condition $g_0S \cong g_iS$ holds, where $1 \le i \le n$, then it can readily be seen that $Mg_0 \ne 0$. Thus the right ideals

$$\bigoplus_{i=1}^{n} g_iS \text{ and } \left((1-e) - \sum_{i=1}^{n} g_i\right)S$$

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of S do not contain isomorphic to simple S-submodules. Since S is a semiartinian regular ring, then $g = \sum_{i=1}^{n} g_i$ is a central idempotent of S and the ring R is isomorphic to the direct product of the regular ring $(1-e-g)S$ and the ring

$$R' = \begin{pmatrix} eRe & eR(1-e) \\ 0 & gR \end{pmatrix}.$$

Inasmuch as $eR = eRe + eR(1-e)$ is a module of finite length and for every $1 \le i \le n$, the idempotent $g_i \in (1-e)R(1-e)$ is primitive, we obtain that the ring R' is Artinian. Thus the ring R' is Artinian serial and $J(R')^2 = 0$ by Theorem 2.7. From Proposition 2.11, we have $(1-e-g)S$ is an SV-ring. Thus, the ring R is a direct product of an SV-ring for which Loewy $(R) \le 2$ and an Artinian serial ring whose squared Jacobson radical vanishes.

(4) ⇒ (1) is followed by Theorem 2.7 and [4, Proposition 2.6].

□

Theorem 2.13 *The following statements are equivalent for a ring R:*

1. *R is a right hereditary CAI-ring.*
2. *R is a right nonsingular CAI-ring.*
3. *R is a direct product of an SV-ring for which Loewy $(R) \le 2$ and a finite direct product of rings of the following form:*

$$\begin{bmatrix} \mathbb{M}_{n_1}(T) & \mathbb{M}_{n_1 \times n_2}(T) \\ 0 & \mathbb{M}_{n_2}(T) \end{bmatrix},$$

where T is a skew-field.

Proof (1) ⇒ (2) is obvious.

(2) ⇒ (3) is followed by Theorem 2.12 and [17, Theorem 8.11].

(3) ⇒ (1) is followed by [18, Proposition 9.6].

□

Corollary 2.14 *Any I-finite right nonsingular right CAI-ring R is isomorphic to a finite direct product of rings of the following form:*

$$\begin{bmatrix} \mathbb{M}_{n_1}(T) & \mathbb{M}_{n_1 \times n_2}(T) \\ 0 & \mathbb{M}_{n_2}(T) \end{bmatrix},$$

where T is a skew-field.

For two-sided CAI-rings, we obtain the important result, that is, they are also the rings for which every (right and left) R-module is almost injective. So, it is natural to ask the following question:
Question. Does the class of rings whose all right R-modules are almost-injective and class of all right CAI-rings coincide?

It is well-known that if M a non-singular indecomposable almost-injective right R-module, then End(M) is an integral domain and every nonzero endomorphism of

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M is a monomorphism. Moreover, if M is a cyclic module over a right Artinian ring, then $\mathrm{End}(M)$ is a skew-field. The following result is obvious.

Lemma 2.15 *Let R be a right Artinian ring and e be a primitive idempotent of R. If eR is a non-singular almost-injective right R-module, then eRe is a skew-field.*

Lemma 2.16 *Let R be a I-finite right nonsingular right CAI-ring and e, e' be any two primitive idempotents in R with $D = eRe$ and $D' = e'Re'$.*

1. *Then eRe' is a left vector space over D with the dimension less than or equal to 1.*
2. *If z is a non-zero element of eRe', there exists embedding $\sigma : D' \to D$ satisfying the property $ze'be' = \sigma(e'be')z$ for all $e'be' \in D'$.*
3. *If $\dim_D(eRe') = 1$, then σ is an isomorphism.*

Proof (1) First we assume that eRe' is non-zero with $D = eRe$ and $D' = e'Re'$. Take any non-zero element ere' in eRe'. We show that $D(ere') = D(eRe')$. In fact, let ese' be an arbitrary nonzero element of eRe'. Consider the mapping $\phi : e'R \to ere'R$ defined by $\phi(x) = erx$ for all $x \in e'R$. One can check that ϕ is a well-defined epimorphism. Since $e'R$ is an indecomposable almost-injective right R-module, $e'R$ is uniform. Assume that $\mathrm{Ker}(\phi)$ is nonzero. Then $e'R/\mathrm{Ker}(\phi)$ is a singular module. But, $\mathrm{Im}(\phi)$ is nonsingular by the nonsingularity of R, which gives a contradiction. It implies $\mathrm{Ker}(\phi) = 0$. It means that $ere'R \cong e'R$. Similarly, $ese'R \cong e'R$. We deduce that there exists an R-isomorphism $\sigma : ere'R \to ese'R$ satisfying $\sigma(ere') = ese'$. Call the homomorphism $\gamma : ere'R \to eR$ such that $\gamma(x) = \sigma(x)$ for all $x \in ere'R$.

Since R is a right CAI-ring, eR is almost eR-injective. Then, we have the following two cases for the homomorphism γ.

Case 1. σ is extended to an endmorphism of eR:

Take $\alpha : eR \to eR$ an endomorphism of eR which is an extension of σ. Then $ese' = \sigma(ere') = \alpha(ere') = e\alpha(e)e(ere') \in D(ere')$

Case 2. σ is not extended to an endmorphism of eR:

There is a homomorphism $\beta : eR \to eR$ such that $\beta \circ \gamma = \iota$ with $\iota : ere'R \to eR$ the inclusion. Then, we have $ere' = (\beta \circ \gamma)(ere') = \beta(ese') = e\beta(e)e(ese')$. Since D is a skew-field, $ese' = [e\beta(e)e]^{-1}ere' \in D(ere')$.

We deduce that $D(ere') = D(eRe')$. Thus, eRe' is a one-dimensional left vector space over D if $eRe' \neq 0$.

(2) Let z be a non-zero element of eRe'. Then, $eRe' = Dz$ by (1). It means that for any $e'be' \in e'Re'$, we have $ze'be' = uz$ for some $u \in D$. This defines a ring monomorphism $\sigma : D' \to D$ such that $\sigma(e'be') = u$. Thus, $\sigma(e'be')z = uz = ze'be'$ for all $e'be' \in D'$.

(3) Assume that R is a right serial ring and $\dim_D(eRe') = 1$. Take any two non-zero elements ere' and ese' in eRe'. By assumption, eR is uniserial, we may suppose $ese'R \leq ere'R$. There is $e'ue'$ in $e'Re'$ such that $ese' = ere'ue'$. We have that $e'Re'$ is a skew-field and obtain $ese'Re' = ere'Re'$. It means that eRe' is a one-dimensional right vector space over D'. Then $eRe' = Dz = zD'$, and so σ is an isomorphism.

□

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Corollary 2.17 *Any I-finite right nonsingular right CAI-ring R is isomorphic to*

$$\begin{bmatrix} \mathbb{M}_{n_1}(e_1Re_1) & \mathbb{M}_{n_1\times n_2}(e_1Re_2) & . & .. & \mathbb{M}_{n_1\times n_k}(e_1Re_k) \\ 0 & \mathbb{M}_{n_2}(e_2Re_2) & . & .. & \mathbb{M}_{n_2\times n_k}(e_2Re_k) \\ 0 & 0 & \mathbb{M}_{n_3}(e_3Re_3) & .. & \mathbb{M}_{n_3\times n_k}(e_3Re_k) \\ . & . & . & .. & . \\ . & . & . & .. & . \\ 0 & 0 & . & .. & \mathbb{M}_{n_k}(e_kRe_k) \end{bmatrix},$$

where e_iRe_i is a division ring, $e_iRe_i \cong e_jRe_j$ for each $1 \le i, j \le k$ and $n_1, \ldots, n_k$ are any positive integers. Furthermore, if $e_iRe_j \neq 0$, then

$$\dim({}_{e_iRe_i}(e_iRe_j)) = 1 = \dim((e_iRe_j)_{e_jRe_j}).$$

3 On Right Noetherian Right Almost V-rings

Firstly, we list some known results related to almost V-ring for the sake of completeness.

Theorem 3.1 [3, Theorem 3.1] *The following statements are equivalent for a ring R.*

1. *R is a right almost V-ring.*
2. *For every simple R-module S, either S is injective or $E(S)$ is projective of length 2.*

Theorem 3.2 [3, Theorem 2.9] *A ring R is a right almost V-ring if and only if every right R-module is simple-extending.*

Theorem 3.3 [5, Theorem 2.4] *The following statements are equivalent for a ring R.*

1. *R is a right Noetherian right almost V-ring.*
2. *Every right R-module is semisimple-extending.*
3. *$R = \oplus_{j=1}^{n} I_j$, where I_j is either a Noetherian V-module with zero socle, or a simple module, or an injective module of length 2.*
4. *$R = I \oplus J$, where I and J are right ideals, I is Noetherian, every semisimple module in $\sigma[I]$ is I-injective, and every module in $\sigma[J]$ is extending.*

The following result provides a characterization of right Noetherian right almost V-rings via almost injective semisimple modules. 2

Theorem 3.4 *The following statements are equivalent for a ring R.*

1. *R is a right Noetherian right almost V-ring.*
2. *Every semisimple right R-module is almost injective and R has finite right Goldie dimension.*
3. *Every semisimple right R-module is almost injective and Soc(R_R) is finitely generated.*

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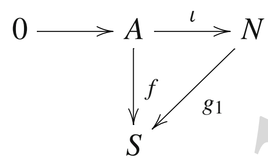
Proof (1) $\Rightarrow$ (2) By hypothesis, R has finite right Goldie dimension. Now we show that every semisimple right R-module S is almost injective. Let N be any module, $0 \to A \to N$ be any monomorphism for a submodule A of N and let $f : A \to S$ be any non-zero homomorphism. Assume $U = E(f(A))$ and $E(S) = U \oplus V$. Since R is a right Noetherian ring,

$$U = \bigoplus_{i \in I} E(S_i).$$

By Theorem 3.1, either $E(S_i)$ is simple or $E(S_i)$ is projective of length 2. Since U is injective, there exists a homomorphism $g : N \to U$ such that $f = g \circ \iota$.

Case 1: $g(N) \leq \oplus_{i \in I} S_i$. Let $\omega : \oplus_{i \in I} S_i \to S$ be the natural embedding and $g_1 = \omega \circ g$. In this case the following diagram commutes



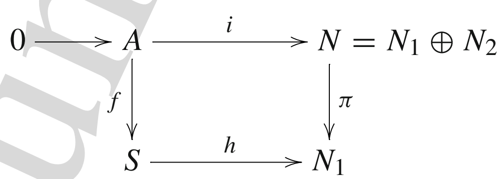
Case 2: $g(N) \nsubseteq \oplus_{i \in I} S_i$. Let $\pi_i : U \to E(S_i)$ be the canonical projection. Then there exists an index $j \in I$ such that $\pi_j(g(N)) \nsubseteq Soc(E(S_j))$. So that $\pi_j(g(N)) = E(S_j)$, since length$(E(S_i) \leq 2$, for any $i \in I$. Hence $\pi_j(g(N))$ is both injective and projective. It follows that there exists a decomposition $N = N_1 \oplus \text{Ker}(\pi_j \circ g)$, and $\varphi = (\pi_j \circ g)|_{N_1}$ is an isomorphism from N_1 to E_j. Set $w_1 = \varphi^{-1}$ and $w_2 = w_1 \pi_j$, $h_1 = w_2|_S$. Then h_1 is a homomorphism from $U \oplus V$ to N_1. Let $h = h_1|_S$ and $\pi : N \to N_1$ be the canonical projection. Let $a \in A$, then $a = a_1 + a_2$ with $a_1 \in N_1$ and $a_2 \in Ker(\pi_j \circ g)$. Therefore

$$\pi_j g(a) = \pi_j g(a_1) + \pi_j g(a_2) = \pi_j g(a_1) = \pi_j f(a_1) \in S_j.$$

Since φ is isomorphic, it follows that $a_1 \in \text{Soc}(N_1)$. Define a homomorphism $\varphi : Soc(N_1) \to S_j$ with $\theta(x) = \pi_j f(x)$. Last, we put $\beta = \pi_j|_S$ and $h = \theta^{-1}\beta$. Then h is a homomorphism from S to N_1. Let $a = x + y \in A$, where $x \in \text{Soc}(N_1)$ and $y \in \text{Ker}(\pi_j g)$. Then $\pi(a) = x$. Hence $\theta(x) = (\pi_j f)(x)$, so that

$$x = \theta^{-1}(\theta(x)) = \theta^{-1}(\pi_j f(x)) = \theta^{-1}(\beta)(f(x)) = (\theta^{-1}\beta)(f(x)) = hf(a).$$

Therefore $\pi \circ \iota = f \circ h$. In this case the following diagram commutes



Thus, S is an almost injective module.

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(2) ⇒ (3) is clear.

(3) ⇒ (1) Assume (3). Then R is an almost right V-ring. Let S be a semisimple right R-module. By [4, Proposition 2.1], S is essentially injective. Then, every semisimple right R-module is essentially injective. It follows that $R/\mathrm{Soc}(R_R)$ is right Noetherian, by [4, Lemma 2.2]. Hence R is a right Noetherian ring since $\mathrm{Soc}(R_R)$ is finitely generated. □

Theorem 3.5 *The following statements are equivalent for a ring R.*

1. *R is an Artinian serial ring with* $J(R)^2 = 0$.
2. *Every semisimple right R-module is almost injective,* R_R *is almost injective and R is a direct sum of indecomposable right ideals.*
3. *Every semisimple right R-module is almost injective,* R_R *is almost injective and* $\mathrm{Soc}(R_R)$ *is finitely generated.*

Proof First we note that if R_R is an almost injective module with finite Goldie dimension then R is a direct sum of uniform right ideals. Hence, it suffices to show that (3) ⇒ (1). Assume (3). By Theorem 3.4, R is a right Noetherian right almost V-ring, and R_R has a decomposition $R_R = e_1R \oplus e_2R \oplus \cdots \oplus e_nR$, where each e_iR is uniform, since R_R is almost injective. Let $e = e_i$, for $1 \le i \le n$. We shall prove that eR is a uniserial module. Let U, V be submodules of eR. Then U and V contain maximal submodules U_1 and V_1, respectively, since R is right Noetherian. Then $eR/(U_1 \oplus V_1)$ has two distinct minimal submodules $(U + V)/(U_1 + V)$ and $(U + V)/(U + V_1)$. This is impossible, since $eR/(U_1 \oplus V_1)$ is an indecomposable module over a right almost V-ring. Therefore eR is uniserial. Assume that eR is not simple, and U is a non-zero proper summodule of eR. Then there exists a maximal submodule U_1 of U. Since eR/U_1 is uniform, its socle is U/U_1. So length$(eR/U_1) = 2$, since R is a right almost V-ring. Hence U is simple and length$(eR) = 2$, and so eR is injective. Last, we get $R_R = e_1R \oplus e_2R \oplus \cdots \oplus e_nR$, where each e_iR is either a simple module or an injective module of length 2. By [11, 13.5, $(e) \Rightarrow (g)$], R is an Artinian serial ring with $J(R)^2 = 0$.

□

We obtain the following result in [4, Theorem 3.1]. 3

Corollary 3.6 *The following statements are equivalent for a ring R.*

1. *R is an Artinian serial ring with* $J(R)^2 = 0$.
2. *Every right R-module is almost injective and R is a direct sum of indecomposable right ideals.*
3. *Every right R-module is almost injective and* $\mathrm{Soc}(R_R)$ *is finitely generated.*

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