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CHARACTERIZATIONS OF QF-RINGS IN TERMS OF PSEUDO C*-INJECTIVITY

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Abstract. As a generalization of quasi-injective modules, an R-module M is pseudo N-c*-injective for every R-module N iff M is injective. In view of this new fact, we can get new generalizations of the following important observations taking the pseudo N-c*-injectivity instead of the continuity and the injectivity, respectively: if R is right continuous, left min-CS and satisfies ACC on its right annihilators then R is quasi Frobenius, and if $R_R^{(\mathbb{N})}$ is injective then R is quasi Frobenius.

1. Introduction

Throughout this paper, R is an associative ring with identity and all modules are unitary. M_R $({}_RM)$ denotes a right (left) R-module. For a module M, we use $E(M)$ and $End(M_R)$ to denote the injective hull and the endomorphism ring of M, respectively. We write $N \leq M$ if N is a submodule of M, $N \leq^{ess} M$ if N is an essential submodule of M and $N \leq^{\oplus} M$ if N is a direct summand of M. We denote by $\mathbb{M}_n(R)$ for the $n \times n$ matrix ring over R.

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We first recall some known notions and facts needed in the sequel.

For any submodule K of M the family of submodules N satisfying $K \cap N = 0$ has a maximal member by Zorn's Lemma, which is called *complement* of K in M. A submodule N of M is called a *complement* in M if N is a complement of a submodule of M. It is well known that a submodule is a complement in M if and only if it has no proper essential extensions in M (namely, a closed submodule). A module is called a *CS-module*, or *extending*, or it satisfies (C1) provided every complement submodule is a direct summand. Note that semi-simple modules, uniform modules and injective modules are CS. Injective modules and CS-modules are very important in algebra because their structures are well known for many classes of rings and each module has a unique injective envelope. There are other generalizations of injectivity;

C2: Every submodule of M that is isomorphic to a direct summand of M is itself a direct summand of M.

C3: If A and B are direct summands of M with $A \cap B = 0$, then $A \oplus B$ is also a direct summand of M.

Clearly, each C2-module is also a C3-module. However, if R is any integral domain which is not a field, then R is C3, but not C2.

A module M is *quasi-injective* in case each homomorphism $g : N \to M$ from a submodule N of M extends to M. For example, each semisimple module is quasi-injective. A module M is *continuous*, if M is both C1 and C2; M is *quasi-continuous* if M is both C1 and C3. We have the following hierarchy for any module M: M is injective $\Rightarrow$ M is quasi-injective $\Rightarrow$ M is continuous $\Rightarrow$ M is quasi-continuous. Let R be any hereditary two-sided noetherian right V-ring. By [4, Proposition 5.19(3)], the classes of all quasi-injective and all injective modules coincide and the class Ci ($i = 2, 3$) is closed under finite direct sums if and only if Ci ($i = 2, 3$) coincides with the class of all injective modules if and only if R is a semisimple artinian ring by [16, Theorem 3.2].

A module M is called continuous (resp., quasi - continuous) if it satisfies C1 and C2 (resp., C1 and C3).

As natural generalizations of quasi-injective modules:

An R-module M is called *GQ-injective* (*generalized quasi-injective*) if, for any submodule N which is isomorphic to a complement K of M, every left R-homomorphism of N into M extends to an endomorphism of M [10].

A module X is called *M-c-injective* if, for every closed submodule K of M, every homomorphism $f : K \to X$ can be lifted to M ([3]). The module M is called self-c-injective if M is M-c-injective.

A module M is called *pseudo-injective* if M is invariant under any monomorphism of its injective hull $E(M)$. By [8], a module is quasi-injective if and only if it is pseudo-injective CS.

A submodule N of M is called an *automorphism-invariant* submodule if $fN \subseteq N$ for every automorphism f of M, and a module is called an *automorphism-invariant module* if it is an automorphism-invariant submodule of its injective hull [12].

A module N is said to be *pseudo M-c*-injective* if for any submodule A of M which is isomorphic to a closed submodule of M, every monomorphism from A to N can be extended to a homomorphism from M to N ([15]). A module M is called *pseudo c*-injective* if M is pseudo M-c*-injective. A ring R is called right (resp., left) pseudo c*-injective if R_R (resp., ${}_RR$) is pseudo c*-injective.

It is easy to see that automorphism-invariant modules are pseudo c*-injective. We have some examples showed that there exist automorphism-invariant modules which are not quasi-injective or self-injective.

In the present paper, we continue to develop properties of these modules. Here we prove that the class of pseudo c*-injective modules is closed under taking direct summands. By [15], the class of pseudo c*-injective modules is a proper extension of the class of continuous modules and it is a proper subclass of modules which satisfy the C2 condition.

A ring R is called *quasi Frobenius* if R is two-sided self injective two-sided Artinian and R is called right *min-CS* if every its minimal right ideal is essential in a direct summand of R. It is easy to see that, if R is right (resp., left) CS then R is right (resp., left) min-CS. The converse is not true in general. For example, let $R = \begin{pmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix}$, then R is right min-CS but it is not right CS ([13, page 86]).

In [14], Nicholson and Yousif proved that, if R is right continuous, left min-CS and satisfies ACC on its right annihilators then R is quasi Frobenius. In Theorem 3.12, we proved that if R is right pseudo c*-injective, two-sided min-CS and satisfies ACC on its right annihilators then R is quasi Frobenius.

In [7], the authors C. Faith and D. V. Huynh proved if $R_R^{(\mathbb{N})}$ is injective then R is quasi Frobenius. In Corollary 3.13, we proved that if $R_R^{(\mathbb{N})}$ is pseudo c*-injective then R is quasi Frobenius.

2. Examples

Recall that quasi-injective or self-injective modules are automorphism invariant and automorphism invariant modules are pseudo c*-injective.

The following two examples give us that there exists an indecomposable module with finite Goldie dimension which is automorphism invariant but not quasi-injective.

Example 2.1. Let $R = \begin{bmatrix} \mathbb{F}_2 & \mathbb{F}_2 & \mathbb{F}_2 \\ 0 & \mathbb{F}_2 & 0 \\ 0 & 0 & \mathbb{F}_2 \end{bmatrix}$ where $\mathbb{F}_2$ is the field of two elements. Take $M := \begin{bmatrix} \mathbb{F}_2 & \mathbb{F}_2 & \mathbb{F}_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = e_{11}R$, where e_{11} is a primitive idempotent. Clearly M is an indecomposable right R-module. Since R is a finite-dimensional $\mathbb{F}_2$-algebra, M is an artinian right R-module and hence it has finite Goldie dimension.

Note that M has two simple submodules $S_1 = e_{12}R = \begin{bmatrix} 0 & \mathbb{F}_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $S_2 = e_{13}R = \begin{bmatrix} 0 & 0 & \mathbb{F}_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, which implies that M is automorphism invariant. But clearly M is not quasi-injective as it is not uniform.

Example 2.2. Let $A = \mathbb{F}_2[x]$ where $\mathbb{F}_2$ is the field of two elements and $R = \begin{bmatrix} A/(x) & 0 \\ A/(x) & A/(x^2) \end{bmatrix}$. Take $M := \begin{bmatrix} 0 & 0 \\ A/(x) & A/(x^2) \end{bmatrix} M = e_{22}R$, where e_{22} is a primitive idempotent. Clearly, M is an indecomposable right R-module. Note that M has two simple submodules $S_1 = \begin{bmatrix} 0 & 0 \\ A/(x) & 0 \end{bmatrix}$ and $S_2 = \begin{bmatrix} 0 & 0 \\ 0 & (x)/(x^2) \end{bmatrix}$ such that $S_1 \oplus S_2$ is essential in M. Clearly, R is a finite-dimensional $\mathbb{F}_2$-algebra. Then M is automorphism invariant. But M is not quasi-injective as M is not uniform.

Example 2.3. Consider the ring R consisting of all eventually constant sequences of elements from $\mathbb{F}_2$. Clearly, R is a commutative automorphism-invariant ring as the only automorphism of its injective envelope is the identity automorphism. But R is not self-injective.

Example 2.4. Let D be a PCI-domain, that is not a division ring. Denote by $E(D)$ the injective hull of D. Then $E(D)/D$ is semisimple, and so $E(D)$ has a maximal submodule M containing D. It follows that M is a continuous right D-module and not injective. Then, M is pseudo c*-injective. Assume that M is automorphism invariant, then M would be injective by [9, Corrolary 3.3], a contradiction. Thus, M is not automorphism invariant.

3. Results

We begin with recalling the basic properties of pseudo M-c*-injective modules.

Lemma 3.1 ([15, Lemma 3.1])**.** *Let M and N be two modules.*

(1) *If N is pseudo M-c*-injective and A is a direct summand of N, A is pseudo M-c*-injective.*

(2) *If N is pseudo M-c*-injective and B is a closed submodule of M, N is pseudo B-c*-injective.*

(3) *If M is pseudo c*-injective, A is pseudo c*-injective for all fully invariant closed submodule A of M.*

Lemma 3.2. *Let M, M', N, N' be modules, $M \cong M'$ and $N \cong N'$. If M is pseudo N-c*-injective then M' is pseudo N'-c*-injective.*

Proof. Let $K \leq M'$. Assume K is isomorphic to a closed submodule of M' and consider the monomorphism $f : K \to N'$. If $\varphi : M' \to M$, $\psi : N' \to N$ is an isomorphism, then $\varphi(K)$ is closed in M and $\psi f : K \to N$ is a monomorphism. Set $g = \psi f \varphi^{-1}_{|\varphi(K)} : \varphi(K) \to N$. By the hypothesis, there exists a homomophism $h : M \to N$ such that it is an extension of g. Now, we show that $\psi^{-1} h \varphi : M' \to N'$ is an extension of f. For every $k \in K$, we get $(\psi^{-1} h \varphi)(k) = \psi^{-1}(h\varphi(k)) = \psi^{-1}(g\varphi(k)) = \psi^{-1}(\psi f(k)) = f(k)$, as desired. ■

Theorem 3.3. *Let M and N be two modules.*

(1) *If M is a pseudo c*-injective module, then*
   (a) *Every direct summand of M is also pseudo c*-injective.*
   (b) *If $N \cong M$, then N is pseudo c*-injective.*

(2) *If $N = \Pi_{i \in I} N_i$ is pseudo M-c*-injective then N_i is pseudo M-c*-injective for all $i \in I$.*

(3) *Let $M = \oplus_{i \in I} M_i$, M_i is uniform module for all $i = 1, 2, ..., n$. Then M is continuous if and only if M is pseudo c*-injective.*

Proof. (1) This follows from Lemmas 3.1 and Lemma 3.2.

(2) Let $N = \Pi_{i \in I} N_i$ be a pseudo M-c*-injective, A be a submodule which is isomorphic to a closed submodule of M and $f_i : A \to N_i$ be a monomorphism. Consider the natural inclusions $\eta_i : N_i \to N$ and the canonical projections $\pi_i : N \to N_i$. Clearly, $g_i = \eta_i \circ f_i : A \to X$ is a monomorphism. Then, there

exists a homomorphism $\varphi_i : M \to X$ which extends to g_i. Set $\psi_i = \pi_i \circ g_i$. It is easy to see that ψ_i is an extension of f_i. Thus, N_i is pseudo M-c*-injective.

(3) This is [15, Theorem 3.4]. ■

Recall that a ring R is called right *hereditary* (resp., *semihereditary*) if every right (resp., finitely generated) ideal of R is projective as R-module. In [11, Corollary 2.28], Lam proved that a ring R is right semihereditary if and only if every right finitely generated projective submodule of R-module is projective. We have:

Theorem 3.4. *The following conditions are equivalent for a ring R:*

(1) *Every right closed ideal of R is projective;*

(2) *Every factor module of a pseudo R_R-c*-injective module is also pseudo R_R-c*-injective;*

(3) *Every factor module of a pseudo R_R-injective module is pseudo R_R-c*-injective;*

(4) *Every factor module of an injective module is pseudo R_R-c*-injective.*

Proof. (2) $\Rightarrow$ (3) $\Rightarrow$ (4) This is clear.

(1) $\Rightarrow$ (2) Let E be a pseudo R_R-c*-injective module and consider the epimorphism $\pi : E \to B$. Let $f : I \to B$ be a monomorphism, where I is a right ideal of R. Consider the following diagram:

$$\begin{tikzcd} & 0 \arrow[d] & \\ 0 \arrow[r] & I \arrow[r, "i"] \arrow[dl, "g"'] \arrow[d, "f"] & R \\ E \arrow[r, "\pi"] & B \arrow[r] & 0 \end{tikzcd}$$

where i is the canonical monomorphism. By (1), I is projective. Then, there exist a homomorphism $g : I \to E$ such that $\pi g = f$. Since E is pseudo R_R-c*-injective, there exists a homomorphism $h : R \to E$ such that $hi = g$. Set $\varphi = \pi g : R \to B$. Then $\varphi i = f$ and so B is pseudo R_R-c*-injective.

(4) $\Rightarrow$ (1) Let I be a closed right ideal of R and consider the epimorphism $h : A \to B$ and the homomorphism $\alpha : I \to B$. Clearly, $\psi : B = h(A) \to$ $\to A/\operatorname{Ker} h$ is an isomorphism defined by $\psi(h(a)) = a + \operatorname{Ker} h$. For the monomorphism $\iota_1 : A/\operatorname{Ker} h \to E(A)/\operatorname{Ker} h$, set $j = \iota_1\psi$ and consider the

following diagram:

$$\begin{array}{ccccc} & & I & \xrightarrow{i} & R \\ & & \downarrow \alpha & & \\ A & \xrightarrow{h} & B & \longrightarrow & 0 \\ & & \downarrow j & & \\ E(A) & \xrightarrow{p} & E(A)/\operatorname{Ker} h & \longrightarrow & 0 \end{array}$$

By (4), $E(A)/\operatorname{Ker} h$ is pseudo R_R-injective. Then, there exists a homomorphism $\alpha' : R \to E(A)/\operatorname{Ker} h$ such that $\alpha' i = j\alpha$. Since R_R is projective, there exists a homomorphic $\alpha'' : R \to E(A)$ such that $p\alpha'' = \alpha'$. Set $h' = \alpha'' i : I \to E(A)$. Clearly, $h'(I) \leq A$, so there exists a homomorphism $\varphi : I \to A$ such that $\varphi(x) = h'(x)$ for all $x \in I$.

Now, we show $h\varphi = \alpha$. For every $x \in I$, we have $j\alpha(x) = \alpha'(i(x)) = \alpha'(x) = = p\alpha''(x) = ph'(x) = p\alpha(x)$. Since, α is an epimorphism, $\alpha(x) = h(a)$ for some $a \in A$. Then $j\alpha(x) = j(h(a)) = a + \operatorname{Ker} h$. Hence, $a + \operatorname{Ker} h = \varphi(x) + \operatorname{Ker} h$, i.e., $h(a - \varphi(x)) = 0$. It follows $\varphi(x) = h(a) = \alpha(x)$. Thus, I is projective. ■

Theorem 3.5 ([15, Theorem 3.3])**.** *If $M \oplus N$ is a pseudo c^*-injective then M is N-injective.*

Corollary 3.6. *A ring R is right quasi injective if and only if $(R \oplus R)_R$ is pseudo c^*-injective.*

From Corollary 3.6 and [13, Theorem 1.50], we have:

Corollary 3.7. *A ring R is quasi Frobenius if and only if R satisfies ACC on right (or left) annihilators and $(R \oplus R)_R$ is pseudo c^*-injective.*

Theorem 3.8. *The following conditions are equivalent:*

(1) *The direct sum of every two pseudo c^*-injective modules is pseudo c^*-injective;*

(2) *Every pseudo c^*-injective module is injective;*

(3) *The direct sum of any family of pseudo c^*-injective modules is pseudo c^*-injective.*

Proof. (1) $\Rightarrow$ (2) Assume M is pseudo c*-injective. By the hypothesis, $M \oplus E(R_R)$ is pseudo c*-injective. By Theorem 3.5, M is $E(R_R)$-injective, so M is R_R-injective. Hence, M is an injective R-module.

(2) $\Rightarrow$ (3) We first prove R is a right Noetherian. Consider a family simple modules $(S_i)_{i\in\mathbb{N}}$ and $E_i = E(S_i)$ be the injective envelopes of S_i. Since $\oplus_{i\in\mathbb{N}} S_i$

is semisimple, it is pseudo c*-injective. By the hypothesis, $\oplus_{i\in\mathbb{N}} S_i$ is injective. Hence, $\oplus_{i\in\mathbb{N}} S_i$ is direct summand of $\oplus_{i\in\mathbb{N}} E_i$. However, $\oplus_{i\in\mathbb{N}} S_i \leq^e \oplus_{i\in\mathbb{N}} E_i$. It follows $\oplus_{i\in\mathbb{N}} S_i = \oplus_{i\in\mathbb{N}} E_i$. So, $\oplus_{i\in\mathbb{N}} E_i$ is injective. By [11, Therem 3.46], R is right Noetherian. Now, assume $(M_i)_{i\in I}$ is a family of pseudo c*-injective R-modules. Since, M_i is injective for all $i \in I$, we get $\oplus_I M_i$ is injective. Hence, $\oplus_I M_i$ is pseudo c*-injective.

(3) $\Rightarrow$ (1) This is clear. ∎

Recall the following hierarchy for any module M: M is injective $\Rightarrow$ M is quasi-injective.

Theorem 3.9. *The following statements are equivalent for an R-module M:*

(1) *M is injective;*

(2) *M is pseudo N-c*-injective for every R-module N.*

Proof. (1) $\Rightarrow$ (2) This is clear.

(2) $\Rightarrow$ (1) Consider the external direct sum $M \oplus E(M)$. Then, $M \oplus 0$ is a closed submodule of $M \oplus E(M)$ and $M \cong 0 \oplus M \cong M \oplus 0$. Consider the homomorphism $\alpha : M \to 0 \oplus M$ defined by $\alpha(m) = (0, m)$ for all $m \in M$. Clearly, α is an isomorphism. By the hypothesis, M is pseudo $M \oplus E(M)$-c*-injective. There exists a homomorphism $\beta : M \oplus E(M) \to M$ such that $\beta j = \alpha^{-1}$, where $j : 0 \oplus M \to M \oplus E(M)$ is the canonical projection. We have $\beta j \alpha = \alpha^{-1}\alpha = 1_M$ and $j\alpha = \iota_2 \iota$ where $\iota : M \to E(M), \iota_2 : E(M) \to M \oplus E(M)$ are inclusions. Hence $(\beta\iota_2)\iota = 1_M$. So, M is a summand of $E(M)$, i.e., M is injective. ∎

For a module M, we use $J(M)$ and $\mathrm{Soc}(M)$ to denote the Jacobson radical and the socle of M, respectively.

Proposition 3.10. *If R is a right pseudo c*-injective ring and $R/\mathrm{Soc}(R_R)$ satisfies ACC on right annihilators, then $J(R)$ is nilpotent.*

Proof. Assume $R/\mathrm{Soc}(R_R)$ has ACC on right annihilators. Set $S = \mathrm{Soc}(R_R)$ and $\overline{R} = R/S$. Take $\overline{a} \in \overline{R}$ such that $\overline{a} = a + S$ where $a \in R$.

For $a_1, a_2, ... \in J(R)$, we have

$$r_{\overline{R}}(\overline{a}_1) \leq r_{\overline{R}}(\overline{a}_2.\overline{a}_1) \leq \cdots \leq r_{\overline{R}}(\overline{a}_n...\overline{a}_2.\overline{a}_1).$$

By the hypothesis, there exists a positive integer m such that

$$r_{\overline{R}}(\overline{a}_n...\overline{a}_2.\overline{a}_1) = r_{\overline{R}}(\overline{a}_m...\overline{a}_2.\overline{a}_1)$$

for all $n > m$. For any $n \in \mathbb{N}$, we have $r(a_{n+1}a_n...a_1) \leq^e R_R$ since $a_{n+1}a_n...a_1 \in$ $\in J(R) = Z(R_R)$. Hence $S \leq r(a_{n+1}a_n...a_1)$. Now we shall prove

$$r_{\overline{R}}(\overline{a}_n...\overline{a}_2.\overline{a}_1) \leq r(a_{n+1}a_n...a_1)/S \leq r_{\overline{R}}(\overline{a}_{n+1}...\overline{a}_2.\overline{a}_1).$$

If $b + S \in r_{\overline{R}}(\overline{a}_n...\overline{a}_2.\overline{a}_1)$, then $a_n...a_1b \in S$. Since $S \leq r(a_{n+1})$, we get $a_{n+1}a_n...a_1b = 0$. Thus $b \in r(a_{n+1}a_n...a_1)$ which implies that $b+S \in r(a_{n+1}a_n...a_1)/S$. Clearly, $r(a_{n+1}a_n...a_1)/S \leq r_{\overline{R}}(\overline{a}_{n+1}...\overline{a}_2.\overline{a}_1)$. Hence,

$$r(a_{m+1}a_n...a_1)/S = r(a_{m+2}a_{m+1}...a_1)/S.$$

Then,

$$r(a_{m+1}a_n...a_1) = r(a_{m+2}a_{m+1}...a_1).$$

So, $a_{m+1}a_m...a_1R \cap r(a_{m+2}) = 0$. As $r(a_{m+2})$ is closed right ideal of R, we have $a_{m+1}a_m...a_1 = 0$ which shows $J(R)$ is right T-nilpotent and $(J(R)+S)/S$ is a right T-nilpotent ideal. By [2, Proposition 29.1], $(J(R)+S)/S$ is nilpotent. There exists a positive integer number k such that $J(R)^k \leq S$. So, $J(R)^{k+1} \leq$ $\leq SJ(R) = 0$, i.e., $J(R)$ is nilpotent. ■

Recall that a family $\{A_i | i \in I\}$ of submodules of a module M is independent if and only if the sum of the A_i is a direct sum. Equivalently, the map $\oplus_{i\in I} A_i \to$ $\to \sum_{i\in I} A_i$ is an isomorphism. A family $\{A_i | i \in I\}$ of independent submodules of a module M is said to be a *local direct summand* if for any finite subset $J \subset I$, $\oplus_{i\in J} A_i$ is a direct summand of M.

Lemma 3.11 ([15, Corollary 3.6]). *If R is right pseudo c^*-injective and satisfies ACC on right annihilators, then R is semiprimary.*

By [13], a ring R is quasi Frobenius if only if R is right continuous, left min-CS and satisfies ACC on its right annihilators.

Theorem 3.12. *The following conditions are equivalent for a ring R:*

(1) *R is quasi Frobenius;*

(2) *R is right pseudo-c^*-injective, two-sided min-CS and satisfies ACC on right annihilators.*

Proof. (1) ⇒ (2) This is clear.

(2) ⇒ (1) Since R is right pseudo-c*-injective and satisfies ACC on right annihilators, by [15, Corollary 3.6], R is semiprimary. Assume $\mathrm{Soc}(R_R) = \oplus_{i\in I} S_i$ where each S_i is a simple. As R is right min-CS, there exists idempotents f_i of R such that $S_i \leq^e f_iR$. On the other hand, $(S_i)_{i\in I}$ is independent, so $(f_iR)_{i\in I}$ is independent and $\mathrm{Soc}(R_R) \leq \oplus_{i\in I} f_iR$. Hence, $\oplus_{i\in I} f_iR \leq^e R_R$. By [15, Theorem 3.1], R_R satisfies the C2 condition. Then $\oplus_{i\in I} f_iR$ is a local direct

summand of R_R. In addition, R satisfies ACC on right annihilators, by [6, Lemma 8.1(1)], $\oplus_{i\in I} f_i R$ is closed submodule of R_R. Since $\oplus_{i\in I} f_i R \leq^e R_R$, we get $R_R = \oplus_{i\in I} f_i R$. So $R_R = \oplus_{i=1}^n f_i R$ (for some positive integer n) and $f_i R$ are uniforms for all $i = 1, 2, ..., n$. By Theorem 3.3, R is right continuous and so R is quasi Frobenius by [13, Theorem 4.22]. ∎

By [7], if $R_R^{(\mathbb{N})}$ is injective, (i.e., R is right countable injective) then R is quasi Frobenius.

Corollary 3.13. *The following conditions are equivalent for a ring R:*

(1) *R is quasi Frobenius;*

(2) *$R_R^{(\mathbb{N})}$ is pseudo c^*-injective.*

Proof. (1) ⇒ (2) This is clear.

(2) ⇒ (1) This follows from Theorem 3.5 and [7, Corollary 9.1]. ∎

Corollary 3.14. *The following conditions are equivalent for a ring R:*

(1) *R is quasi Frobenius;*

(2) *R is left Noetherian, right pseudo c^*-injective and two-sided min-CS.*

Proof. (1) ⇒ (2) This is clear.

(2) ⇒ (1) As R is left Noetherian, $R/J(R)$ is also a left Noetherian ring. By [15, Corollary 3.4], $R/J(R)$ is a von Neumann regular ring, so $R/J(R)$ is a semisimple Artinian ring. By Proposition 3.10, $J(R)$ is nilpotent and so R is semiprimary. Thus R is a left Artinian ring which implies that R satisfies ACC on right annihilators. By Theorem 3.12, R is QF. ∎

We finish this part with a question: Is there a right pseudo c*-injective and right min-CS ring but it is not right continuous?

4. On rings in which every cyclic module is pseudo c*-injective

In this section, we study rings R in which every cyclic right R-module is pseudo c*-injective.

An R-module M is called a C4-module if, whenever A_1 and A_2 are submodules of M with $M = A_1 \oplus A_2$ and $f : A_1 \to A_2$ is an R-homomorphism with $ker(f) \leq^{\oplus} A_1$, we have $Im(f) \leq^{\oplus} A_2$ [5].

Proposition 4.1. *Let R be a ring in which every cyclic right R-module is pseudo c^*-injective and let e and f be orthogonal idempotents of R. Then the following conditions holds:*

(1) *If* $eaf \neq 0$ *for some* $a \in R$*, then* $eafR \subseteq^{\oplus} eR$*.*

(2) *If* $fR \cong eR$*, then for every* $0 \neq b \in eR$*,* bR *contains a nonzero idempotent of* R*. In particular* $rad(eR) = rad(fR) = 0$*.*

(3) *If* e, f *are indecomposable and* $eaf \neq 0$ *for some* $a \in R$*, then* $eR \cong fR$ *and they are minimal right ideals of* R*.*

Proof. Let e and f be orthogonal idempotents of R. Then, we have that eR and fR are orthogonal summands and obtain $eR \oplus fR = (e+f)R$. Hence o $eR \oplus fR$ is a summand of R.

(1) We define $g : fR \longrightarrow eR$ by $g(fr) = eafr$. Clearly, g is a well-defined non-zero homomorphism with $Im(g) = eafR$. Set $K = Ker(g)$ and consider the monomorphism $h : fR/K \longrightarrow eR$ defined by $h(a+K) = g(a)$, for all $a \in fR$. Since every cyclic right R-module is pseudo c*-injective, $(e+f)R/K \cong$ $\cong fR/K \oplus eR$ is a pseudo c*-injective module. So $eafR = Im(g) = Im(h)$ is a direct summand of eR.

(2) Let $fR \cong eR$, and $b \in eR$ with $b \neq 0$. One can check that $b = eb$. Now, if $eb(1-e) \neq 0$, then, by (1), $eb(1-e)R \subseteq^{\oplus} eR$. Since $eb(1-e)R \subseteq ebR = bR$, we get bR contains a non-zero idempotent, as required. If $eb(1-e) = 0$, then $b = eb = ebe$. We see $ebeR \oplus eR \cong ebeR \oplus fR = (ebe+f)R$ and so, by hypothesis, $ebeR \oplus eR$ is a $C4$-module. Consequently, $ebeR \subseteq^{\oplus} eR$ and bR contains a non-zero idempotent, since $ebeR = ebR = bR$. Now, if $K \subseteq eR$ is a small submodule of eR and $0 \neq k \in K$, then kR contains a non-zero idempotent $g \in R$ by the first part of the proof, and so gR is small in eR, a contradiction. Hence $rad(eR) = 0$. Therefore, $rad(fR) = 0$.

(3) By (1), we get $eafR$ a direct summand of R, and so $eafR = eR$ is projective. Therefore, the epimorphism $g : fR \to eafR$ given by $g(fr) = eafr$ splits by the projectivity of $eafR$. Thus, $eR = eafR \cong fR$. Now, if $0 \neq$ $\neq b \in eR$, then bR contains a nonzero idempotent of R by (2) and since eR is indecomposable $bR = eR$. Hence eR as well as fR is minimal. ■

Corollary 4.2. *Let* R *be a ring in which every cyclic right* R*-module is pseudo c***-injective such that* $R = C \oplus A \oplus B$ *where* $A \cong B$ *and* C *embeds in* $A \oplus B$*. Then* $rad(R) = 0$*.*

In particular, if every cyclic right R*-module is pseudo c***-injective such that* $R = A \oplus B$ *where* $A \cong B$*, then* $rad(R) = 0$*.*

A ring is called an *I-finite ring* if it contains no infinite sets of orthogonal idempotents.

Theorem 4.3. *Let* $R/J(R)$ *be an I-finite ring. Then every cyclic right* R*-module is pseudo c***-injective if and only if* $R = S \oplus T$*, where* S *is semisimple artinian and* T *is a finite direct sum of semilocal rings with no nontrivial idempotents in which every cyclic right module is pseudo c***-injective.*

Proof. Assume that $R/J(R)$ is an I-finite ring. Then R is an I-finite ring, and so the ring R has an indecomposable decomposition $R = e_1R \oplus e_2R \oplus \cdots \oplus e_nR$, where e_i are pairwise orthogonal primitive idempotents of R. Denote

$$[e_tR] = \sum_i \{e_iR : e_iR \cong e_tR\}.$$

Renumbering if necessary, we may write $R = [e_1R] \oplus [e_2R] \oplus \cdots \oplus [e_kR]$. By Proposition 4.1, each $[e_iR]$ is an ideal of R. If $[e_iR]$ contains more than one direct summands, then $[e_iR]$ is a simple artinian ring by Proposition 4.1. If $[e_jR]$ consists of exactly one direct summand, then $T_j := [e_jR] = e_jR = e_jRe_j$ is a rings with no nontrivial idempotents in which every cyclic right module is pseudo c*-injective. Next, we show that each T_j is a semilocal ring. In fact, we have that $R/J(R)$ is an I-finite ring and obtain that the ring $T_j/J(T_j)$ is too. Note that e_jR is a pseudo c*-injective module. It follows that $T_j/J(T_j)$ is a regular ring. We deduce that T_j is a semilocal ring. ■

Corollary 4.4. *Let R be a semiperfect ring. Then every cyclic right R-module is pseudo c*-injective if and only if $R = S \oplus T$, where S is semisimple artinian and T is a finite direct sum of local rings with no nontrivial idempotents in which every cyclic right module is pseudo c*-injective.*

We denote by $\mathbb{M}_n(R)$ for the $n \times n$ matrix ring over R.

Lemma 4.5. *Let $n \geq 2$. The following are equivalent for a ring R:*

(1) *Every n-generated R-module is a pseudo c*-injective module.*

(2) *Every cyclic $\mathbb{M}_n(R)$-module is a pseudo c*-injective module.*

Proof. Let $P = (R^n)_R$ and $S = \mathrm{End}(P_R)$. Then

$$Hom_R(P, -) : N_R \mapsto Hom_R({}_SP_R, N_R)$$

defines a Morita equivalence between Mod-R and Mod-S with the inverse equivalence $-\otimes_S P : M_S \mapsto M \otimes P$. For any n-generated R-module N, $Hom_R(P, N)$ is a cyclic S-module, and, for any cyclic S-module M, $M \otimes_S P$ is an n-generated R-module. Moreover, a Morita equivalence preserves the pseudo c*-injectivity for modules. Thus, every cyclic S-module is a pseudo c*-injective module if and only if every n-generated R-module is a pseudo c*-injective module. ■

Corollary 4.6. *The following are equivalent for a ring R:*

(1) *Every cyclic $\mathbb{M}_2(R)$-module is a pseudo c*-injective module.*

(2) *Every 2-generated R-module is a pseudo c*-injective module.*

(3) *R is semisimple.*

Proof. (1) $\Leftrightarrow$ (2) This follows from Lemma 4.5

(3) $\Rightarrow$ (1) & (2) They are obvious.

(1) & (2) $\Rightarrow$ (3) First we show that every cyclic right R-module is quasi-injective. In fact, let $M = mR$ be a cyclic right R-module with $m \in M$. By hypothesis, the 2-generated right R-module $mR \oplus mR$ is pseudo c*-injective, and so $M = mR$ is quasi-injective, as required. Now, we show that $rad(R) = 0$. Clearly, by (1), every cyclic $\mathbb{M}_2(R)$-module is a pseudo c*-injective module. We have $M_2(R) = \begin{bmatrix} R & R \\ R & R \end{bmatrix} = \begin{bmatrix} R & R \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ R & R \end{bmatrix}$ is a direct sum of two isomorphic right ideals. By Corollary 4.2, $rad(M_2(R)) = 0$ Consequently, $rad(R) = 0$ since $rad(M_2(R)) = M_2(rad(R))) = 0$. Inasmuch as R has the property that every cyclic right R-module is quasi-injective and $rad(R) = 0$, we infer from [1, Corollary], that R is semisimple. ■

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