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On the weak Lefschetz property for almost complete intersections generated by uniform powers of general linear forms

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A B S T R A C T

In [10], Conjecture 6.6, Migliore, the first author, and Nagel conjectured that, for all $n \geq 4$, the artinian ideal $I = (L_0^d, \dots, L_{2n+1}^d) \subset R = k[x_0, \dots, x_{2n}]$ generated by the d-th powers of $2n+2$ general linear forms fails to have the weak Lefschetz property if and only if $d > 1$. This paper is entirely devoted to prove partially this conjecture. More precisely, we prove that R/I fails to have the weak Lefschetz property, provided $4 \leq n \leq 8$, $d \geq 4$ or $d = 2r$, $1 \leq r \leq 8$, $4 \leq n \leq 2r(r+2) - 1$.

1. Introduction

Ideals generated by powers of linear forms have attracted great deal of attention recently. For instance, their Hilbert function has been the focus of the papers [2,8,16];

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and the presence or failure of the weak Lefschetz property has been deeply studied in [8,10,12–14], among others.

Let k be a field of characteristic zero and $R = k[x_0, \dots, x_n]$ be the standard graded polynomial ring over k in $n+1$ variables. A graded artinian k-algebra $A := R/I$ is said to have the *weak Lefschetz property* (WLP for short) if there is a linear form $\ell \in [A]_1$ such that the multiplication

$$\times \ell : [A]_i \longrightarrow [A]_{i+1}$$

has maximal rank for all i, i.e., $\times \ell$ is either injective or surjective, for all i. On the contrary, we say that A *fails to have the WLP* if there is an integer i such that the above multiplication does not have maximal rank for any linear form ℓ. There has been a long series of papers determining classes of algebras holding/failing the WLP but much more work remains to be done.

The first result in this direction is due to Stanley [15] and Watanabe [17] and it asserts that the WLP holds for *any* artinian complete intersection ideal I generated by powers of linear forms. In fact, they showed that there is a linear form $\ell \in [A]_1$ such that the multiplication

$$\times \ell^s : [A]_i \longrightarrow [A]_{i+s}$$

has maximal rank for all i, s. When this property holds, the algebra is said to have the *strong Lefschetz property* (briefly SLP). In [14], Schenck and Seceleanu gave the nice result that *any* artinian ideal $I \subset R = k[x, y, z]$ generated by powers of linear forms has the WLP. Moreover, when these linear forms are general, the SLP of R/I has also been studied, in particular, the multiplication by the square ℓ^2 of a general linear form ℓ induces a homomorphism of maximal rank in any graded component of R/I, see [1,11]. However, Migliore, the first author, and Nagel showed by examples that in 4 variables, an ideal generated by the d-th powers of five general linear forms fails to have the WLP for $d = 3, \dots, 12$ [10]. Therefore, it is natural to ask when the WLP holds for artinian ideals $I \subset k[x_0, \dots, x_n]$ generated by powers of $\geq n+2$ general linear forms. In [10], Migliore, the first author, and Nagel studied this question where the ideal is an almost complete intersection and they also proposed the following conjecture in order to complete this investigation.

Conjecture 1.1. *[10, Conjecture 6.6] Let $R = k[x_0, \dots, x_{2n}]$ be the polynomial ring over a field of characteristic zero. Consider an artinian ideal $I = (L_0^d, \dots, L_{2n+1}^d) \subset R$ generated by the d-th powers of general linear forms. If $n \geq 4$, then the ring R/I fails to have the WLP if and only if $d > 1$. Furthermore, if $n = 3$, then R/I fails to have the WLP when $d = 3$.*

The first author has shown that R/I fails to have the WLP when $d = 2$ [12] and in the recent paper [13], Nagel and Trok have established Conjecture 1.1 for $n \gg 0$ and

$d \gg 0$. The last part of the conjecture was proved by Di Gennaro, Ilardi, and Vallès in [3, Proposition 5.5]. Unfortunately, there was a gap in their proof. However, it was corrected in [4] and then in [9], the last part of Conjecture 1.1 was proved by Ilardi and Vallès. The goal of this note is to solve partially the conjecture. More precisely, we prove the following (see Corollaries 3.3–3.10, Theorem 4.1 and Remark 4.2).

Theorem. *Let $R = k[x_0, \dots, x_{2n}]$ be the polynomial ring over a field of characteristic zero and consider an artinian ideal $I = (L_0^d, \dots, L_{2n+1}^d) \subset R$ generated by the d-th powers of general linear forms.*

(1) *If $d = 2r$, $2 \le r \le 8$ and $4 \le n \le 2r(r+2) - 1$, then R/I fails to have the WLP.*
(2) *If $4 \le n \le 8$ and $d \ge 4$, then R/I fails to have the WLP.*

Therefore, Theorem answers partially Conjecture 1.1 for $4 \le n \le 8$, missing only the case $d = 3$, since the case $d = 2$ is shown by the first author [12]. Our approach is based on the connection between computing the dimension of $R/(I, \ell)$, where ℓ is a general linear form and the dimension of linear system of fat points. More precisely, we prove the following result (see Theorem 3.1).

Theorem. *If ℓ is a general linear form and $j = \lfloor \frac{(2n^2-1)(d-1)}{2n-1} \rfloor$, then*

$$\begin{aligned}\dim_k [R/(I,\ell)]_j &= \dim_k \mathfrak{L}_{2n-1}\big(j; (j+1-d)^{2n+2}\big) \\ &= \begin{cases} \dim_k \mathfrak{L}_{2n-1}\big(e; 0^{2n+2}\big) & \text{if } d = (2n-1)e+1 \\ \dim_k \mathfrak{L}_{2n-1}\big(e+n-r+1; (n-r)^{2n+2}\big) & \text{if } d = (2n-1)e+2r \\ \dim_k \mathfrak{L}_{2n-1}\big(e+2n-r+1; (2n-r-1)^{2n+2}\big) & \text{if } d = (2n-1)e+2r+1 \end{cases}\end{aligned}$$

where e, r are non-negative integers such that $1 \le r \le n-1$.

2. Preparatory results

Throughout this paper R denotes a polynomial ring $k[x_0, \dots, x_n]$ over a field k of characteristic zero, with its standard grading where $\deg(x_i) = 1$. If $I \subset R$ is a homogeneous ideal, then the k-algebra $A = \oplus_{j \ge 0}[A]_j$ is standard graded. Its Hilbert function is a map $h_A : \mathbb{N} \longrightarrow \mathbb{N}, h_A(j) = \dim_k[A]_j$.

For any artinian ideal $I \subset R$ and a general linear form $\ell \in R$, the exact sequence

$$[R/I]_{j-1} \longrightarrow [R/I]_j \longrightarrow [R/(I,\ell)]_j \longrightarrow 0$$

gives, in particular, that the multiplication by ℓ will fail to have maximal rank exactly when

$$\dim_k[R/(I,\ell)]_j \neq \max\{\dim_k[R/I]_j - \dim_k[R/I]_{j-1}, 0\}. \tag{2.1}$$

In this case, we will say that R/I fails to have the WLP in degree $j-1$.

We first recall a result of Emsalem and Iarrobino giving a duality between powers of linear forms and ideals of fat points in $\mathbb{P}^n$. We quote it in the form that we need.

Lemma 2.1. *[6, Theorem I] Let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be the ideals of s distinct points in $\mathbb{P}^n$ that are dual to the linear forms $\ell_1, \dots, \ell_s \in R$ and choose the positive integers $a_1, \dots, a_s$. Then, for each integer $j \geq -1 + \max\{a_1, \dots, a_s\}$,*

$$\dim_k \left[R/(\ell_1^{a_1}, \dots, \ell_s^{a_s})\right]_j = \dim_k \big[\bigcap_{a_i \leq j} \mathfrak{p}_i^{j+1-a_i} \big]_j.$$

If the points defined by the ideals $\mathfrak{p}_i$ are general points, then the dimension of the linear system $[\mathfrak{p}_1^{b_1} \cap \dots \cap \mathfrak{p}_s^{b_s}]_j \subset R_j$ depends only on the numbers $n, j, b_1, \dots, b_s$. In order to simplify notation, in this case we denote by

$$\mathfrak{L}_n(j; b_1, \dots, b_s)$$

the linear system $[\mathfrak{p}_1^{b_1} \cap \dots \cap \mathfrak{p}_s^{b_s}]_j \subset R_j$. We use superscripts to indicate repeated entries. For example, $\mathfrak{L}_4(j; 2^4, 4^2) = \mathfrak{L}_4(j; 2, 2, 2, 2, 4, 4)$. Notice that, for every linear system $\mathfrak{L}_n(j; b_1, \dots, b_s)$, one has

$$\dim_k \mathfrak{L}_n(j; b_1, \dots, b_s) \geq \max \left\{0, \binom{n+j}{n} - \sum_{i=1}^{s} \binom{n+b_i-1}{n} \right\}.$$

To study the WLP, the following is useful to compute the left-hand side of (2.1).

Lemma 2.2. *[10, Proposition 3.4] Let $(\ell_1^{a_1}, \dots, \ell_s^{a_s})$ be an ideal of R generated by powers of s general linear forms, and let ℓ be a general linear form. Then, for each integer $j \geq -1 + \max\{a_1, \dots, a_s\}$,*

$$\dim_k \left[R/(\ell_1^{a_1}, \dots, \ell_s^{a_s}, \ell)\right]_j = \dim_k \mathfrak{L}_{n-1}(j; j+1-a_1, \dots, j+1-a_s).$$

Using Cremona transformations, one can relate two different linear systems. This is often stated only for general points.

Lemma 2.3. *[5, Theorem 3] Let $s > n \geq 2$ and $j, b_1, \dots, b_s$ be non-negative integers, with $b_1 \geq \dots \geq b_s$. Set $t = (n-1)j - (b_1 + \dots + b_{n+1})$. If $b_i + t \geq 0$ for all $i = 1, \dots, n+1$, then*

$$\dim_k \mathfrak{L}_n(j; b_1, \dots, b_s) = \dim_k \mathfrak{L}_n(j+t; b_1+t, \dots, b_{n+1}+t, b_{n+2}, \dots, b_s).$$

In this note, we are interested in certain almost complete intersections. Then one can compute the right-hand side of (2.1). For any integer m, we denote

$$[m]_+ = \max\{m, 0\}.$$

Lemma 2.4. *[10, Lemma 3.7] Let $I = (L_0^{a_0}, \dots, L_{n+1}^{a_{n+1}}) \subset R$ be an almost complete intersection generated by powers of $n+2$ general linear forms. Set $A = R/(L_0^{a_0}, \dots L_n^{a_n})$. Then, for each integer j,*

$$\dim_k[R/I]_j - \dim_k[R/I]_{j-1} = [h_A(j) - h_A(j-a_{n+1})]_+ - [h_A(j-1) - h_A(j-a_{n+1}-1)]_+.$$

Furthermore, if $j \leq \frac{1}{2}a_{n+1} + \frac{1}{2}\sum_{i=0}^{n}(a_i - 1)$, then the formula simplifies to

$$\dim_k[R/I]_j - \dim_k[R/I]_{j-1} = [h_A(j) - h_A(j-1)] - [h_A(j-a_{n+1}) - h_A(j-a_{n+1}-1)].$$

3. Almost uniform powers of general linear forms

Throughout this section, we always denote $R = k[x_0, \dots, x_{2n}]$ and consider an artinian ideal $I = (L_0^d, \dots, L_{2n+1}^d)$ of R generated by the d-th powers of general linear forms and fix $j = \lfloor \frac{(2n^2-1)(d-1)}{2n-1} \rfloor$.

Theorem 3.1. *If ℓ is a general linear form, then*

$$\begin{aligned}\dim_k[R/(I,\ell)]_j &= \dim_k \mathfrak{L}_{2n-1}\big(j;(j+1-d)^{2n+2}\big)\\ &= \begin{cases}\dim_k \mathfrak{L}_{2n-1}\big(e;0^{2n+2}\big) & \text{if } d = (2n-1)e+1\\ \dim_k \mathfrak{L}_{2n-1}\big(e+n-r+1;(n-r)^{2n+2}\big) & \text{if } d = (2n-1)e+2r\\ \dim_k \mathfrak{L}_{2n-1}\big(e+2n-r+1;(2n-r-1)^{2n+2}\big) & \text{if } d = (2n-1)e+2r+1\end{cases}\end{aligned}$$

where e, r are non-negative integers such that $1 \leq r \leq n-1$ and

$$\dim_k[R/I]_j - \dim_k[R/I]_{j-1} = \sum_{k=0}^{n}(-1)^k\binom{2n+2}{k}\binom{2n-1+j-kd}{2n-1}.$$

Proof. It follows from Lemma 2.2 that

$$D := \dim_k[R/(I,\ell)]_j = \dim_k \mathfrak{L}_{2n-1}\big(j;(j+1-d)^{2n+2}\big).$$

Set

$$t = (2n-2)j - 2n(j+1-d) = -2j + 2n(d-1).$$

As $j = \lfloor \frac{(2n^2-1)(d-1)}{2n-1} \rfloor$, we get

$$j+1-d+t = -j + (2n-1)(d-1) \geq \frac{2(n-1)^2(d-1)}{2n-1} \geq 0.$$

Using Lemma 2.3 $(n+1)$ times, in each step the Cremona transformation changes the multiplicities of linear system $\mathfrak{L}_{2n-1}\big(j;(j+1-d)^{2n+2}\big)$ by t, we obtain

$$\begin{aligned}
D &= \dim_k \mathfrak{L}_{2n-1}\big(j; (j+1-d)^{2n+2}\big) \\
&= \dim_k \mathfrak{L}_{2n-1}\big(-j+2n(d-1); (j+1-d)^2, (-j+(2n-1)(d-1))^{2n}\big) \\
&= \dim_k \mathfrak{L}_{2n-1}\big(-3j+4n(d-1); (-j+(2n-1)(d-1))^4, \\
&\quad (-3j+(4n-1)(d-1))^{2n-2}\big) \\
&\quad \vdots \\
&= \dim_k \mathfrak{L}_{2n-1}\big(-(2n+1)j+2n(n+1)(d-1); (-(2n-1)j+(2n^2-1)(d-1))^{2n+2}\big).
\end{aligned}$$

These computations are correct and has a chance of resulting in a non-empty linear system if

$$-(2n+1)j+2n(n+1)(d-1) > -(2n-1)j+(2n^2-1)(d-1) \geq 0,$$

which is true since $j = \lfloor \frac{(2n^2-1)(d-1)}{2n-1} \rfloor$. Thus

$$D = \dim_k \mathfrak{L}_{2n-1}\big(2n(n+1)(d-1)-(2n+1)j; ((2n^2-1)(d-1)-(2n-1)j)^{2n+2}\big). \tag{3.1}$$

Now we consider three cases:

Case 1: $d = (2n-1)e+1$, hence $j = (2n^2-1)e$. By (3.1) and a simple computation shows that

$$D = \dim_k \mathfrak{L}_{2n-1}\big(e; 0^{2n+2}\big).$$

Case 2: $d = (2n-1)e+2r,\ 1 \leq r \leq n-1$. A straightforward computation shows that

$$j = (2n^2-1)e+2nr+r-n-1.$$

Therefore, by (3.1), we obtain

$$D = \dim_k \mathfrak{L}_{2n-1}\big(e+n-r+1; (n-r)^{2n+2}\big).$$

Case 3: $d = (2n-1)e+2r+1,\ 1 \leq r \leq n-1$. It is easy to show that

$$j = (2n^2-1)e+2nr+r-1.$$

By (3.1) we get that

$$D = \dim_k \mathfrak{L}_{2n-1}\big(e+2n-r+1; (2n-r-1)^{2n+2}\big).$$

Finally, let $A = R/(L_0^d, \dots, L_{2n}^d)$, hence A is a complete intersection and it has the SLP (see, e.g., [15] or [17]), one has

$$\dim_k[R/I]_j - \dim_k[R/I]_{j-1} = [h_A(j) - h_A(j-1)] - [h_A(j-d) - h_A(j-d-1)],$$

by Lemma 2.4 since $j \le \frac{d}{2} + \frac{(2n+1)(d-1)}{2}$. Resolving A over R using the Koszul resolution, we get for the Hilbert function of A

$$h_A(j) = \sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k}\binom{2n+j-kd}{2n}.$$

As $j = \lfloor \frac{(2n^2-1)(d-1)}{2n-1} \rfloor$, hence $j - kd < 0$ if $k \ge n+1$. It follows that

$$h_A(j) = \sum_{k=0}^{n}(-1)^k \binom{2n+1}{k}\binom{2n+j-kd}{2n}.$$

A straightforward computation gives

$$\dim_k[R/I]_j - \dim_k[R/I]_{j-1} = \sum_{k=0}^{n}(-1)^k \binom{2n+2}{k}\binom{2n-1+j-kd}{2n-1}. \quad \square$$

Proposition 3.2. *Assume that $d = (2n-1)e + 2r$, e and r are non-negative integers such that $1 \le r \le n$. If $n \le 2r(r+2) - 1$ then*

$$\dim_k[R/(I,\ell)]_j > 0,$$

where ℓ is a general linear form in R.

Proof. As $d = (2n-1)e + 2r$, $1 \le r \le n$, by Theorem 3.1 we get that

$$\dim_k[R/(I,\ell)]_j = \dim_k \mathfrak{L}_{2n-1}\big(e+n-r+1; (n-r)^{2n+2}\big),$$

where e is a non-negative integer. It is enough to show that

$$\dim_k \mathfrak{L}_{2n-1}\big(n-r+1; (n-r)^{2n+2}\big) > 0.$$

Lemma 2.1 shows that

$$\dim_k \mathfrak{L}_{2n-1}\big(n-r+1; (n-r)^{2n+2}\big) = \dim_k \Big[k[x_0,\dots,x_{2n-1}]/(\ell_0^2,\dots,\ell_{2n+1}^2)\Big]_{n-r+1},$$

where $\ell_0,\dots,\ell_{2n+1}$ are general linear forms in $k[x_0,\dots,x_{2n-1}]$. Setting

$$P = k[x_0,\dots,x_{2n-1}]/(\ell_0^2,\dots,\ell_{2n}^2)$$

then, by [12, Proposition 3.4], for every $0 \le t \le n$,

$$\dim_k [P]_t = \binom{2n}{t} - \binom{2n}{t-2}.$$

It follows that

$$\begin{aligned}\dim_k \mathfrak{L}_{2n-1}\big(n-r+1; (n-r)^{2n+2}\big) &\geq \dim_k [P]_{n-r+1} - \dim_k [P]_{n-r-1} \\ &= \binom{2n}{n-r+1} - 2\binom{2n}{n-r-1} + \binom{2n}{n-r-3}.\end{aligned}$$

We have

$$\begin{aligned}&\binom{2n}{n-r+1} - 2\binom{2n}{n-r-1} + \binom{2n}{n-r-3} > 0 \\ &\Leftrightarrow \frac{(2n)!}{(n-r+1)!(n+r-1)!} - \frac{(2n)!}{(n-r-1)!(n+r+1)!} \\ &> \frac{(2n)!}{(n-r-1)!(n+r+1)!} - \frac{(2n)!}{(n-r-3)!(n+r+3)!} \\ &\Leftrightarrow \frac{2r(2n+1)}{(n-r+1)!(n+r+1)!} > \frac{2(r+2)(2n+1)}{(n-r-1)!(n+r+3)!} \\ &\Leftrightarrow \frac{r}{(n-r)(n-r+1)} > \frac{r+2}{(n+r+2)(n+r+3)} \\ &\Leftrightarrow n^2 - (2r^2+4r-1)n - (2r^2+4r) < 0 \\ &\Leftrightarrow -1 < n < 2r(r+2).\end{aligned}$$

Thus $\dim_k [R/(I,\ell)]_j$ for any $r \leq n \leq 2r(r+2) - 1$. □

Corollary 3.3. *If* $2 \leq n \leq 15$ *and* $d = 4$, *then* R/I *fails to have the WLP.*

Proof. In this case, we have $j = 3n+1$. By Proposition 3.2, for any $2 \leq n \leq 15$,

$$\dim_k [R/(I,\ell)]_{3n+1} > 0,$$

where ℓ is a general linear form in R.

On other hand, by Theorem 3.1, we have

$$\dim_k [R/I]_{3n+1} - \dim_k [R/I]_{3n} = \sum_{k=0}^{n} (-1)^k \binom{2n+2}{k} \binom{5n-4k}{2n-1}.$$

Using `Macaulay2` [7], we see that for any $2 \leq n \leq 15$,

$$\dim_k [R/I]_{3n+1} - \dim_k [R/I]_{3n} \leq 0.$$

It follows that R/I fails to have the WLP since the surjectivity does not hold. □

Remark 3.4. Set

$$S_n = \sum_{k=0}^{n} (-1)^k \binom{2n+2}{k} \binom{5n-4k}{2n-1}, n \geq 2.$$

Examples suggest that the sequence $(S_n)_{n\geq 2}$ of integers is strictly decreasing with $S_2 = 0$, and so all these are non-positive.

Corollary 3.5. *If $3 \leq n \leq 29$ and $d = 6$, then R/I fails to have the WLP.*

Proof. In this case, we have $j = 5n + 2$. Let ℓ be a general linear form in R. By Proposition 3.2 we get that

$$\dim_k [R/(I, \ell)]_{5n+2} > 0,$$

for any $3 \leq n \leq 29$.

On other hand, by Theorem 3.1, we have

$$\dim_k [R/I]_{5n+2} - \dim_k [R/I]_{5n+1} = \sum_{k=0}^{n} (-1)^k \binom{2n+2}{k} \binom{7n+1-6k}{2n-1}.$$

Using `Macaulay2` [7], we see that for any $3 \leq n \leq 29$,

$$\dim_k [R/I]_{5n+2} - \dim_k [R/I]_{5n+1} < 0,$$

which shows that R/I fails to have the WLP since the surjectivity does not hold. □

Corollary 3.6. *If $4 \leq n \leq 47$ and $d = 8$, then R/I fails to have the WLP.*

Proof. Let ℓ be a general linear form. In this case, we have $j = 7n+3$. By Proposition 3.2, for any $4 \leq n \leq 47$,

$$\dim_k [R/(I, \ell)]_{7n+3} > 0.$$

On other hand, by Theorem 3.1, we have

$$\dim_k [R/I]_{7n+3} - \dim_k [R/I]_{7n+2} = \sum_{k=0}^{n} (-1)^k \binom{2n+2}{k} \binom{9n+2-8k}{2n-1}.$$

Using `Macaulay2` [7], we see that for any $4 \leq n \leq 47$,

$$\dim_k [R/I]_{7n+3} - \dim_k [R/I]_{7n+2} < 0,$$

which shows that R/I fails to have the WLP since the surjectivity does not hold. □

Corollary 3.7. *If $5 \leq n \leq 69$ and $d = 10$, then R/I fails to have the WLP.*

Proof. Let ℓ be a general linear form in R. In this case, one has $j = 9n + 4$. By Proposition 3.2, for any $5 \leq n \leq 69$,

$$\dim_k [R/(I, \ell)]_{9n+4} > 0.$$

On other hand, by Theorem 3.1, we have

$$\dim_k [R/I]_{9n+4} - \dim_k [R/I]_{9n+3} = \sum_{k=0}^{n} (-1)^k \binom{2n+2}{k} \binom{11n+3-10k}{2n-1}.$$

Using `Macaulay2` [7], we see that for any $5 \leq n \leq 69$,

$$\dim_k [R/I]_{9n+4} - \dim_k [R/I]_{9n+3} < 0,$$

which shows that R/I fails to have the WLP since the surjectivity does not hold. □

Corollary 3.8. *If $6 \leq n \leq 95$ and $d = 12$, then R/I fails to have the WLP.*

Proof. In this case, one has $j = 11n + 5$. For any $6 \leq n \leq 95$, one has

$$\dim_k [R/(I, \ell)]_{11n+5} > 0,$$

by Proposition 3.2, where ℓ is a general linear form in R.

On other hand, by Theorem 3.1, we have

$$\dim_k [R/I]_{11n+5} - \dim_k [R/I]_{11n+4} = \sum_{k=0}^{n} (-1)^k \binom{2n+2}{k} \binom{13n+4-12k}{2n-1}.$$

Using `Macaulay2` [7], we see that for any $6 \leq n \leq 95$,

$$\dim_k [R/I]_{11n+5} - \dim_k [R/I]_{11n+4} < 0,$$

which shows that R/I fails to have the WLP since the surjectivity does not hold. □

Corollary 3.9. *If $7 \leq n \leq 125$ and $d = 14$, then R/I fails to have the WLP.*

Proof. Let ℓ be a general linear form in R. In this case, one has $j = 13n + 6$. Proposition 3.2 follows that

$$\dim_k [R/(I, \ell)]_{13n+6} > 0,$$

for any $7 \leq n \leq 125$.

On other hand, by Theorem 3.1, we have

$$\dim_k[R/I]_{13n+6} - \dim_k[R/I]_{13n+5} = \sum_{k=0}^{n}(-1)^k \binom{2n+2}{k}\binom{15n+5-14k}{2n-1}.$$

Using `Macaulay2` [7], we see that for any $7 \le n \le 125$,

$$\dim_k[R/I]_{13n+6} - \dim_k[R/I]_{13n+5} < 0,$$

which shows that R/I fails to have the WLP since the surjectivity does not hold. □

Corollary 3.10. *If $8 \le n \le 159$ and $d = 16$, then R/I fails to have the WLP.*

Proof. Let ℓ be a general linear form in R. Computation shows that $j = 15n + 7$. Proposition 3.2 follows that

$$\dim_k[R/(I,\ell)]_{15n+7} > 0,$$

for any $8 \le n \le 159$.

On other hand, by Theorem 3.1, we have

$$\dim_k[R/I]_{15n+7} - \dim_k[R/I]_{15n+6} = \sum_{k=0}^{n}(-1)^k \binom{2n+2}{k}\binom{17n+6-16k}{2n-1}.$$

Using `Macaulay2` [7], we see that for any $7 \le n \le 159$,

$$\dim_k[R/I]_{15n+7} - \dim_k[R/I]_{15n+6} < 0,$$

which shows that R/I fails to have the WLP since the surjectivity does not hold. □

Proposition 3.11. *Assume that $n, d \ge 2$ and ℓ is a general linear form in R. Then*

$$\dim_k[R/(I,\ell)]_j > 0$$

if one of the following conditions is satisfied

(i) $2n-1$ *or* $2n+1$ *divides* $d-1$.
(ii) $2n-1$ *divides* $d+1$.
(iii) $2n-1$ *divides* $d+3$.
(iv) $2n-1$ *divides* $d+5$.
(v) $d \ge 4n^2 - 2n + 2$.

Proof. Set $t = \lfloor \frac{2n(n+1)(d-1)}{2n+1} \rfloor$. It is easy to show that $j \le t$. It follows from [13, Proposition 4.1] that

$$\dim_k [R/(I,\ell)]_t > 0$$

if $2n+1$ divides $d-1$ or $d \ge 4n^2 - 2n + 2$. Hence

$$\dim_k [R/(I,\ell)]_j > 0$$

if $2n+1$ divides $d-1$ or $d \ge 4n^2 - 2n + 2$ as claimed in the item (v) and the last part of the item (i). Now if $2n-1$ divides $d-1$, then, by Theorem 3.1,

$$\dim_k [R/(I,\ell)]_j = \dim_k \mathfrak{L}_{2n-1}\big(e; 0^{2n+2}\big) > 0,\ \forall e \ge 1,$$

which complete the proof of the item (i).

If $d+1 = (2n-1)e$, $e \ge 1$, then $d = (2n-1)(e-1) + 2(n-1)$. By Theorem 3.1,

$$\dim_k [R/(I,\ell)]_j = \dim_k \mathfrak{L}_{2n-1}\big(e+1; 1^{2n+2}\big) > 0,\ \forall e \ge 1, n \ge 2$$

as claimed in the item (ii).

If $d+3 = (2n-1)e$, $e \ge 1$, then $d = (2n-1)(e-1)+2(n-2)$. If $n \ge 3$, by Theorem 3.1,

$$\dim_k [R/(I,\ell)]_j = \dim_k \mathfrak{L}_{2n-1}\big(e+2; 2^{2n+2}\big) \ge \dim_k \mathfrak{L}_{2n-1}\big(3; 2^{2n+2}\big).$$

As

$$\begin{aligned}\dim_k \mathfrak{L}_{2n-1}\big(3; 2^{2n+2}\big) &\ge \binom{2n+2}{2n-1} - (2n+2)\binom{2n}{2n-1} \\ &= \frac{2n(n+1)(2n-5)}{3} > 0.\end{aligned}$$

If $n = 2$, then $d+3 = 3e$, $e \ge 2$. It follows that

$$\dim_k [R/(I,\ell)]_j = \dim_k \mathfrak{L}_3\big(e+2; 2^6\big) > 0.$$

Thus (iii) is proved.

It remains to show (iv). Since $d+5 = (2n-1)e$ with $e \ge 1$ if $n \ge 4$, one has

$$d = (2n-1)(e-1) + 2(n-3).$$

If $n \ge 5$, by Theorem 3.1,

$$\begin{aligned}\dim_k[R/(I,\ell)]_j &= \dim_k \mathfrak{L}_{2n-1}\big(e+3;3^{2n+2}\big)\\ &\geq \dim_k \mathfrak{L}_{2n-1}\big(4;3^{2n+2}\big)\\ &\geq \binom{2n+3}{2n-1} - (2n+2)\binom{2n+1}{2n-1}\\ &= \frac{n(n+1)(2n+1)(2n-9)}{6} > 0.\end{aligned}$$

Note that $e \geq \begin{cases} 3 & \text{if } n=2\\ 2 & \text{if } n=3\end{cases}$. Thus

$$\dim_k[R/(I,\ell)]_j = \begin{cases} \dim_k \mathfrak{L}_3\big(e-2;0^6\big) & \text{if } n=2, e\geq 3\\ \dim_k \mathfrak{L}_5\big(e+3;3^8\big) & \text{if } n=3, e\geq 2\\ \dim_k \mathfrak{L}_7\big(e+3;3^{10}\big) & \text{if } n=4, e\geq 1.\end{cases}$$

Therefore, $\dim_k[R/(I,\ell)]_j > 0$ if $n \in \{2,3\}$. If $n=4$, then

$$\dim_k[R/(I,\ell)]_j = \dim_k \mathfrak{L}_7\big(e+3;3^{10}\big) \geq \dim_k \mathfrak{L}_7\big(4;3^{10}\big).$$

Set $P = k[x_0,\ldots,x_7]/(x_0^2,\ldots,x_7^2)$. We have

$$\dim_k \mathfrak{L}_7\big(4;3^{10}\big) \geq h_P(4) - 2h_P(2) + h_P(0) = 15 > 0.$$

This completes the argument. □

We close this section by giving the following result. It is similar to a result of Nagel and Trok in [13, Proposition 6.3].

Proposition 3.12. *Given integers $n \geq 2$ and $0 \leq q \leq 2(n-1)$, define a polynomial function $P_{n,q}: \mathbb{R} \longrightarrow \mathbb{R}$ by*

$$P_{n,q}(t) = \sum_{k=0}^{n}(-1)^k\binom{2n+2}{k}\binom{n-1+\lfloor\frac{(n-1)q}{2n-1}\rfloor+(q+1)(n-k)+t[2n^2-1-(2n-1)k]}{2n-1}.$$

Then one has:

(a) *If for some q with $1 \leq q \leq 2(n-1)$, $P_{n,q}(t) \leq 0$ for every $t \geq 0$, then Conjecture 1.1 is true for every $d \geq 4n^2-2n+2$ such that $d-1-q$ is divisible by $2n-1$.*
(b) *If $P_{n,0}(t) \leq 0$ for every $t \geq 1$, then Conjecture 1.1 is true for every d such that $d-1$ is divisible by $2n-1$.*
(c) *If $\sum_{k=0}^{n}(-1)^k\binom{2n+2}{k}[2n^2-1-(2n-1)k]^{2n-1} < 0$ for each integer $n \geq 4$, then Conjecture 1.1 is true for every $d \gg 0$.*

Proof. Let ℓ be a general linear form in R. It follows from Proposition 3.11 that

$$\dim_k[R/(I,\ell)]_j > 0,$$

provided $d \geq 4n^2 - 2n + 2$ or $d+i$ is divisible by $2n-1$ with $i \in \{-1,1,3,5\}$. It follows that under these assumptions the multiplication

$$\times\ell : [R/I]_{j-1} \longrightarrow [R/I]_j$$

fails to have maximal rank if and only if it fails surjectivity. It is enough to show that

$$\dim_k[R/I]_j - \dim_k[R/I]_{j-1} \leq 0.$$

Now write $d - 1 = (2n-1)t + q$ with integers t and q where $0 \leq q \leq 2(n-1)$. Then a straightforward computation gives

$$j = (2n^2 - 1)t + nq + \lfloor \frac{(n-1)q}{2n-1} \rfloor.$$

By Theorem 3.1,

$$\begin{aligned}
&\dim_k[R/I]_j - \dim_k[R/I]_{j-1} = \sum_{k=0}^{n}(-1)^k \binom{2n+2}{k}\binom{2n-1+j-kd}{2n-1} \\
&= \sum_{k=0}^{n}(-1)^k \binom{2n+2}{k}\binom{n-1+\lfloor\frac{(n-1)q}{2n-1}\rfloor + (q+1)(n-k) + t[2n^2-1-(2n-1)k]}{2n-1} \\
&= P_{n,q}(t).
\end{aligned}$$

Now, if for some integer $t \geq 0$ we have $P_{n,q}(t) \leq 0$, then

$$\dim_k[R/(I,\ell)]_j \neq \max\{\dim_k[R/I]_j - \dim_k[R/I]_{j-1}, 0\}.$$

This proves assertions (a) and (b).

Note that $P_{n,q}(t)$ is a polynomial in t of degree $2n-1$ and

$$c_n := \sum_{k=0}^{n}(-1)^k \binom{2n+2}{k}[2n^2 - 1 - (2n-1)k]^{2n-1}$$

is the coefficient of t^{2n-1} in $P_{n,q}(t)$. Since $c_n < 0$ by assumption, it follows that $P_{n,q}(t) < 0$ for all $t \gg 0$ independent of q, and thus the claim (c) is proved. □

Based on computations, we conjecture that

$$c_n := \sum_{k=0}^{n}(-1)^k \binom{2n+2}{k}[2n^2 - 1 - (2n-1)k]^{2n-1} < 0, \text{ for any } n \geq 2.$$

In facts that computations suggest that the sequence $(c_n)_{n\geq 2}$ of integers is strictly decreasing with $c_2 = -26$, and so all these are negatives. Thank to `Macaulay2` [7], we can check it $c_n < 0$ for any $2 \leq n \leq 400$. This conjecture implies that Conjecture 1.1 is true for $d \gg 0$.

4. Almost uniform powers of general linear forms in a few variables

Our main result of this section is the following.

Theorem 4.1. *Let $R = k[x_0, \ldots, x_{2n}]$ and $I = (L_0^d, \ldots, L_{2n+1}^d)$, where $L_0, \ldots, L_{2n+1}$ are general linear forms in R. If $4 \leq n \leq 8$ and $d \geq 4$, then R/I fails to have the WLP.*

Proof. Let $\ell \in R$ be a general linear form and we will show that the multiplication

$$\times\ell : [R/I]_{j-1} \longrightarrow [R/I]_j$$

fails to have maximal rank with $j = \lfloor \frac{(2n^2-1)(d-1)}{2n-1} \rfloor$, provided $4 \leq n \leq 8$ and $d \geq 4$. To do this, we will show

$$\dim_k [R/(I, \ell)]_j \neq \max\{\dim_k [R/I]_j - \dim_k [R/I]_{j-1}, 0\}.$$

First, we prove the following assertion.

Claim 1. *$D := \dim_k [R/(I, \ell)]_j > 0$ for any $4 \leq n \leq 8$ and $d \geq 4$.*

Indeed, Theorem 3.1 shows that

$$D = \begin{cases} \dim_k \mathfrak{L}_{2n-1}(e; 0^{2n+2}) & \text{if} \quad d = (2n-1)e + 1 \\ \dim_k \mathfrak{L}_{2n-1}(e + n - r + 1; (n-r)^{2n+2}) & \text{if} \quad d = (2n-1)e + 2r \\ \dim_k \mathfrak{L}_{2n-1}(e + 2n - r + 1; (2n-r-1)^{2n+2}) & \text{if} \quad d = (2n-1)e + 2r + 1 \end{cases}$$

where e, r are non-negative integers and $1 \leq r \leq n-1$. Note that the dimension of linear systems satisfies

$$\dim_k \mathfrak{L}_{2n-1}(i; a^{2n+2}) \geq \binom{2n-1+i}{2n-1} - (2n+2)\binom{2n-2+a}{2n-1}. \tag{4.1}$$

We now consider the following cases:

Case 1: n=4. Using (4.1), computations show that these linear systems are not empty for every

$$e \geq \begin{cases} 1 & \text{if} \quad d-1 \equiv 0, 1, 2 (\text{mod} 7) \\ 0 & \text{if} \quad d-1 \equiv 3, 4, 5, 6 (\text{mod} 7). \end{cases}$$

In other words, $D \neq 0$ for any $d \geq 4$.

Case 2: n=5. Using (4.1), computations show that these linear systems are not empty, provided

$$e \geq \begin{cases} 1 & \text{if} \quad d-1 \equiv 0,1,2,4(\text{mod}9) \\ 0 & \text{if} \quad d-1 \equiv 3,5,6,7,8(\text{mod}9). \end{cases}$$

In other words, $D \neq 0$ for all $d \geq 4$ and $d \neq 5$.

Let $\ell_0, \ldots, \ell_{2n+1}$ be general linear forms in $k[x_0, \ldots, x_{2n-1}]$ and set

$$P_{n,s} = k[x_0, \ldots, x_{2n-1}]/(\ell_0^s, \ldots, \ell_{2n-1}^s), \quad Q_{n,s} = k[x_0, \ldots, x_{2n-1}]/(\ell_0^s, \ldots, \ell_{2n}^s)$$

and $R_{n,s} = k[x_0, \ldots, x_{2n-1}]/(\ell_0^s, \ldots, \ell_{2n+1}^s)$. The exact sequence

$$[Q_{n,s}]_{i-s} \xrightarrow{\times \ell_{2n+1}^s} [Q_{n,s}]_i \longrightarrow [R_{n,s}]_i \longrightarrow 0$$

deduces that

$$\begin{aligned} h_{R_{n,s}}(i) &\geq h_{Q_{n,s}}(i) - h_{Q_{n,s}}(i-s) \\ &= h_{P_{n,s}}(i) - 2h_{P_{n,s}}(i-s) + h_{P_{n,s}}(i-2s) \end{aligned}$$

where the last equality deduces from the fact that $P_{n,s}$ is a complete intersection and has the SLP (see [15] or [17]).

Now we need to show $D \neq 0$ for $d = 5$. Indeed, in this case, one has

$$\begin{aligned} D = \dim_k \mathfrak{L}_9(9; 7^{12}) &= \dim_k [R_{5,3}]_9 \\ &\geq h_{P_{5,3}}(9) - 2h_{P_{5,3}}(6) + h_{P_{5,3}}(3) \\ &= 8350 - 2 \times 2850 + 210 \\ &= 2860 \end{aligned}$$

which shows $D \neq 0$ for $d = 5$.

Case 3: n=6. Using (4.1), computations show that these linear systems are not empty for any $d \geq 6$ and $d \neq 7, 9$. We need to show $D \neq 0$ for $d = 4, 5, 7, 9$. If $d = 4$, then $D \neq 0$, by Proposition 3.2. With the notations as in the case 2, one has

$$D = \begin{cases} \dim_k \mathfrak{L}_{11}(11; 9^{14}) & = \dim[R_{6,3}]_{11} \quad \text{if} \quad d = 5 \\ \dim_k \mathfrak{L}_{11}(10; 8^{14}) & = \dim[R_{6,3}]_{10} \quad \text{if} \quad d = 7 \\ \dim_k \mathfrak{L}_{11}(9; 7^{14}) & = \dim[R_{6,3}]_9 \quad \text{if} \quad d = 9. \end{cases}$$

The h-vector of $P_{6,3}$ is

$$\begin{aligned}h_{P_{6,3}} =&(1,12,78,352,1221,3432,8074,16236,28314,43252,58278,69576,73789,\\&69576,58278,43252,28314,16236,8074,3432,1221,352,78,12,1).\end{aligned}$$

It is easy to see

$$\dim[R_{6,3}]_i \geq h_{P_{6,3}}(i) - 2h_{P_{6,3}}(i-3) + h_{P_{6,3}}(i-6) > 0,$$

for each $i \in \{9, 10, 11\}$.
Thus, $D > 0$ for every $d \geq 4$.
Case 4: n=7. Using (4.1), computations show that these linear systems are not empty for $d \geq 4$ and $d \neq 5, 6, 7, 9, 11, 16$. We need to show $D \neq 0$ for $d = 5, 6, 7, 9, 11, 16$. By Proposition 3.2 we get that $D \neq 0$ for $d = 6$. With the notations as in the case 2, one has

$$D = \begin{cases} \dim_k \mathfrak{L}_{13}(13; 11^{16}) = \dim[R_{7,3}]_{13} & \text{if} \quad d = 5 \\ \dim_k \mathfrak{L}_{13}(12; 10^{16}) = \dim[R_{7,3}]_{12} & \text{if} \quad d = 7 \\ \dim_k \mathfrak{L}_{13}(11; 9^{16}) \ \ = \dim[R_{7,3}]_{11} & \text{if} \quad d = 9 \\ \dim_k \mathfrak{L}_{13}(10; 8^{16}) \ \ = \dim[R_{7,3}]_{10} & \text{if} \quad d = 11 \\ \dim_k \mathfrak{L}_{13}(15; 12^{16}) = \dim[R_{7,4}]_{15} & \text{if} \quad d = 16. \end{cases}$$

As h-vector of $Q_{7,3}$ is

$$\begin{aligned}h_{Q_{7,3}} = &(1,14,105,545,2170,6993,18837,43290,85995,148785,224796,295659,\\&334425,315420,227475,83097)\end{aligned}$$

we get $D > 0$ for $d = 5, 7, 9, 11$. Similarly, one can easy show that $D > 0$ for $d = 16$. Thus, $D > 0$ for every $d \geq 4$.
Case 5: n=8. By Proposition 3.2, we have $D \neq 0$ for $d = 15e+2r$, e and r are non-negative integers such that $2 \leq r \leq 8$. Using (4.1), we can also show that $D \neq 0$ for $d \geq 4$ and $d \neq 5, 7, 9, 11, 13, 18, 20$. We now need to prove $D \neq 0$ for $d \neq 5, 7, 9, 11, 13, 18, 20$. With the notations as in the case 2, one has

$$D = \begin{cases} \dim_k \mathfrak{L}_{15}(15; 13^{18}) = \dim[R_{8,3}]_{15} & \text{if} \quad d = 5 \\ \dim_k \mathfrak{L}_{15}(14; 12^{18}) = \dim[R_{8,3}]_{14} & \text{if} \quad d = 7 \\ \dim_k \mathfrak{L}_{15}(13; 11^{18}) = \dim[R_{8,3}]_{13} & \text{if} \quad d = 9 \\ \dim_k \mathfrak{L}_{15}(12; 10^{18}) = \dim[R_{8,3}]_{12} & \text{if} \quad d = 11 \\ \dim_k \mathfrak{L}_{15}(11; 9^{18}) \ \ = \dim[R_{8,3}]_{11} & \text{if} \quad d = 13 \\ \dim_k \mathfrak{L}_{15}(17; 14^{18}) = \dim[R_{8,4}]_{17} & \text{if} \quad d = 18 \\ \dim_k \mathfrak{L}_{15}(16; 13^{18}) = \dim[R_{8,4}]_{16} & \text{if} \quad d = 20. \end{cases}$$

As h-vector of $Q_{8,3}$ is

$$h_{Q_{8,3}} = (1, 16, 136, 799, 3604, 13192, 40528, 106828, 245242, 495312, 885768, 1406886, 1983696, 2469624, 2677704, 2448816, 1730787, 625992)$$

we get $D > 0$ for $d = 5, 7, 9, 11, 13$. Similarly, the Hilbert functions of $Q_{8,4}$ up to degree 17 are

$$h_{Q_{8,4}}(t) = (1, 16, 136, 816, 3859, 15232, 51952, 156672, 424558, 1046112, 2364768, 4937888, 9574978, 17312256, 29277264, 46411904, 69063979, 96521904)$$

which show $D > 0$ for $d = 18, 20$. Thus, $D > 0$ for every $d \geq 4$.

Therefore, Claim 1 is completely proved.

Second, to prove failure of the WLP in degree j it remains to show the following assertion.

Claim 2. $E := \dim_k[R/I]_j - \dim_k[R/I]_{j-1} \leq 0$ *for all* $4 \leq n \leq 8$ *and* $d \geq 4$.

Theorem 3.1 gives

$$E := \dim_k[R/I]_j - \dim_k[R/I]_{j-1} = \sum_{k=0}^{n}(-1)^k \binom{2n+2}{k}\binom{2n-1+j-kd}{2n-1}.$$

We consider the following cases:

Case 1: n=4. We consider seven cases for $d-1 = 7e+m, 0 \leq m \leq 6$. Thank to `Macaulay2` [7], we can show that $E < 0$ for any $d \geq 4$.

Subcase 1: If $d = 7e+1$, then $j = 31e$ and hence

$$\begin{aligned} E &= \binom{31e+7}{7} - 10\binom{24e+6}{7} + 45\binom{17e+5}{7} - 120\binom{10e+4}{7} + 210\binom{3e+3}{7} \\ &= \frac{1}{7!}(-1086400574e^7 - 914853422e^6 - 328170248e^5 - 60270140e^4 - 5015486e^3 \\ &\quad + 102442e^2 + 60228e + 5040) < 0 \quad \text{for any} \quad e \geq 1. \end{aligned}$$

Subcase 2: If $d = 7e+2$, then $j = 31e+4$ and we have

$$\begin{aligned} E &= \binom{31e+11}{7} - 10\binom{24e+9}{7} + 45\binom{17e+7}{7} - 120\binom{10e+5}{7} + 210\binom{3e+3}{7} \\ &= \frac{1}{7!}(-1086400574e^7 - 1829706844e^6 - 1272885740e^5 - 457929640e^4 \\ &\quad - 84318206e^3 - 5316556e^2 + 535080e + 75600) < 0 \quad \text{for any} \quad e \geq 1. \end{aligned}$$

Subcase 3: If $d = 7e + 3$, then $j = 31e + 8$. It follows that

$$\begin{aligned} E &= \binom{31e+15}{7} - 10\binom{24e+12}{7} + 45\binom{17e+9}{7} - 120\binom{10e+6}{7} + 210\binom{3e+3}{7} \\ &= \frac{1}{7!}(-1086400574e^7 - 2744560266e^6 - 2847411560e^5 - 1530367860e^4 \\ &\quad - 431507006e^3 - 50737554e^2 + 1747620e + 680400) < 0 \quad \text{for any} \quad e \geq 1. \end{aligned}$$

Subcase 4: If $d = 7e + 4$, then $j = 31e + 13$. One has

$$\begin{aligned} E &= \binom{31e+20}{7} - 10\binom{24e+16}{7} + 45\binom{17e+12}{7} - 120\binom{10e+8}{7} + 210\binom{3e+4}{7} \\ &= \frac{1}{7!}(-1086400574e^7 - 4059690376e^6 - 6472447730e^5 - 5696621560e^4 \\ &\quad - 2981962256e^3 - 925181824e^2 - 156720480e - 11088000) < 0 \quad \text{for any} \quad e \geq 0. \end{aligned}$$

Subcase 5: If $d = 7e + 5$, then $j = 31e + 17$ and hence

$$\begin{aligned} E &= \binom{31e+24}{7} - 10\binom{24e+19}{7} + 45\binom{17e+14}{7} - 120\binom{10e+9}{7} + 210\binom{3e+4}{7} \\ &= \frac{1}{7!}(-1086400574e^7 - 4974543798e^6 - 9666743618e^5 - 10305716610e^4 \\ &\quad - 6484301936e^3 - 2393744472e^2 - 475568352e - 38586240) < 0 \quad \text{for any} \quad e \geq 0. \end{aligned}$$

Subcase 6: If $d = 7e + 6$, then $j = 31e + 22$ and therefore

$$\begin{aligned} E &= \binom{31e+29}{7} - 10\binom{24e+23}{7} + 45\binom{17e+17}{7} - 120\binom{10e+11}{7} + 210\binom{3e+5}{7} \\ &= \frac{1}{7!}(-1086400574e^7 - 6289673908e^6 - 15592053428e^5 - 21447402760e^4 \\ &\quad - 17672567486e^3 - 8719279492e^2 - 2383703952e - 278359200) < 0 \text{ for any } e \geq 0. \end{aligned}$$

Subcase 7: If $d = 7e + 7$, then $j = 31e + 26$. It follows that

$$\begin{aligned} E &= \binom{31e+33}{7} - 10\binom{24e+26}{7} + 45\binom{17e+19}{7} - 120\binom{10e+12}{7} + 210\binom{3e+5}{7} \\ &= \frac{1}{7!}(-1086400574e^7 - 7204527330e^6 - 20406119384e^5 - 31980364500e^4 \\ &\quad - 29926695806e^3 - 16705543050e^2 - 5144220396e - 673001280) < 0 \text{ for any } e \geq 0. \end{aligned}$$

Thus $E < 0$ for any $d \geq 4$. Claim 2 is proved for $n = 4$.

Case 2: n=5. We write $d - 1 = 9e + m, 0 \leq m \leq 8$. We will prove that $E < 0$ for any $d \geq 4$. Thank to `Macaulay2` [7], a straightforward computation gives

Subcase 1: If $d = 9e + 1$, then $j = 49e$. It follows that

$$E = \sum_{k=0}^{5} (-1)^k \binom{12}{k} \binom{(49-9k)e + 9 - k}{9}$$
$$= \frac{1}{9!}(-32649547827918e^9 - 29495874488598e^8 - 11942585863236e^7$$
$$- 2793889960092e^6 - 406323342558e^5 - 35868202902e^4 - 1535113104e^3$$
$$+ 29687112e^2 + 7209216e + 362880) < 0 \text{ for any } e \geq 1.$$

Analogously we can check the another cases.
Subcase 2: If $d = 9e + 2r,\ 1 \leq r \leq 4$, then $j = 49e + 11r - 6$ and

$$E = \sum_{k=0}^{5} (-1)^k \binom{12}{k} \binom{(49-9k)e + (11-2k)r + 3}{9}.$$

We compute with `Macaulay2` to show that if $r = 1$, then $E < 0$ for any $e \geq 1$ and if $r \in \{2, 3, 4\}$ then $E < 0$ for any $e \geq 0$.
Subcase 3: If $d = 9e + 2r + 1,\ 1 \leq r \leq 4$, then $j = 49e + 11r - 1$ and

$$E = \sum_{k=0}^{5} (-1)^k \binom{12}{k} \binom{(49-9k)e + (11-2k)r - k + 8}{9}.$$

Similarly, we can show that if $r = 1$, then $E < 0$ for any $e \geq 1$ and if $r \in \{2, 3, 4\}$ then $E < 0$ for any $e \geq 0$.

It follows that $E < 0$ for any $d \geq 4$. Claim 2 is proved for $n = 5$.
Case 3: n=6. Write $d - 1 = 11e + m, 0 \leq m \leq 10$. Thank to `Macaulay2` [7], we will show that $E < 0$ for any $d \geq 2$.
Subcase 1: If $d = 11e + 1$, then $j = 71e$ and

$$E = \sum_{k=0}^{6} (-1)^k \binom{14}{k} \binom{(71-11k)e + 11 - k}{11} = \frac{1}{11!}(-2310696921327619572e^{11}$$
$$- 2159206229822458212e^{10} - 925836626096405100e^9 - 238845827273630940e^8$$
$$- 40895244843536556e^7 - 4822097086873836e^6 - 390251062386900e^5$$
$$- 20387890763460e^4 - 526999267872e^3 + 8455070448e^2$$
$$+ 1189900800e + 39916800) < 0 \text{ for any } e \geq 1.$$

Analogously we can check the another cases.
Subcase 2: If $d = 11e + 2r,\ 1 \leq r \leq 5$, then $j = 71e + 13r - 7$. For each $1 \leq r \leq 5$, computations with `Macaulay2` show that

$$E = \sum_{k=0}^{6} (-1)^k \binom{14}{k} \binom{(71-11k)e + (13-2k)r + 4}{11} < 0 \quad \text{for any } e \geq 0.$$

Subcase 3: If $d = 11e + 2r + 1,\ 1 \leq r \leq 5$, then $j = 71e + 13r - 1$. For each $1 \leq r \leq 5$, computations with Macaulay2 show that

$$E = \sum_{k=0}^{6} (-1)^k \binom{14}{k} \binom{(71-11k)e + (13-2k)r - k + 10}{11} < 0 \quad \text{for any } e \geq 0.$$

It follows that $E < 0$ for any $d \geq 2$. Claim 2 is proved for $n = 6$.

Case 4: n=7. We write $d - 1 = 13e + m, 0 \leq m \leq 12$. Thank to Macaulay2 [7], we will show that $E < 0$ for any $d \geq 2$.

Subcase 1: If $d = 13e + 1$, then $j = 97e$ and hence

$$\begin{aligned} E &= \sum_{k=0}^{7} (-1)^k \binom{16}{k} \binom{(97-13k)e + 13 - k}{13} \\ &= \frac{1}{13!}(-33468841461064989051029 1e^{13} - 318779633066827110608001e^{12} \\ &- 14132994371496075952090 5e^{11} - 38495945182007845679433e^{10} \\ &- 7165747937184385180203e^{9} - 958746457198704734703e^{8} \\ &- 94270438988259145755e^{7} - 6819523889292264579e^{6} \\ &- 354359614333473606e^{5} - 12260161531299396e^{4} - 210757791455640e^{3} \\ &+ 2848164688512e^{2} + 260089315200e + 6227020800) < 0 \text{ for any } e \geq 1. \end{aligned}$$

Analogously we can check the another cases.

Subcase 2: If $d = 13e + 2r,\ 1 \leq r \leq 6$, then $j = 97e + 15r - 8$. For each $1 \leq r \leq 6$, computations with Macaulay2 show that

$$E = \sum_{k=0}^{7} (-1)^k \binom{16}{k} \binom{(97-13k)e + (15-2k)r + 5}{13} < 0 \text{ for any } e \geq 0.$$

Subcase 3: If $d = 13e + 2r + 1,\ 1 \leq r \leq 6$, then $j = 97e + 15r - 1$. For each $1 \leq r \leq 6$, computations with Macaulay2 show that

$$E = \sum_{k=0}^{7} (-1)^k \binom{16}{k} \binom{(97-13k)e + (15-2k)r - k + 12}{13} < 0 \text{ for any } e \geq 0.$$

It follows that $E < 0$ for all $d \geq 2$ as desired.

Case 5: n=8. Write $d - 1 = 15e + m, 0 \leq m \leq 14$. Thank to Macaulay2 [7], we will show that $E < 0$ for any $d \geq 2$.

Subcase 1: If $d = 15e + 1$, then $j = 127e$ and hence

$$\begin{aligned}
E &= \sum_{k=0}^{8} (-1)^k \binom{18}{k} \binom{(127-15k)e + 15 - k}{15} \\
&= \frac{1}{15!}(-894161807620841305974330316700e^{15} - 861892646000120903654158306920e^{14} \\
&- 390530284485072995291476748300e^{13} - 110154896946958692275699151900e^{12} \\
&- 215977126172169884124585983000e^{11} - 311249224672723122942089934e^{10} \\
&- 3397943759406996655552411 0e^{9} - 285141602709214448379897 0e^{8} \\
&- 184330466550469812352780e^{7} - 907638168575042945640 6e^{6} \\
&- 329488524726287066140e^{5} - 8109990225233736840e^{4} - 98835121150056720e^{3} \\
&+ 1138217439820032e^{2} + 71328551374080e + 1307674368000) < 0 \text{ for any } e \geq 1.
\end{aligned}$$

Analogously we can check the another cases.

Subcase 2: If $d = 15e + 2r,\ 1 \leq r \leq 7$, then $j = 127e + 17r - 9$. For each $1 \leq r \leq 7$, computations with `Macaulay2` show that

$$E = \sum_{k=0}^{8} (-1)^k \binom{18}{k} \binom{(127-15k)e + (17-2k)r + 6}{15} < 0 \text{ for any } e \geq 0.$$

Subcase 3: If $d = 15e + 2r + 1,\ 1 \leq r \leq 7$, then $j = 127e + 17r - 1$. For each $1 \leq r \leq 7$, computations with `Macaulay2` show that

$$E = \sum_{k=0}^{8} (-1)^k \binom{18}{k} \binom{(127-15k)e + (17-2k)r - k + 14}{15} < 0 \text{ for any } e \geq 0.$$

It follows that $E < 0$ for all $d \geq 2$ and $n = 8$.

Thus Claim 2 is completely proved.

Finally, Theorem 4.1 follows from the above two claims. □

Remark 4.2.

(1) The first author has shown that an artinian ideal $I = (L_0^2, \dots, L_{2n+1}^2) \subset R$ generated by the quadratic powers of general linear forms fails to have the WLP [12]. Therefore, Theorem 4.1 answers partially Conjecture 1.1 for $4 \leq n \leq 8$, missing only the case $d = 3$.

(2) Theorem 4.1 together with Corollaries 3.3–3.10 says that R/I fails to have the WLP for all $d = 2r, 2 \leq r \leq 8$ and $4 \leq n \leq 2r(r+2) - 1$.

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