Computation tool for the q-deformed quasi-shuffle algebras and representations of structure of MZVs

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Abstract

We would like to introduce our package using Maple to compute within the q-deformed[1] quasi-shuffle algebras and to represent structure of multiple zeta values (MZVs). For this package, we can define an arbitrary alphabet from which the letters associated with indices totally ordered and then carry out computations on words, that will complement functions for the package *StringTools* in Maple. In the vector space of (noncommutative) polynomials which is equipped q-deformed quasi-shuffle products and concatenation product [1, 2], we compute the bases in duality and express an arbitrary homogeneous polynomial in terms of these bases. Moreover, due to our algorithms, we can represent structure of MZVs on the transcendence bases in terms of irreducible elements [4]. We used this package to compute all examples and verify the results in the paper [4] which was present at the conference ISSAC 2015.

1 Introduction

This package is written as a module and stored in the file *QuasiShuffleAlgrbraPackage.txt*. In order to use it, we first call the two following sequences

[> *read "QuasiShuffleAlgebraPackage.txt";*

[> *with(QuasiShuffleAlgebra);*

In the followings, we will show the popular commands.

1.1 Lists of particular words

We assume here computing on the alphabet[2] $Y = (y_s)_{s\in\mathbb{N}_+}$ in totally ordered by $y_1 > y_2 > \ldots$.

Calling sequences:

[> *ListLength(alphabet,order,length);*

[> *ListSameWeight(alphabet,weight);*

[> *LyndonBasis(alphabet,order,length);*

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[1] q belongs to any field extension of the field of rational numbers.

[2] Y^* denotes the set of words generated by Y. In the Maple program, we denote $Y_{s_1,\ldots,s_k}$ instead of the word $y_{s_1}\cdots y_{s_k}$.

Parameters:

alphabet - a name of alphabet.

order - a sequence of numbers corresponding the order of letters.

length, weight - integers.

Description:

- The commands *ListLength* and *ListSameWeight*, respectively, list all words of length *length*, or of weight[3] *weight*, of the alphabet *alphabet*.
- The *LyndonBasis* command lists all Lyndon words of length less than or equal to *length*.

Example 1 [> $ListLength(Y, [3, 2, 1], 2)$;

$$Y_{3,3}, Y_{3,2}, Y_{3,1}, Y_{2,3}, Y_{2,2}, Y_{2,1}, Y_{1,3}, Y_{1,2}, Y_{1,1}$$

[> $ListOfWeight(Y, 3)$;

$$Y_3, Y_{2,1}, Y_{1,2}, Y_{1,1,1}$$

[> $LyndonBasis(Y, [3, 2, 1], 2)$;

$$Y_3, Y_2, Y_1, Y_{3,2}, Y_{3,1}, Y_{2,1}$$

1.2 Computations for q-deformed quasi-shuffle algebras

We constructed the bialgebras [1, 2, 3, 4, 5, 6] in duality denoted by $(\mathbb{Q}\langle Y\rangle, \bullet, 1_{Y^*}, \Delta_{\uplus}, \mathtt{e})$ and $(\mathbb{Q}\langle Y\rangle, \uplus, 1_{Y^*}, \Delta_{\bullet}, \mathtt{e})$. $(\Pi_w)_{w\in Y^*}, (\Sigma_w)_{w\in Y^*}$ respectively denote the Poincaré-Birkhoff-Witt basis and the Schützenberger basis.

Calling sequences:

[> *q_StuffleProduct(poly1, poly2, q, alphabet);*

[> *DeltaStuffle_ q(poly1,q);*

[> *PiOfWord(w,order,q);*

[> *SigmaOfWord(w,order,q);*

Parameters:

poly1, poly2 - polynomials.

q - a rational number or a formal parameter[4].

w - a word.

Description:

- The *q_StuffleProduct* command computes the q-deformed quasi-shuffle product of the polynomials *poly1, poly2*.

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[3] The weight of a word is sum of indices of letters of that word.

[4] $q = 0$, this is the case of shuffle product. $q = 1$ is the case of stuffle product, or more cases, one can see in [3]

- The *DeltaStuffle_q* command computes the coproduct[5], which is the dual law of the q-deformed quasi-shuffle product, of the polynomial *poly1*.
- The *PiOfWord* command computes Π_w.
- The *SigmaOfWord* command computes Σ_w.

Example 2 $[> q\_StuffleProduct(Y[3,1], Y[2], q, Y);$

$$qY_{3,3} + qY_{5,1} + Y_{2,3,1} + Y_{3,1,2} + Y_{3,2,1}$$

$[> DeltaStuffle\_q(Y[2,1], q);$

$$qL_1R_{1,1} + qL_{1,1}R_1 + L_{[]}R_{2,1} + L_1R_2 + L_2R_1 + L_{2,1}R_{[]}$$

$[> PiOfWord(Y[1,2], [2,1], q);$

$$Y_{1,2} - \frac{1}{2}qY_{1,1,1}$$

$[> SigmaOfWord(Y[2,1,3], [3,2,1], q);$

$$qY_{2,4} + qY_{5,1} + Y_{2,1,3} + Y_{2,3,1} + Y_{3,2,1} + \frac{1}{2}q^2Y_6 + qY_{3,3}$$

1.3 Representation of polynomials on bases

Calling sequences:

[> *ExpressByPi(poly,order);*

[> *ExpressBySigma(poly);*

Description: These commands respectively represent the polynomial *poly* in terms of the basis[6] $(\Pi_w)_{w\in Y^*}$ and $(\Sigma_w)_{w\in Y^*}$.

Example 3 $[> ExpressByPi(2Y[1,2] - 1/2Y[3], [3,2,1]);$

$$-\frac{1}{2}\Pi_3 - \frac{1}{4}\Pi_{2,1} + \frac{3}{2}\Pi_{1,2} + \frac{11}{12}\Pi_{1,1,1}$$

$[> ExpressBySigma(2Y[1,2] - 1/2Y[3]);$

$$2\Sigma_{1,2} - 2\Sigma_{2,1} - \frac{3}{2}\Sigma_3$$

[5] In the Maple program, we denote L the word on the left of tensor, R the word on the right of tensor. Example $L_2R_1 + L_{2,1}R_{[]}$ means $y_2 \otimes y_1 + y_2y_1 \otimes 1_{Y^*}$.

[6] In the Maple program, $\Pi_{2,1}$ or $\Sigma_{2,1}$ are used instead of $\Pi_{y_2y_1}$ or $\Sigma_{y_2y_1}$.

1.4 Structure of MZVs

Calling sequences:

[> *StructurePolyzetaOnSigma(weight);*

[> *StructurePolyzetaOnS(weight);*

Description: These commands respectively represent structures, in terms of irreducible elements, of MZVs on the bases[7] $(\Sigma_l)_{l\in\mathcal{L}ynY}$ and $(S_l)_{l\in\mathcal{L}ynX}$ until the weight *weight*. Here we use one more alphabet $X=\{x_0,x_1\}$ ordered $x_0<x_1$.

Example 4 [> $StructurePolyzetaOnSigma(5)$;

$$Relations\ among\ MZVs\ of\ weight\ 3$$
$$\left\{\zeta_{\Sigma_{2,1}}=\tfrac{3}{2}\zeta_{\Sigma_3}\right\}$$
$$Relations\ among\ MZVs\ of\ weight\ 4$$
$$\left\{\zeta_{\Sigma_4}=\tfrac{2}{5}\zeta_{\Sigma_2}^2,\zeta_{\Sigma_{3,1}}=\tfrac{3}{10}\zeta_{\Sigma_2}^2,\zeta_{\Sigma_{2,1,1}}=\tfrac{2}{3}\zeta_{\Sigma_2}^2\right\}$$
$$Relations\ among\ MZVs\ of\ weight\ 5$$
$$\{\zeta_{\Sigma_{3,2}}=3\,\zeta_{\Sigma_3}\zeta_{\Sigma_2}-5\,\zeta_{\Sigma_5},\zeta_{\Sigma_{4,1}}=-\zeta_{\Sigma_3}\zeta_{\Sigma_2}+5/2\,\zeta_{\Sigma_5},\zeta_{\Sigma_{2,2,1}}=3/2\,\zeta_{\Sigma_3}\zeta_{\Sigma_2}-\tfrac{25}{12}\zeta_{\Sigma_5},\zeta_{\Sigma_{3,1,1}}=\tfrac{5}{12}\zeta_{\Sigma_5},$$
$$\zeta_{\Sigma_{2,1,1,1}}=1/4\,\zeta_{\Sigma_3}\zeta_{\Sigma_2}+5/4\,\zeta_{\Sigma_5}\}$$

[> $StructurePolyzetaOnS(5)$;

$$Relations\ among\ MZVs\ of\ weight\ 3$$
$$\left\{\zeta_{S_{0,1,1}}=\zeta_{S_{0,0,1}}\right\}$$
$$Relations\ among\ MZVs\ of\ weight\ 4$$
$$\left\{\zeta_{S_{0,0,0,1}}=\tfrac{2}{5}\zeta_{S_{0,1}}^2,\zeta_{S_{0,0,1,1}}=\tfrac{1}{10}\zeta_{S_{0,1}}^2,\zeta_{S_{0,1,1,1}}=\tfrac{2}{5}\zeta_{S_{0,1}}^2\right\}$$
$$Relations\ among\ MZVs\ of\ weight\ 5$$
$$\{\zeta_{S_{0,0,0,1,1}}=-\zeta_{S_{0,1}}\zeta_{S_{0,0,1}}+2\,\zeta_{S_{0,0,0,0,1}},\zeta_{S_{0,0,1,0,1}}=-3/2\,\zeta_{S_{0,0,0,0,1}}+\zeta_{S_{0,1}}\zeta_{S_{0,0,1}},$$
$$\zeta_{S_{0,0,1,1,1}}=-\zeta_{S_{0,1}}\zeta_{S_{0,0,1}}+2\,\zeta_{S_{0,0,0,0,1}},\zeta_{S_{0,1,0,1,1}}=1/2\,\zeta_{S_{0,0,0,0,1}},\zeta_{S_{0,1,1,1,1}}=\zeta_{S_{0,0,0,0,1}}\}$$

2 Conclusion

The package gives an effective computing tool on the q-deformed quasi-shuffle algebras and represent the structure of MZVs. Although it is limited to our world, it can be extended to help someone who want to study algebraic combinatorics on words devoted to shuffle algebras and similar laws.

References

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[7] $\mathcal{L}ynX,\mathcal{L}ynY$ denote the sets of Lyndon words of X^*,Y^* respectively.