Separation of Convex Sets via Barrier Cones

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Dedicated to Professor Hoang Tuy

Abstract
A closed convex subset of a normed linear space is said to have the strong separation property if it can be strongly separated from every other disjoint, closed, and convex set by a closed hyperplane. In this paper, we give some results on the separation of convex sets noticing the role of barrier cones, develop some characterizations of subsets having the strong separation property, and apply them to consider a class of convex optimization problems.

Keywords Convex set · Separation theorem · Barrier cone · Recession cone · Set having the strong separation property

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1 Introduction

Let C and D be convex subsets of a real normed linear space X with dual space X^*. If there exists $x^* \in X^* \setminus \{0\}$ such that

$$\sup \left\{ \langle x^*, c \rangle \mid c \in C \right\} \leq \inf \left\{ \langle x^*, d \rangle \mid d \in D \right\},$$

then we say that C and D are separated. Furthermore, if

$$\sup \left\{ \langle x^*, c \rangle \mid c \in C \right\} < \inf \left\{ \langle x^*, d \rangle \mid d \in D \right\},$$

then C and D are said to be strongly separated.

A convex subset of X is said to have the (strong) separation property if it can be (strongly) separated from every other disjoint closed convex subset.

Let C be a closed convex subset of X. We denote by $\mathrm{rec}(C)$ and $\mathrm{bar}(C)$, respectively, the recession cone and the barrier cone of C, i.e.,

$$\mathrm{rec}(C) := \{ v \in X \mid c + v \in C, \forall c \in C \};$$
$$\mathrm{bar}(C) := \{ x^* \in X^* \mid \sigma_C(x^*) < +\infty \},$$

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where $\sigma_C : X^* \to \overline{\mathbb{R}}$ is the support function of C, defined by

$$\sigma_C(x^*) = \sup \left\{ \langle x^*, c \rangle \; : \; c \in C \right\}, x^* \in X^*.$$

The set C is called *linearly bounded* if $\mathrm{rec}(C) = \{0\}$. It is obvious that a bounded subset is also linearly bounded. The set C is said to be *locally compact* if there exist $c_0 \in C$ and $r > 0$ such that

$$\overline{B(c_0; r)} \cap C \text{ is compact,} \tag{1}$$

where $\overline{B(c_0; r)}$ denotes the closed ball of radius r around c_0. It should be noted that, since C is convex and closed, this definition does not depend on both c_0 and r, i.e., if (1) holds, then for every $c \in C$ and $s > 0$, $\overline{B(c; s)} \cap C$ is also compact.

The following results are well known (see, for instance, [2, 4, 7–9, 11]) in convex analysis.

Theorem 1 *Let C and D be disjoint convex subsets of X. Then, they are separated if at least one of the following conditions holds:*

(a) $\mathrm{int}(C) \cup \mathrm{int}(D) \neq \emptyset$;
(b) $\dim(X) < \infty$.

Theorem 2 *Let C and D be the convex subsets of X. The following statements are equivalent:*

(a)   *C and D are strongly separated;*
(b) $d(C; D) := \inf\{\|c - d\| \mid c \in C, d \in D\} > 0.$

Theorem 3 *Let C and D be disjoint convex subsets in X. If*

$$C \text{ or } D \text{ is weakly compact,} \tag{2}$$

and the other is closed, then they are strongly separated.

Corollary 1 *Let C and D be disjoint closed convex subsets of a reflexive Banach space X. If one of the sets is bounded, then they are strongly separated.*

Theorem 4 *Let C and D be disjoint closed convex subsets satisfying:*

$$\mathrm{rec}(C) \cap \mathrm{rec}(D) = \{0\}. \tag{3}$$

If, in addition, C or D is locally compact, then they are strongly separated.

It is evident that any closed set in a finite-dimensional space is locally compact. Thus, a locally compact set may still be unbounded and, hence, may not be weakly compact.

Since a convex set is linearly bounded whenever it is bounded, (3) is much weaker than (2). Therefore, to compensate for that weakness, in Theorem 4 one of the sets is required to be locally compact for a strong separation.

Remark 1 For the strong separation, (3) seems to be essential even in the case of finite-dimensional spaces. Indeed, it is obvious that the following subsets of $\mathbb{R}^2$

$$C = \left\{ (x, y) \in \mathbb{R}^2 \mid x > 0, y \geq \frac{1}{x} \right\} \quad \text{and} \quad D = \left\{ (x, 0) \in \mathbb{R}^2 \mid x \in \mathbb{R} \right\} \tag{4}$$

are convex, closed, and disjoint, but are not strongly separated. The reason for this is that

$$\text{rec}(C) \cap \text{rec}(D) = \{(u, 0) \mid u \geq 0\} \neq \{(0, 0)\}.$$

Remark 2 In Corollary 1, if the underlying space is infinte-dimensional, then the boundedness (or weak compactness) hypothesis of one of the subsets cannot be substituted by (3). Indeed, consider the following subsets of the Hilbert space l^2:

$$C = \left\{ \xi = (x_n) \in l^2 \; \middle| \; \sum_{n=1}^{\infty} \frac{x_n}{n} = 1; x_n \geq 0, \forall n \right\},$$

$$D = \left\{ \zeta = (y_n) \in l^2 \; \middle| \; \sum_{n=1}^{\infty} \frac{y_n}{n+1} = 1; y_n \geq 0, \forall n \right\}.$$

Obviously, C and D are disjoint unbounded closed convex subsets of l^2. Let $(\xi^k) \subset C$ and $(\zeta^k) \subset D$ be sequences defined by

$$\xi^k = \left(0, \ldots, 0, k^{k-\text{th}}, 0, \ldots\right); \quad \zeta^k = \left(\frac{2}{k+1}, 0, \ldots, 0, k^{k-\text{th}}, 0, \ldots\right); \; k \in \mathbb{N}.$$

Since $\|\xi^k - \zeta^k\|_2 = \frac{2}{k+1} \to 0$, C and D are not strongly separated. It should be noted that, although being unbounded, both C and D are linearly bounded; hence, $\text{rec}(C) \cap \text{rec}(D) = \{0\}$.

Remark 3 The local compactness assumption on the sets in Theorem 4 seems a bit strong in the case of infinite-dimensional spaces. For example, consider the following subsets of l^2

$$C = \left\{ x = (x_n) \in l^2 \mid x_1 \geq \left(\sum_{i \neq 1} x_i^2\right)^{\frac{1}{2}} \right\},$$

$$D = \left\{ x = (x_n) \in l^2 \mid x_2 \geq 1 + 2\left(\sum_{i \neq 2} x_i^2\right)^{\frac{1}{2}} \right\}.$$

Firstly, we have $\text{rec}(C) \cap \text{rec}(D) = \{0\}$ because

$$\text{rec}(C) = \left\{ v \in l^2 \mid v_1 \geq \left(\sum_{i \neq 1} v_i^2\right)^{\frac{1}{2}} \right\},$$

$$\text{rec}(D) = \left\{ v \in l^2 \mid v_2 \geq 2\left(\sum_{i \neq 2} v_i^2\right)^{\frac{1}{2}} \right\}.$$

It is easy to check that C and D are disjoint closed convex sets and are strongly separated by the vector $x^* = (1, -1, 0, 0, \ldots) \in l^2$.

On the other hand, by setting $e_1 = (1, 0, 0, \ldots)$, $e_2 = (0, 1, 0, 0, \ldots)$, we have $e_1 \in \text{int } C$ and $2e_2 \in \text{int } D$. Thus, C and D are not locally compact; hence, Theorem 4 cannot be applied to establish a strong separation for them.

Our first aim in this paper is *to develop a new result on the strong separation of convex sets by imposing an assumption on the barrier cones of the sets in place of weak compactness or local compactness assumptions.*

From Theorems 1, 3, and 4, it follows that if C has a nonempty interior or X is finite-dimensional, then C has the separation property; and if C is weakly compact or it is locally compact and linearly bounded, then it has the strong separation property. Some further features of subsets having (strong) separation property have been established in the literature (for instance, see [5, 6]). Especially, in the case of Hilbert spaces, we have the interesting result below. For a convex set $C \subset X$, let ri C denote its relative interior; that is

$$\text{ri } C := \{x \in C \mid \exists \epsilon > 0, B(x; \epsilon) \cap C \subset \text{aff}(C)\},$$

where aff(C) is the affine hull of C and $B(x; \epsilon)$ denotes the open ball with radius ϵ around x.

Theorem 5 [5, Theorem 2] *An unbounded closed convex subset C of a Hilbert space X has the separation property if and only if* aff(C) *is a finite-codimensional closed affine subspace and* ri C *is nonempty.*

Our second aim is *to provide some necessary and/or sufficient conditions for a closed convex subset in a normed space to have the strong separation property.*

Recall that if $M \subset \mathbb{R}^n$ is a nonempty closed convex set and $f : \mathbb{R}^n \to \overline{R}$ is a convex, lower semicontinuous and coercive function, then the optimization problem

$$\mathcal{P}(M; f) : \begin{cases} f(x) \to \inf, \\ x \in M \end{cases}$$

has a nonempty compact solution set.

The third aim is *to prove a similar result for the convex programming problem with the constraint set having the strong separation property and the coerciveness assumption of the objective function is replaced by a weaker one.*

The rest of the paper is organized as follows: The next section will present a characterization for the interior of the barrier cones of convex sets in normed linear spaces. In Section 3, we develop a new result on strong separation noticing the role of barrier cones. In Section 4, we provide some conditions for a closed convex set to have the strong separation property. Finally, Section 5 is devoted to considering convex optimization problems with constraint set having the strong separation property.

2 A Characterization of the Interior of the Barrier Cone

In this section, we try to characterize the interior of the barrier cone of a closed convex subset C in a normed linear space X. We first note that, since the support function σ_C is sublinear and $\sigma_C(0) = 0$, the barrier cone of C is a convex cone containing the origin.

With the sets given in (4), we have

$$\text{bar}(C) = \left\{(u, v) \in \mathbb{R}^2 \mid u \leq 0, v \leq 0\right\}; \quad \text{bar}(D) = \{(0, v) \mid v \in \mathbb{R}\}.$$

Thus, int bar$(C) \neq \emptyset$ and int bar$(D) = \emptyset$.

It is well known that the weak*-closure of bar(C) coincides with the polar cone of rec(C); i.e.,

$$\overline{\text{bar}(C)}^* = \text{rec}(C)^0 = \left\{x^* \in X^* \mid \langle x^*, v \rangle \leq 0, \forall v \in \text{rec}(C)\right\}.$$

If X is a reflexive Banach space then the norm-closure and the weak*-closure of bar(C) coincide. Thus, we have

$$\overline{\text{bar}(C)} = \text{rec}(C)^0.$$

However, this relation may fail in a general normed linear space. In [1], the authors have given a complete description of the norm-closure of bar(C) when C is a closed convex subset of a normed linear space X

$$\overline{\text{bar}(C)} = \left\{ x^* \in X^* \;\middle|\; \lim_{r \to \infty} \left(\inf_{c \in C; \langle x^*, c \rangle \geq r} \frac{\|c\|}{r} \right) = \infty \right\}.$$

In fact, $\overline{\text{bar}(C)}$ can be represented in another form as stated below.

Theorem 6

$$\overline{\text{bar}(C)} = \left\{ x^* \in X^* \;\middle|\; \limsup_{c \in C; \|c\| \to \infty} \frac{\langle x^*, c \rangle}{\|c\|} \leq 0 \right\}.$$

Proof We need to show that, for every $x^* \in X^*$

$$\lim_{r \to \infty} \left(\inf_{c \in C; \langle x^*, c \rangle \geq r} \frac{\|c\|}{r} \right) = \infty \Leftrightarrow \limsup_{c \in C; \|c\| \to \infty} \frac{\langle x^*, c \rangle}{\|c\|} \leq 0.$$

Since both sides of the relation hold for $x^* = 0$, we may assume $x^* \neq 0$ and prove that the statements below are equivalent:

(i) $\forall M > 0, \exists N > 0, \forall r \geq N, \forall c \in C, \langle x^*, c \rangle \geq r \Rightarrow \frac{\|c\|}{r} > M$;

(ii) $\forall \epsilon > 0, \exists K > 0, \forall c \in C, \|c\| \geq K \Rightarrow \frac{\langle x^*, c \rangle}{\|c\|} < \epsilon$.

(i) $\Rightarrow$ (ii) For every $\epsilon > 0$, we set $M = \frac{1}{\epsilon}$. Then, there exists $N > 0$ satisfying (i). Let $K = \frac{N}{\epsilon} > 0$. For every $c \in C$ such that $\|c\| \geq K$, by letting $r := \langle x^*, c \rangle$, we have

• If $\langle x^*, c \rangle = r \geq N$, then $\frac{\|c\|}{\langle x^*, c \rangle} = \frac{\|c\|}{r} > M$; hence, $\frac{\langle x^*, c \rangle}{\|c\|} < \epsilon$.

• If $\langle x^*, c \rangle < N$, then $\frac{\langle x^*, c \rangle}{\|c\|} < \frac{N}{K} = \epsilon$.

(ii) $\Rightarrow$ (i) For every $M > 0$, we set $\epsilon = \frac{1}{M} > 0$ again. Then, there exists $K > 0$ satisfying (ii). Let now $N = K\|x^*\|$. For every $r \geq N$ and $c \in C$ such that $\langle x^*, c \rangle \geq r$, we have:

$$K\|x^*\| = N \leq r \leq \langle x^*, c \rangle \leq \|x^*\|\|c\|.$$

This shows that $\|c\| \geq K$, which, by (ii), implies $\frac{r}{\|c\|} \leq \frac{\langle x^*, c \rangle}{\|c\|} < \epsilon$ and hence $\frac{\|c\|}{r} > M$. $\square$

Inspired by this result, we derive a characterization for the interior of bar(C) as below:

$$\text{int bar}(C) = \left\{ x^* \in X^* \;\middle|\; \limsup_{c \in C; \|c\| \to \infty} \frac{\langle x^*, c \rangle}{\|c\|} < 0 \right\}.$$

We state this fact in the following result.

Theorem 7 *Let $x^* \in$ bar(C). The following statements are equivalent:*
(a) $x^* \in$ int bar(C);
(b) *There exists $\gamma > 0$ such that*

$$\sup_{c \in C \setminus B(0; \gamma)} \langle x^*, c \rangle < \sigma_C(x^*); \tag{5}$$

(c) *There exist positive numbers α, R such that*

$$\langle x^*, c \rangle \leq -\alpha \|c\|, \forall c \in C \setminus B(0; R); \tag{6}$$

(d) $\limsup_{c \in C; \|c\| \to \infty} \frac{\langle x^*, c \rangle}{\|c\|} < 0,$

where, $B(0; \gamma)$ and $B(0; R)$ denote, respectively, the open balls of radii γ and R around the origin.

Proof Since the equivalence between (c) and (d) is rather obvious, we only need to prove (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b) Suppose that (5) fails to hold for every $\gamma > 0$, or equivalently,

$$\sup_{c \in C \setminus B(0;\gamma)} \langle x^*, c \rangle = \sigma_C(x^*), \forall \gamma > 0. \tag{7}$$

Then, there is a sequence $(c_n) \subset C$ such that $\|c_n\| \to \infty$ and

$$\lim_{n \to \infty} \langle x^*, c_n \rangle = \sigma_C(x^*).$$

Since the sequence (c_n) is unbounded, by virtue of Banach-Steinhaus theorem, there exists $u^* \in X^*$ such that

$$\limsup_{n \to \infty} \langle u^*, c_n \rangle = \infty.$$

It implies that

$$\sigma_C(x^* + \lambda u^*) \geq \limsup_{n \to \infty} \langle x^* + \lambda u^*, c_n \rangle = \infty, \forall \lambda > 0.$$

In other words, $x^* + \lambda u^* \notin \text{bar}(C)$, for every $\lambda > 0$. Thus, $x^* \notin \text{int bar}(C)$.

(b) $\Rightarrow$ (c) By (5), there exists $c_0 \in C \cap B(0; \gamma)$ such that, for some $\varepsilon > 0$

$$\sup_{c \in C \setminus B(0;\gamma)} \langle x^*, c \rangle < \langle x^*, c_0 \rangle - \varepsilon.$$

Choose R large enough such that $R > \gamma$ and $\langle x^*, c_0 \rangle - \frac{\varepsilon}{4\gamma} R \leq 0$. We shall prove that (6) holds for such R and $\alpha := \frac{\varepsilon}{4\gamma}$.

Take $c \in C \setminus B(0; R)$ arbitrarily. Since $\|c\| \geq R > \gamma > \|c_0\|$, there exists $\lambda \in (0, 1)$ such that $\|u\| = \gamma$ with $u = \lambda c + (1 - \lambda)c_0 \in C$. We have

$$\gamma = \|\lambda c + (1 - \lambda)c_0\| \geq \lambda \|c\| - (1 - \lambda)\|c_0\|,$$

which implies that

$$\lambda \leq \frac{\gamma + \|c_0\|}{\|c\| + \|c_0\|} \leq \frac{2\gamma}{\|c\|}. \tag{8}$$

Since $u \in C \setminus B(0; \gamma)$, we have

$$\langle x^*, c_0 \rangle - \varepsilon > \langle x^*, u \rangle = \lambda \langle x^*, c \rangle + (1 - \lambda)\langle x^*, c_0 \rangle,$$

which together with (8) implies that

$$\varepsilon < \frac{2\gamma}{\|c\|} \left(\langle x^*, c_0 \rangle - \langle x^*, c \rangle \right),$$

or,

$$\langle x^*, c \rangle \leq -\frac{\varepsilon}{2\gamma} \|c\| + \langle x^*, c_0 \rangle.$$

Noting that $\|c\| \geq R$ and $-\frac{\varepsilon}{4\gamma} R + \langle x^*, c_0 \rangle \leq 0$, we have

$$\langle x^*, c \rangle \leq -\frac{\varepsilon}{4\gamma} \|c\| - \frac{\varepsilon}{4\gamma} R + \langle x^*, c_0 \rangle \leq -\alpha \|c\|.$$

(c) $\Rightarrow$ (a) If (6) fulfills, then for every $u^* \in B(x^*; \alpha)$, we have

$$\langle u^*, c \rangle \leq \langle x^*, c \rangle + \|u^* - x^*\| \|c\| \leq \langle x^*, c \rangle + \alpha \|c\| \leq 0, \forall c \in C \setminus B(0; R),$$

and hence,

$$\sigma_C(u^*) \leq \max \left\{ 0, \sigma_{C \cap B(0; R)}(u^*) \right\} \leq R \|u^*\| < \infty.$$

Thus, $B(x^*; \alpha) \subset \mathrm{bar}(C)$, from which (a) follows. $\qquad \square$

Corollary 2 *C is bounded if and only if* $\mathrm{bar}(C) = X^*$.

Proof Since $\mathrm{bar}(C)$ is a cone, $\mathrm{bar}(C) = X^*$ if and only if $0 \in \mathrm{int}\,\mathrm{bar}(C)$. On the other hand, it follows from Theorem 7 that $0 \in \mathrm{int}\,\mathrm{bar}(C)$ if and only if there exists $\gamma > 0$ such that $C \setminus B(0; \gamma) = \emptyset$, or equivalently, C is bounded. $\qquad \square$

3 Separation Theorems via Recession Cone and Barrier Cone

As we have seen in Theorem 4, for the strong separation of unbounded subsets, besides (3), the assumption of local compactness is also required. In the following discussion, instead of using local compactness assumption on the sets, we require one of their barrier cones to have a nonempty interior. The main result of the section is stated below.

Theorem 8 *Let C and D be disjoint closed convex subsets of a reflexive Banach space, satisfying (3). If, in addition,*

$$(\mathrm{int}\,\mathrm{bar}(C)) \cup (\mathrm{int}\,\mathrm{bar}(D)) \neq \emptyset, \tag{9}$$

then C and D are strongly separated.

Before proceeding to the proof, we prove the following lemmas.

Lemma 1 *Let C be a closed convex subset of X and (c_n) be a sequence in C such that $\|c_n\| \to \infty$ and*

$$\frac{c_n}{\|c_n\|} \xrightarrow{w} u \in X.$$

Then, $u \in \mathrm{rec}(C)$.

Proof Take $c \in C$ we prove that $c + u \in C$. Since $\|c_n\| \to \infty$,

$$v_n := \left(1 - \frac{1}{\|c_n\|} \right) c + \frac{1}{\|c_n\|} c_n \in C$$

for n large enough (such that $1 < \|c_n\|$). On the other hand, (v_n) weakly converges to $c + u$. By noting that a closed convex set is also weakly closed, we deduce $c + u \in C$. Since this inclusion holds for every $c \in C$, it follows that $u \in \mathrm{rec}(C)$. $\qquad \square$

Lemma 2 *Let (c_n) and (d_n) be sequences in X such that $\|c_n\| \to \infty$, and for some $r > 0$, $\|c_n - d_n\| \leq r$ for every n. If*

$$\frac{c_n}{\|c_n\|} \longrightarrow u, \quad or \quad \frac{c_n}{\|c_n\|} \xrightarrow{w} u,$$

with $u \in X$, then

$$\frac{d_n}{\|d_n\|} \longrightarrow u, \quad or \quad \frac{d_n}{\|d_n\|} \xrightarrow{w} u, \ respectively.$$

Proof Since

$$\left\| \frac{c_n}{\|c_n\|} - \frac{d_n}{\|d_n\|} \right\| \le \frac{\|c_n - d_n\|}{\|c_n\|} + \left| \frac{1}{\|c_n\|} - \frac{1}{\|d_n\|} \right| \|d_n\| \le \frac{2r}{\|c_n\|} \to 0,$$

we have

$$\frac{c_n}{\|c_n\|} - \frac{d_n}{\|d_n\|} \longrightarrow 0,$$

from which the lemma follows.   □

Proof of Theorem 8 Assume int bar(C) is nonempty. We prove d(C; D) > 0 by contradiction. Suppose that there exist sequences $(c_n) \subset C$, $(d_n) \subset D$ such that $\|c_n - d_n\| \to 0$. There are two cases depending on whether or not $\|c_n\|$ tends to ∞.

• $\|c_n\| \to \infty$. Since the space is reflexive, without loss of generality, we may assume that

$$\frac{c_n}{\|c_n\|} \xrightarrow{w} u \in X,$$

and hence, from Lemma 2,

$$\frac{d_n}{\|d_n\|} \xrightarrow{w} u.$$

Thus, by Lemma 1, $u \in \mathrm{rec}(C) \cap \mathrm{rec}(D)$.

Choose $x^* \in \mathrm{int\, bar}(C)$ such that $x^* \ne 0$. By Theorem 7, for some $\alpha > 0$, we have

$$\left\langle x^*, \frac{c_n}{\|c_n\|} \right\rangle \le -\alpha$$

for n large enough. By letting $n \to \infty$, we obtain

$$\langle x^*, u \rangle \le -\alpha < 0,$$

which implies $u \ne 0$, contradicting (3).

• $\|c_n\| \not\to \infty$. By restricting to a subsequence if necessary, we may assume that (c_n) weakly converges to $u \in X$. However, in this situation, (d_n) also weakly converges to u. Since C and D are convex and closed, they are weakly closed. Thus, $u \in C \cap D$, contradicting to the assumption that C and D are disjoint.   □

Example 1 Let C and D be the sets given in Remark 3. For each $x \in C$, we have $x_1 \ge 0$ and

$$\|x\|_2^2 = \sum_{i=1}^{\infty} x_i^2 \le 2x_1^2.$$

It implies that

$$\langle -e_1, x \rangle = -x_1 \le -\frac{1}{\sqrt{2}} \|x\|_2; \ \forall x \in C.$$

Therefore, by Theorem 7, $-e_1 \in \mathrm{int\, bar}(C)$. Applying Theorem 8, we deduce that C and D are strongly separated. While, as mentioned in Remark 3, Theorem 4 cannot be applied to establish a strong separation here.

Remark 4 The sets C and D given in Remark 2 satisfy (3), but are not strongly separated. It is not difficult to verify that

$$\mathrm{bar}(C) = \mathrm{bar}(D) = \left\{ (y_n) \in l^2 \ \middle| \ \sup_{n \ge 1}(ny_n) < \infty \right\},$$

and hence,

$$\text{int bar}(C) = \text{int bar}(D) = \emptyset.$$

This fact shows that (9) is crucial even in the case where X is a Hilbert space.

Example 2 In Theorem 8, the assumption about reflexivity of the space is essential. Consider two subsets of the nonreflexive space l^1:

$$C = \left\{ \xi = (x_n) \in l^1 \; \middle| \; \sum_{n=1}^{\infty} x_n = 1; \; x_n \geq 0, \forall n \right\},$$

$$D = \left\{ \zeta = (y_n) \in l^1 \; \middle| \; \sum_{n=1}^{\infty} \frac{ny_n}{n+1} = 1; \; y_n \geq 0, \forall n \right\}.$$

Obviously, C and D are disjoint bounded closed convex subsets of l^1. Thus, (3) is fulfilled. In addition, since C is bounded, $\text{int bar}(C) = l^\infty$. However, by letting $(\xi^k) \subset C$ and $(\zeta^k) \subset D$ be the sequences defined by

$$\xi^k = \left(0, \ldots, 0, 1^{k-\text{th}}, 0, \ldots \right); \quad \zeta^k = \frac{k+1}{k} \xi^k; \; k \in \mathbb{N},$$

we have $\|\xi^k - \zeta^k\|_1 = \frac{1}{k} \to 0$. Thus, C and D are not strongly separated.

As we have seen, in a finite-dimensional space, any pair of disjoint closed convex sets satisfying (3) is strongly separated. In the case of infinite-dimensional spaces, besides the assumption of local compactness or (9), (3) is also required for the strong separation of convex sets.

Thus, (3) plays an important role in the strong separation. However, it should be noted that this condition alone is not enough to yield even the (weak) separation of two disjoint closed convex subsets. The following example illustrates this point.

Example 3 Let X be a real Hilbert space, in which there exist two closed subspaces M and N such that $M \cap N = \{0\}$, $M + N$ is dense but not closed in X, i.e., $M + N \neq X$ (see, [3, Problem 2, p. 129]).

Take $x_0 \in X \setminus (M + N)$ and let $C = x_0 - M$, $D = N$. Thus, C and D are disjoint closed convex subsets. Furthermore, since $\text{rec}(C) = M$ and $\text{rec}(D) = N$, $\text{rec}(C) \cap \text{rec}(D) = \{0\}$. We shall show that C and D are not separated. Suppose the contrary. Take $v \in X \setminus \{0\}$ such that

$$\langle v, x_0 - m \rangle \leq \langle v, n \rangle, \forall m \in M, \, n \in N.$$

It implies that

$$\langle v, x_0 \rangle \leq \langle v, x \rangle, \forall x \in M + N.$$

Since $M + N$ is dense in X, it follows that $v = 0$, a contradiction. Hence, C and D are not separated.

4 Subsets Having the Strong Separation Property

In this section, we are interested in properties of subsets having the strong separation property. As usual, let S and S^* denote the unit spheres in X and X^*, respectively. Let C be a closed convex subset of X. In some cases, the following conditions are needed:

(A) X is reflexive and int bar$(C) \neq \emptyset$.

(B) C is locally compact.

(C) For some $r > 0$ and finite-dimensional subspace Z, we have

$$C \subset B(0; r) + Z.$$

Remark 5 The conditions (A), (B), and (C) are strongly independent in the sense that each of them cannot be followed from the two remaining ones. This fact will be shown by the examples below.

• $C = \{(x, y) \in \mathbb{R}^2 \mid y \leq 0\}$ satisfies (B) and (C) but fails (A).

• $C = \{x = (x_n) \in l^2 \mid \|x\|_2 \leq 1\}$ satisfies (A) and (C) but fails (B).

• Let

$$C = \left\{ x := (x_n) \in l^2 \mid 0 \leq x_{n+1} \leq \frac{n}{n+1} x_n, \forall n \geq 1 \right\}.$$

Then, C is a closed convex cone. For every $x \in C$, we have

$$0 \leq x_1, \ 0 \leq x_2 \leq \frac{x_1}{2}, \ 0 \leq x_3 \leq \frac{2x_2}{3} \leq \frac{x_1}{3}, \dots, 0 \leq x_n \leq \frac{x_1}{n}, \dots$$

It implies that

$$0 \leq x_1 \leq \|x\|_2 = \sqrt{\sum_{i=1}^{\infty} x_i^2} \leq x_1 \sqrt{\sum_{i=1}^{\infty} \frac{1}{i^2}} = \frac{\pi x_1}{\sqrt{6}}.$$

Taking $x_0^* = (-1, 0, 0, \dots) \in l^2$, we obtain

$$\langle x_0^*, x \rangle = -x_1 \leq -\frac{\sqrt{6}}{\pi} \|x\|_2, \forall x \in C,$$

which, by Theorem 7, implies $x_0^* \in$ int bar(C). Thus, C satisfies (A).

We have

$$C \cap \overline{B(0; 1)} = \left\{ x \in l^2 \mid \|x\|_2 \leq 1; 0 \leq x_{n+1} \leq \frac{n}{n+1} x_n, \forall n \geq 1 \right\}$$

$$\subset E := \left\{ x \in l^2 \mid 0 \leq x_n \leq \frac{1}{n}, \forall n \geq 1 \right\}.$$

Since E is compact, C satisfies (B). Finally, we show that C does not satisfy (C). Indeed, if (C) holds then, since C is a closed convex cone, $C = \text{rec}(C) \subset Z$. But this is impossible because Z is finite-dimensional while C contains the following infinite set of linearly independent vectors:

$$V = \left\{ (1, 0, 0, \dots), \left(1, \frac{1}{2}, 0, \dots \right), \left(1, \frac{1}{2}, \frac{1}{3}, 0, \dots \right), \dots \right\}.$$

Theorem 9 below will provide a necessary condition for a closed convex subset of X to have the strong separation property.

Lemma 3 *Let C be a closed convex subset of X and $(c_n) \subset C$ is a sequence such that $\|c_n\| \to +\infty$. If one of the conditions (A), (B), or (C) is satisfied, then there exists a subsequence (c_{n_k}) of (c_n) such that, for some $x_0^* \in S^*$ and $\rho > 0$, we have*

$$\lim_{n_k \to \infty} \left\langle x_0^*, \frac{c_{n_k}}{\|c_{n_k}\|} \right\rangle = -\rho. \tag{10}$$

Proof We prove the lemma under each of the conditions (A), (B), or (C).

(A) Take $x_0^* \in S^* \cap \operatorname{int} \operatorname{bar}(C)$. By Theorem 7, for some $\alpha > 0$ and $R > 0$, we have

$$\langle x_0^*, c \rangle \leq -\alpha \|c\|, \forall c \in C \setminus B(0; R). \tag{11}$$

Since X is reflexive, there exists a subsequence (c_{n_k}) of (c_n) such that

$$\frac{c_{n_k}}{\|c_{n_k}\|} \xrightarrow{w} s \in X.$$

This, together with (11), implies (10) with $\rho = -\langle x_0^*, s \rangle \geq \alpha > 0$.

(B) Since C is locally compact, there exists a subsequence (c_{n_k}) of (c_n) such that

$$\frac{c_{n_k}}{\|c_{n_k}\|} \longrightarrow s \in S.$$

By choosing $x_0^* \in S^*$ such that $\langle x_0^*, s \rangle = -1$, we obtain (10) with $\rho = 1$.

(C) Since $C \subset B(0; r) + Z$, there exists a sequence $(z_n) \subset Z$ such that $\|z_n - c_n\| < r$ for all n; hence, $\|z_n\| \to \infty$. Since $\dim Z < \infty$, there exists a subsequence (z_{n_k}) of (z_n) such that

$$\frac{z_{n_k}}{\|z_{n_k}\|} \longrightarrow s \in S.$$

From Lemma 2, we also have

$$\frac{c_{n_k}}{\|c_{n_k}\|} \longrightarrow s \in S,$$

and by choosing x_0^* as in the case of (B) we obtain (10). $\qquad\square$

Theorem 9 *Let $C \subset X$ be a closed convex subset having the strong separation property. In addition, at least one of the conditions* (A), (B), *or* (C) *is satisfied. Then*

$$\operatorname{bar}(C) = \operatorname{int} \operatorname{bar}(C) \cup \{0\}, \tag{12}$$

that is to say, $x^* \in \operatorname{int} \operatorname{bar}(C)$ whenever $x^* \in \operatorname{bar}(C) \setminus \{0\}$.

Proof The proof is by contradiction. Suppose that there exists $x^* \in \operatorname{bar}(C) \setminus \{0\}$ so that $x^* \notin \operatorname{int} \operatorname{bar}(C)$. Since $\operatorname{bar}(C)$ is a cone, we may assume $\|x^*\| = 1$. By Theorem 7, (7) holds; hence, there exists a sequence $(c_n) \subset C$ such that $\|c_n\| \to +\infty$ and

$$\langle x^*, c_n \rangle \to \beta := \sigma_C(x^*) < \infty.$$

Without loss of generality, we may assume that

$$\beta - \frac{1}{n} < \langle x^*, c_n \rangle \leq \beta; \forall n.$$

From Lemma 3, without loss of generality, we may assume that the sequence $\left(\frac{c_n}{\|c_n\|}\right)$ converges (strongly or weakly) to $s \in X$ and

$$\lim_{n \to \infty} \left\langle x_0^*, \frac{c_n}{\|c_n\|} \right\rangle = \langle x_0^*, s \rangle = -\rho < 0$$

for some $x_0^* \in S^*$ and $\rho > 0$.

Choose $v \in S$ such that $\langle x^*, v \rangle > \frac{1}{2}$ and put $d_n := c_n + \frac{4}{n}v$, for each integer $n \geq 1$. We now show that the following subset

$$D = \overline{\operatorname{co}}\{d_n \mid n \geq 1\}$$

is convex, closed, and disjoint from C.

Clearly, D is convex and closed. We prove $C \cap D = \emptyset$ by contradiction. Suppose that there exists $c_0 \in C \cap D$. Since the sequence $\left(\frac{c_n}{\|c_n\|}\right)$ converges (strongly or weakly) to $s \in X$, by virtue of Lemma 2, the sequence $\left(\frac{c_n-c_0}{\|c_n-c_0\|}\right)$ also converges to s. Therefore, by setting

$$s_n := \frac{c_n - c_0}{\|c_n - c_0\|}, \quad t_n := \|c_n - c_0\|,$$

we have $s_n \in S$, (s_n) converges (strongly or weakly) to s, $t_n \to +\infty$,

$$c_n = c_0 + t_n s_n; \ \forall n \geq 1,$$

and

$$\lim_{n \to \infty} \langle x_0^*, s_n \rangle = \langle x_0^*, s \rangle = -\rho < 0.$$

Take $k \in \mathbb{N}$ large enough such that

$$\langle x_0^*, s_n \rangle < -\frac{\rho}{2}, t_n > 12; \forall n > k, \tag{13}$$

and then set

$$\gamma := \max\{t_1, t_2, \dots, t_k\} + 1; \ \varepsilon := \min\left\{\frac{1}{2k\gamma}, \frac{2\rho}{5+k\gamma}\right\} < \frac{1}{2k}. \tag{14}$$

Since $c_0 \in D = \overline{\text{co}}\{d_n \mid n \geq 1\}$, there exist nonnegative numbers $\lambda_1, \lambda_2, \dots, \lambda_m$, with $m > k$, such that

$$\sum_{n=1}^{m} \lambda_n = 1; \quad \left\|\sum_{n=1}^{m} \lambda_n d_n - c_0\right\| < \varepsilon.$$

Noting that $\|x^*\| = 1$, we have

$$\varepsilon > \left\|\sum_{n=1}^{m} \lambda_n d_n - c_0\right\| \geq \left\langle x^*, \sum_{n=1}^{m} \lambda_n d_n - c_0 \right\rangle = \sum_{n=1}^{m} \lambda_n \left\langle x^*, c_n + \frac{4}{n}v \right\rangle - \langle x^*, c_0 \rangle$$

$$\geq \sum_{n=1}^{m} \lambda_n \left(\beta - \frac{1}{n}\right) + \sum_{n=1}^{m} \lambda_n \frac{2}{n} - \beta = \sum_{n=1}^{m} \frac{\lambda_n}{n}. \tag{15}$$

It follows that

$$\varepsilon > \sum_{n=1}^{k} \frac{\lambda_n}{n} \geq \frac{1}{k}\sum_{n=1}^{k}\lambda_n,$$

which, together with (14), gives

$$\sum_{n=1}^{k} \lambda_n < k\varepsilon \leq \frac{1}{2}, \tag{16}$$

and hence,

$$\sum_{n=k+1}^{m} \lambda_n > \frac{1}{2}. \tag{17}$$

On the other hand, we also have

$$\varepsilon > \left\|\sum_{n=1}^{m} \lambda_n d_n - c_0\right\| = \left\|\sum_{n=1}^{m} \lambda_n (d_n - c_0)\right\| = \left\|\sum_{n=1}^{m} \lambda_n \left(t_n s_n + \frac{4}{n}v\right)\right\|$$

$$\geq \left\|\sum_{n=k+1}^{m} \lambda_n t_n s_n\right\| - \left\|\sum_{n=1}^{k} \lambda_n t_n s_n\right\| - \left\|\left(\sum_{n=1}^{m} \frac{4\lambda_n}{n}\right)v\right\|.$$

Noting that $v, s_n \in S$, $x_0^* \in S^*$ and $\langle x_0^*, s_n \rangle < -\frac{\rho}{2}$ for $n > k$, one has

$$\varepsilon > \left\langle -x_0^*, \sum_{n=k+1}^{m} \lambda_n t_n s_n \right\rangle - \sum_{n=1}^{k} \lambda_n t_n - 4 \sum_{n=1}^{m} \frac{\lambda_n}{n}$$

$$> \frac{\rho}{2} \sum_{n=k+1}^{m} \lambda_n t_n - \sum_{n=1}^{k} \lambda_n t_n - 4 \sum_{n=1}^{m} \frac{\lambda_n}{n}.$$

It follows that

$$\frac{\rho}{2} \sum_{n=k+1}^{m} \lambda_n t_n < \varepsilon + \sum_{n=1}^{k} \lambda_n t_n + 4 \sum_{n=1}^{m} \frac{\lambda_n}{n}. \tag{18}$$

Since (13) and (17), we have

$$\frac{\rho}{2} \sum_{n=k+1}^{m} \lambda_n t_n > \frac{\rho}{2} \frac{1}{2} 12 = 3\rho. \tag{19}$$

On the other hand, from (14) and (16), it follows that

$$\sum_{n=1}^{k} \lambda_n t_n < \gamma \sum_{n=1}^{k} \lambda_n < k\gamma\varepsilon. \tag{20}$$

Combining (15), (18), (19), (20), and the definition of ε, we obtain

$$3\rho < \varepsilon + k\gamma\varepsilon + 4\varepsilon = (5 + k\gamma)\varepsilon \le 2\rho$$

which is clearly absurd. Consequently, $C \cap D = \emptyset$.

Consequently, D is a closed convex subset disjoint from C. On the other hand, since $\|c_n - d_n\| = \frac{4}{n} \to 0$, $\mathrm{d}(C; D) = 0$; hence, C and D are not strongly separated. Thus, C does not have the strong separation property. This completes the proof of the theorem. $\square$

Proposition 1 *If C is unbounded and $\overline{\mathrm{aff}(C)} \ne X$, then*

$$\mathrm{bar}(C) \ne \mathrm{int}\, \mathrm{bar}(C) \cup \{0\}. \tag{21}$$

Proof Indeed, since $\overline{\mathrm{aff}(C)} \ne X$, there exists $x^* \in X^* \setminus \{0\}$ such that

$$\langle x^*, c \rangle = \alpha := \sigma_C(x^*), \forall c \in C.$$

Hence, $x^* \in \mathrm{bar}(C)$. On the other hand, since C is unbounded, (7) holds. It now follows from Theorem 7 that $x^* \notin \mathrm{int}\, \mathrm{bar}(C)$ and (21) is derived. $\square$

From Theorem 9 and Proposition 1, we deduce the next corollary.

Corollary 3 *Let C be an unbounded closed convex subset of X having the strong separation property. In addition, suppose that at least one of the conditions* (A), (B), *or* (C) *is satisfied. Then, $\overline{\mathrm{aff}(C)} = X$. Furthermore, if $\dim(C) < \infty$, then $\mathrm{aff}(C) = X$ and $\mathrm{int}\, C \ne \emptyset$.*

As a converse of Theorem 9, we have the following.

Theorem 10 *Let X be a reflexive Banach space and $C \subset X$ be a closed convex subset. If C has the separation property and* (12) *holds, then C has the strong separation property.*

Proof We shall prove that, if C has the separation property but does not have the strong separation property, then (12) fails to hold.

Let D be a closed convex subset of X, disjoint from C, but cannot be strongly separated from C. That is $\mathrm{d}(C; D) = 0$, i.e., there exist sequences $(c_n) \subset C$, $(d_n) \subset D$ such that $\|c_n - d_n\| \to 0$. If $\|c_n\| \nrightarrow +\infty$ then, since X is reflexive, by restricting to a subsequence if necessary, we may assume that $c_n \overset{w}{\to} \bar{x}$; hence, $d_n \overset{w}{\to} \bar{x}$ too. Since C and D are (weakly) closed, $\bar{x}$ must belong to both of them, contradicting the assumption that they are disjoint. Consequently,

$$\|c_n\| \to +\infty. \tag{22}$$

On the other hand, by the separation property of C, there is a hyperplane $H(x^*; \alpha)$ $(x^* \neq 0)$ separating C and D, i.e.,

$$\langle x^*, c \rangle \leq \alpha \leq \langle x^*, d \rangle; \forall c \in C, \forall d \in D. \tag{23}$$

It implies that $x^* \in \mathrm{bar}(C)$ and

$$\langle x^*, c_n \rangle \leq \alpha \leq \langle x^*, d_n \rangle; \forall n.$$

Noting that $\langle x^*, d_n - c_n \rangle \leq \|x^*\| \|d_n - c_n\| \to 0$, we derive the equalities

$$\lim_{n \to \infty} \langle x^*, c_n \rangle = \lim_{n \to \infty} \langle x^*, d_n \rangle = \alpha,$$

which, together with (22)–(23), implies (7). Thus, $x^* \in \mathrm{bar}(C) \setminus \mathrm{int}\,\mathrm{bar}(C)$. □

Theorem 11 *Let X be an infinite-dimensional real Hilbert space and C be an unbounded closed convex subset of X. If, in addition, C is locally compact, then it does not have the strong separation property.*

Proof Suppose the contrary that C has the strong separation property. By virtue of Theorem 5, $\mathrm{aff}(C)$ is a finite-codimensional closed affine subspace and $\mathrm{ri}\,C \neq \emptyset$. On the other hand, by Corollary 3, $\mathrm{aff}(C) = \overline{\mathrm{aff}(C)} = X$; hence, $\mathrm{int}\,C = \mathrm{ri}\,C \neq \emptyset$. But this is impossible because C is a locally compact subset in an infinite-dimensional space. □

Theorem 12 *Let X be a real Hilbert space and $C \subset X$ be an unbounded closed convex subset satisfying either condition (A) or (C). Then, C has the strong separation property if and only if $\mathrm{int}\,C$ is nonempty and (12) holds.*

Proof If C has the strong separation property then, by Theorems 5, Theorem 9, and Corollary 3, we deduce that $\mathrm{int}\,C$ is nonempty and (12) holds.

Conversely, if $\mathrm{int}\,C$ is nonempty and (12) holds then, by Theorems 1 and 10, C has the strong separation property. □

Corollary 4 *Let C be a closed convex subset in a finite-dimensional space X. Then, C has the strong separation property if and only if (12) holds. Furthermore, if C is unbounded and C has the strong separation property, then $\mathrm{int}\,C$ is nonempty.*

Proof If X is finite-dimensional then it is reflexive and every closed convex subset of X is locally compact and has the separation property. The conclusion of the corollary therefore follows directly from Theorems 9 and 10. □

Remark 6 Corollary 4 shows that, in finite-dimensional spaces, apart from bounded subsets, every unbounded closed convex subset also has the strong separation property whenever (12) is fulfilled. The example below presents a set of this type.

Example 4 The following subset

$$C = \left\{ (x, y) \in \mathbb{R}^2 \mid y \geq x^2 \right\}$$

is convex, closed, and unbounded. It is not hard to verify that

$$\sigma_C(u, v) = \begin{cases} +\infty & \text{if } (v > 0) \text{ or } ((v = 0) \text{ and } (u \neq 0)), \\ 0 & \text{if } u = v = 0, \\ -\dfrac{u^2}{4v} & \text{if } v < 0. \end{cases}$$

Consequently, $\operatorname{bar}(C) = \{(0, 0)\} \cup \{(u, v) \mid v < 0\}$, $\operatorname{int} \operatorname{bar}(C) = \{(u, v) \mid v < 0\}$. Thus, (12) holds, and C has the strong separation property.

Example 5 Consider the subset of $\mathbb{R}^2$

$$C = \left\{ (x, y) \in \mathbb{R}^2 \mid \exp(x) - y \leq 0 \right\}.$$

Since

$$\sigma_C(u, v) = \begin{cases} +\infty, & (u < 0 \text{ or } v \geq 0) \text{ and } ((u, v) \neq (0, 0)), \\ u \ln\left(-\dfrac{u}{v}\right) - u, & u > 0 > v, \\ 0, & v \leq 0 = u, \end{cases}$$

$\operatorname{bar}(C) = \{(u, v) \mid u \geq 0 > v\} \cup \{(0, 0)\}$. Thus,

$$\operatorname{int} \operatorname{bar}(C) \cup \{(0, 0)\} = \{(u, v) \mid u > 0 > v\} \cup \{(0, 0)\} \neq \operatorname{bar}(C).$$

It implies that C does not have the strong separation property.

5 Application to a Convex Optimization Problem

In this section, we shall establish some results for a convex optimization problem whose constraint set has the strong separation property. We assume throughout the section that $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a proper convex, lower semicontinuous function, and $M \subset \mathbb{R}^n$ is a nonempty closed convex set. Consider the optimization problem

$$\mathcal{P}(M; f) : \begin{cases} f(x) \to \inf, \\ x \in M, \end{cases}$$

in which we seek $\bar{x} \in M$ such that

$$f(\bar{x}) = \bar{f} := \inf\{f(x) : x \in M\}.$$

The solution set of $\mathcal{P}(M; f)$ is denoted by $\operatorname{Sol}(M; f)$, that is

$$\operatorname{Sol}(M; f) = \{\bar{x} \in M \mid f(\bar{x}) = \bar{f}\}.$$

The horizon function $f^\infty : \mathbb{R}^n \to \overline{\mathbb{R}}$, associated with f, is defined by

$$f^\infty(v) := \lim_{\lambda \to +\infty} \frac{f(x_0 + \lambda v) - f(x_0)}{\lambda},$$

with some $x_0 \in \mathrm{dom}\, f$. In fact, such a limit is independent of $x_0 \in \mathrm{dom}\, f$. The function f^∞ is proper, sublinear, and lower semicontinuous (see, for example, [10]). f is said to be coercive if

$$\lim_{\|x\| \to \infty} f(x) = +\infty.$$

Since f is convex on a finite-dimensional space, it is not difficult to verify that f is coercive if and only if

$$\liminf_{\|x\| \to \infty} \frac{f(x)}{\|x\|} > 0,$$

or, equivalently,

$$\forall v \neq 0,\; f^\infty(v) > 0. \tag{24}$$

It is well known that, if the objective function f is coercive and the constraint set M is closed, then $\mathrm{Sol}(M; f)$ is nonempty and compact. In the following, we show that if M has the strong separation property then, in order for the solution set to be compact, f need not be coercive, instead, it is required to satisfy the next weaker condition

$$\forall 0 \neq v \in C(f^\infty; 0),\, \exists \tilde{x} \in \mathrm{dom}\, f,\; \lim_{\lambda \to +\infty} f(\tilde{x} + \lambda v) = -\infty, \tag{25}$$

where

$$C(f^\infty; 0) := \{ v \in \mathbb{R}^n \mid f^\infty(v) \leq 0 \}.$$

This fact is stated in the following theorem.

Theorem 13 *If M has the strong separation property, f is bounded below on M and satisfies* (25)*, then the solution set of $\mathcal{P}(M; f)$ is nonempty and compact.*

Proof Since f is convex and lower semicontinuous, $\mathrm{Sol}(M; f)$ is a closed convex set. Suppose that $\mathrm{Sol}(M; f)$ is not compact or empty. Then, there exists a sequence $(x_n) \subset M$ such that $\|x_n\| \to +\infty$ and

$$\lim_{n \to \infty} f(x_n) = \bar{f}.$$

By an argument analogous to the proof of Lemma 3 (under condition (C)), we may assume that

$$\frac{x_n}{\|x_n\|} \to s \in S.$$

Take $x_0 \in M$. By Lemmas 1 and 2, we have $s \in \mathrm{rec}(M)$,

$$s_n := \frac{x_n - x_0}{\|x_n - x_0\|} \to s,$$

and $x_n = x_0 + t_n s_n$ with $t_n = \|x_n - x_0\| \to +\infty$.

Fix a number $\lambda > 0$. For n large enough, one has $\lambda < t_n$ and

$$\frac{f(x_0 + \lambda s_n) - f(x_0)}{\lambda} \leq \frac{f(x_0 + t_n s_n) - f(x_0)}{t_n} = \frac{f(x_n) - f(x_0)}{t_n}.$$

Since $f(x_n) \to \bar{f}$, the right-hand side of the inequality tends to 0 while the left-hand side tends to $\frac{f(x_0 + \lambda s) - f(x_0)}{\lambda}$, when $n \to +\infty$. Consequently,

$$\frac{f(x_0 + \lambda s) - f(x_0)}{\lambda} \leq 0;\; \forall \lambda > 0,$$

and hence, $f^\infty(s) \leq 0$. Because f satisfies (25), there exists $\tilde{x} \in \mathrm{dom}\, f$ such that

$$\lim_{\lambda \to \infty} f(\tilde{x} + \lambda s) = -\infty. \tag{26}$$

We shall show that the following straight line

$$L = \{\tilde{x} + \lambda s \mid \lambda \in \mathbb{R}\}$$

does not intersect M. Assume the contrary. Let $\lambda_0 \in \mathbb{R}$ such that $\tilde{x} + \lambda_0 s \in M$. Since $s \in \text{rec}(M)$, $\tilde{x} + \lambda s \in M$ for all $\lambda \geq \lambda_0$. This together with (26) implies that $\bar{f} = -\infty$, contradicting the fact that f is bounded below on M.

Since L is convex and disjoint from M, there exists a vector $x_0^* \in \mathbb{R}^n \setminus \{0\}$ separating L and M; that is to say

$$\sup\{\langle x_0^*, x \rangle \mid x \in M\} \leq \inf\{\langle x_0^*, y \rangle \mid y \in L\}.$$

Thus, $x_0^* \in \text{bar}(M) \setminus \{0\}$. Since M has the strong separation property, it follows from Theorems 7 and 9 that

$$\lim_{\substack{x \in M \\ \|x\| \to \infty}} \langle x_0^*, x \rangle = -\infty.$$

Since $x_0 + \lambda s \in M$, for all $\lambda > 0$, it implies that $\langle x_0^*, s \rangle < 0$. On the other hand, because $\langle x_0^*, \cdot \rangle$ is bounded below on L, we have $\langle x_0^*, s \rangle = 0$. This contradiction completes the proof. □

Example 6 Let consider the problem $\mathcal{P}(M; f)$ with

$$M = \{(x, y) \in \mathbb{R}^2 \mid y \geq x^2\},$$

and

$$f(x, y) = y + x^2, (x, y) \in \mathbb{R}^2.$$

The set M has the strong separation property as shown in Example 4. The function f is bounded below on M by 0. On the other hand,

$$
\begin{aligned}
f^\infty(u, v) &= \lim_{\lambda \to +\infty} \frac{f((0, 0) + \lambda(u, v)) - f(0, 0)}{\lambda} \\
&= \lim_{\lambda \to +\infty} \frac{\lambda v + \lambda^2 u^2}{\lambda} = \begin{cases} +\infty, & u \neq 0, \\ v, & u = 0. \end{cases}
\end{aligned}
$$

Therefore,

$$C(f^\infty; 0) = \{(0, v) \mid v \leq 0\}.$$

For every $(0, 0) \neq (u, v) \in C(f^\infty; 0)$, that is, $u = 0$ and $v < 0$, we have

$$\lim_{\lambda \to +\infty} f((0, 0) + \lambda(0, v)) = -\infty.$$

Thus, f satisfies (25). By virtue of Theorem 13, $\text{Sol}(M; f)$ is nonempty and compact. In fact, by solving directly we can derive the solution set $\text{Sol}(M; f) = \{(0, 0)\}$. It should be noticed that the function f is not coercive since $f^\infty(0, v) < 0$ for all $v < 0$.

Example 7 In Theorem 13, if f is not coercive then the assumption that M has the strong separation property is essential and cannot be dropped. Let consider the optimization problem $\mathcal{P}(M; f)$ with

$$M = \{(x, 0) \mid x \in \mathbb{R}\} \subset \mathbb{R}^2,$$

and

$$f(x, y) = \begin{cases} \exp(-x) - \sqrt{xy} & \text{if } x \geq 0 \text{ and } y \geq 0, \\ +\infty & \text{if } x < 0 \text{ or } y < 0. \end{cases}$$

We can verify that f is proper, convex, and lower semicontinuous on $\mathbb{R}^2$. Besides,

$$f^\infty((u, v)) = \lim_{\lambda \to +\infty} \frac{f(\lambda u, \lambda v) - f(0, 0)}{\lambda} = \begin{cases} -\sqrt{uv} & \text{if } u \geq 0 \text{ and } v \geq 0, \\ +\infty & \text{if } u < 0 \text{ or } v < 0. \end{cases}$$

So, if $(0, 0) \neq (u, v) \in C(f^\infty; 0)$ then $u \geq 0$, $v \geq 0$, $u + v > 0$; hence, by taking $(\tilde{x}, \tilde{y}) = (1, 1)$, we have

$$\lim_{\lambda \to +\infty} f((\tilde{x}, \tilde{y}) + \lambda(u, v)) = \lim_{\lambda \to +\infty} \left[\exp(-1 - \lambda u) - \sqrt{(1 + \lambda u)(1 + \lambda v)} \right] = -\infty.$$

That means (25) holds. Furthermore, f is bounded below (by 0) on M. However, it is easy to see that $\bar{f} = 0$ and $\text{Sol}(M; f) = \emptyset$. This happens because M does not have the strong separation property and f is not coercive.

Sometimes, the constraint set M is defined by a system of convex inequalities as follows

$$M = \{x \in \mathbb{R}^n \mid f_i(x) \leq 0, 1 \leq i \leq m\}, \tag{27}$$

where f_i, $1 \leq i \leq m$, are convex functions on $\mathbb{R}^n$. In order for the set given in (27) to have the strong separation property, each constraint function is required to satisfy the following condition:

$$\forall 0 \neq v \in C(f_i^\infty; 0), \forall x \in \text{dom } f_i, \lim_{\lambda \to +\infty} f_i(x + \lambda v) = -\infty. \tag{28}$$

Theorem 14 *Assume that $f_i : \mathbb{R}^n \to \mathbb{R}$, $1 \leq i \leq m$ are convex functions satisfying (28). Then the set M defined as (27) has the strong separation property.*

Proof Suppose the contrary. Let $D \subset \mathbb{R}^n$ be a closed convex subset, disjoint from M, but cannot be strongly separated from M. It follows from Theorem 4 that there exists $0 \neq v \in \text{rec}(M) \cap \text{rec}(D)$. Take $x_0 \in M$ and $y_0 \in D$. Since $x_0 + \lambda v \in M$ for all $\lambda > 0$, we have

$$f_i^\infty(v) = \lim_{\lambda \to +\infty} \frac{f_i(x_0 + \lambda v) - f_i(x_0)}{\lambda} \leq \lim_{\lambda \to +\infty} \frac{-f_i(x_0)}{\lambda} = 0; 1 \leq i \leq m.$$

Because f_i satisfies (28), we have

$$\lim_{\lambda \to +\infty} f_i(y_0 + \lambda v) = -\infty; 1 \leq i \leq m.$$

Consequently, there exists $\lambda > 0$ such that $f_i(y_0 + \lambda v) \leq 0$, $1 \leq i \leq m$, or $y_0 + \lambda v \in M$. On the other hand, since $v \in \text{rec}(D)$, $y_0 + \lambda v \in D$. Thus, $M \cap D \neq \emptyset$, contradicting the fact that M and D are disjoint. $\qquad\square$

Corollary 5 *Let $f_i : \mathbb{R}^n \to \mathbb{R}$, $1 \leq i \leq m$ be convex functions satisfying (28), $f_0 : \mathbb{R}^n \to \mathbb{R}$ be a proper, convex, and lower semicontinuous function satisfying (25). Then, the solution set of the following optimization problem*

$$\mathcal{P}(f_1, f_2, \ldots, f_m; f_0) : \begin{cases} f_0(x) \to \inf, \\ x \in \mathbb{R}^n, \\ f_i(x) \leq 0, \ 1 \leq i \leq m \end{cases}$$

is nonempty and compact.

Remark 7 From assumptions imposed on convex functions, we observe that (24) $\Rightarrow$ (28) and (28) $\Rightarrow$ (25). However, the converses are not true. For example, the function f given in

Example 7 satisfies (25), while, by taking $(1, 0) \in C(f^\infty; 0)$ and $(x, y) = (0, 0) \in \operatorname{dom} f$ we have:

$$\lim_{\lambda \to +\infty} f((0, 0) + \lambda(1, 0)) = 0 > -\infty.$$

Thus, f does not satisfy (28). Also, it is not hard to verify that the following function

$$f(x) = \begin{cases} -\sqrt{x} & \text{if } x \ge 0, \\ +\infty & \text{if } x < 0 \end{cases}$$

is proper, convex, and lower semicontinuous on $\mathbb{R}$ satisfying (28), but it is not coercive.

6 Conclusion

In this paper, we have studied strong separation of convex sets and characterization of sets having the strong separation property by using results on the barrier cones of convex sets. We provide a full description of the interior of the barrier cone of a convex set, and prove a new strong separation theorem under an assumption on the barrier cones instead of local compactness or weak compactness assumptions on the sets. We also develop some necessary and/or sufficient conditions for a closed convex set to have the strong separation property. The nonemptiness and compactness of the solution set in a convex optimization problem whose constraint set has the strong separation property are also considered in the paper.

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