Journal of Algebra and Its Applications
(2021) 2150228 (15 pages)

DOI: 10.1142/S0219498821502285

On rings with envelopes and covers regarding to $C3$, $D3$ and flat modules

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Received 22 August 2019
Accepted 27 August 2020
Published 19 September 2020

Communicated by A. Leroy

In this paper, by taking the class of all $C3$ (or $D3$) right R-modules for general envelopes and covers, we characterize a semisimple artinian ring (or a right perfect ring) via $D3$-covers (or $D3$-envelopes) and a right V-ring (or a right noetherian V-ring) via $C3$-covers (or $C3$-envelopes). By using isosimple-projective preenvelope, we obtained that if R is a semiperfect right FGF ring (or left coherent ring), then every isosimple right R-module has a projective cover. Moreover, we also characterize semihereditary serial rings (respectively, hereditary artinian serial rings) in terms of epic flat (respectively, projective) envelopes.

Keywords: Flat; cover; envelope; semihereditary serial ring; hereditary serial ring.

Mathematics Subject Classification 2020: 16D50, 16U60, 16W20

1. Introduction

Motivated by injective envelopes and projective covers, many other varied notions of envelopes and covers have been defined and investigated in various setting. Aiming

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at categorical definitions, many authors used envelopes and covers of modules under the general setting by using a class $\mathcal{X}$ of right R-modules (for examples, $\mathcal{X}$ is the class of injective modules, or the class of pure-injective modules, or the class of projective modules, or the class of flat modules...). They obtained many properties of general envelopes and covers related to classes $\mathcal{X}$ and in some special cases of $\mathcal{X}$, some known results were obtained, for example, see [1, 4, 11, 16–18, 23]. In this paper, we take $\mathcal{X}$ the class of all $D3$-modules (or $C3$-modules) and then we can characterize a semisimple artinian ring via $D3$-covers or $D3$-envelopes as follows: A ring R is semisimple artinian if and only if every right R-module has a $D3$-cover ($D3$-envelope). Moreover, we can prove that a ring R is a right V-ring if and only if every finitely cogenerated right R-module has a $C3$-cover ($C3$-envelope).

It is well known that every hereditary artinian serial ring is Morita equivalent to a finite direct sum of upper triangular matrix rings over division rings [9]. Moreover, a ring R is hereditary artinian serial if and only if every nonsingular module is a direct sum of a projective module and a singular module [10] if and only if every nonsingular projective module is extending [14]. Motivated by these results of Goldie and Oshiro, we characterize semihereditary serial rings (respectively, hereditary artinian serial rings) in terms of epic flat (respectively, projective) envelopes.

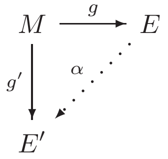
Throughout this paper, all rings are associative rings with unit and all modules are right unital modules. We use $N \leq M$ ($N < M$) means that N is a submodule of M (respectively, proper submodule), and we write $N \leq^e M$ to indicate that N is an essential submodule of M. Let M be an arbitrary module and $\mathrm{ann}(m), m \in M$, the annihilator of $\{m\}$ in R. Recall that $Z(M) = \{m \in M \mid \mathrm{ann}(m) \leq^e R_R\}$ is called the *singular submodule* of M, and if $Z(M) = M$ ($Z(M) = 0$, respectively), then M is called *singular* (*nonsingular, respectively*). A submodule N of M is called *small* in M, whenever for every submodule L of M, $N + L = M$ implies $L = M$. A module M is called a *noncosmall* module, if M is a homomorphic image of a projective module P whose kernel is not essential in P. It is equivalent to $M \neq Z(M)$ (see, for example [14, 22]). M is called a *uniserial* module, if the set of submodules of M is linear ordered by inclusion. M is called 2-*generated* if it is a homomorphic image of R^2. We denote by $J(R)$ the Jacobson radical of R. A ring R is said to be I-*finite* if R contains no infinite orthogonal sets of idempotents. An ideal I of R is called *right* T-*nilpotent* in case for each $a_1, a_2, \dots$ in I, $a_n, \dots, a_1 = 0$, for some n. A ring R is called *semiperfect* in case $R/J(R)$ is semisimple and idempotents lift modulo $J(R)$. It is equivalent to every finitely generated right (left) R-module has a projective cover. A ring R is called a right *perfect* ring in case $R/J(R)$ is semisimple and $J(R)$ is right T-nilpotent. It is equivalent to every right R-module has a projective cover. A ring R is called *right FGF* if every finitely generated right R-module can be embedded into a free right R-module. A ring R is called right *semihereditary* if every finitely generated right ideal of R is projective, a right *hereditary* ring if every right ideal of R is projective, a right *nonsingular* ring if R_R is nonsingular, a right *serial* if R_R is a direct sum of uniserial modules, a right *coherent* if every finitely generated right ideal is finitely presented. Left-sided for these notations are defined similarly.

All terms such as "artinian", "semihereditary", "hereditary", ... when applied to a ring will apply all both sided. For any terms not defined here the reader is referred to [3, 6, 13, 25, 26].

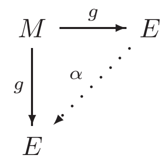
2. Cover and Envelope of Modules

Following ideas of Enochs and Jenda in [7], we recall the notions of envelopes and covers in general setting, and then in special cases: for examples, the classes of flat, projective, $D3$, injective, $C3, \ldots$, modules, we will have the respective envelopes and covers.

Let $\mathcal{X}$ be a class of right R-modules closed under isomorphisms. An R-homomorphism $g : M \to E$ is an $\mathcal{X}$*-preenvelope* of a module M provided that $E \in \mathcal{X}$ and each diagram with $E' \in \mathcal{X}$ can be completed by a homomorphism $\alpha : E \to E'$ to a commutative diagram:

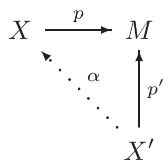


If, moreover, the diagram

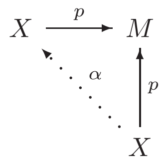


can be only completed by an automorphism α, we call g an $\mathcal{X}$*-envelope* of M. It is easy to see that the $\mathcal{X}$-envelope of a module M is unique up to isomorphisms. It is well known that in the case when $\mathcal{X}$ is the class of all injective modules, then an $\mathcal{X}$-(pre)envelope of each module exists. In this case, an $\mathcal{X}$-envelope of a module coincides with its (injective) envelope, see [26, Theorem 1.2.11].

An R-homomorphism $p : X \to M$ is an $\mathcal{X}$*-precover* of a module M provided that $X \in \mathcal{X}$ and each diagram



with $X' \in \mathcal{X}$ can be completed by a homomorphism $\alpha : X' \to X$ to a commutative diagram. If, moreover, the diagram



can be only completed by an automorphism α, we call p an $\mathcal{X}$*-cover* of M.

It is also easy to see that the $\mathcal{X}$-cover of a module M is unique up to isomorphisms. If $\mathcal{X}$ is the class of all flat modules, then an $\mathcal{X}$-cover of each module exists. It is well known that in the case when $\mathcal{X}$ is the class of all injective modules, then an $\mathcal{X}$-(pre)cover of each module exists, if and only if R is a right noetherian ring (see, e.g. [7, Theorem 5.4.1]). If $\mathcal{X}$ is the class of all projective right R-modules, then an $\mathcal{X}$-cover of a module coincides with its projective cover (see, e.g. [26, Theorem 1.2.12]).

Let us consider the following conditions:

($C1$-condition) Every submodule of M is essential in a direct summand of M.

($C2$-condition) If a submodule A of M is isomorphic to a direct summand of M, then A is a direct summand of M.

($C3$-condition) If M_1 and M_2 are direct summands of M and $M_1 \cap M_2 = 0$ then $M_1 \oplus M_2$ is a direct summand of M.

A module M is called *continuous* if it satisfies ($C1$) and ($C2$); and M is called *quasi-continuous* if it enjoys ($C1$) and ($C3$). A module M satisfying ($C1$) is usually called an extending (or CS) module.

Dually, we have the following conditions:

($D1$-condition) For any submodule A of M, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$ and $A \cap M_2$ is small in M_2.

($D2$-condition) If for every submodule N of M and M/N is isomorphic to a direct summand of M, then N is a direct summand of M.

($D3$-condition) If for two summands A, B of M, $M = A + B$ holds, then $A \cap B$ is a direct summand of M.

A module M is called *discrete* if it satisfies ($D1$) and ($D2$); and M is called *quasi-discrete* if it enjoys ($D1$) and ($D3$).

Now, we take $\mathcal{X}$ the class of all $C3$ right R-modules, then $\mathcal{X}$-envelopes and $\mathcal{X}$-covers are called $C3$-*envelopes* and $C3$-*covers*, respectively. If $\mathcal{X}$ is the class of all $D3$ right R-modules, then $\mathcal{X}$-envelopes and $\mathcal{X}$-covers are called $D3$-*envelopes* and $D3$-*covers*, respectively. In [27], Yousif *et al.* introduced the notion of $D3$-covers, where an R-homomorphism $\phi : P \to M$ is called a $D3$-cover of the right R-module M if P is a $D3$-module, ϕ is an epimorphism, and Kerϕ is small in P. Of course, a projective cover is a $D3$-cover, so this notion is a generalization of the projective cover and the authors used it to characterize a (semi)perfect ring that extend Bass' results. But our notion is categorical. If we take $\mathcal{X}$ the class of all projective right R-modules, then an $\mathcal{X}$-cover coincides with a projective cover and then it follows that it is a $D3$-cover in sense of Yousif, Amin and Ibrahim. But if we take $\mathcal{X}$ class of all $D3$ right R-modules then the class of $D3$-covers in our sense is different from the class of projective covers. So the class of $D3$-covers in our sense is different from the class of $D3$-covers in sense of Yousif, Amin and Ibrahim.

It is well known that a ring R is semisimple artinian if and only if every right (left) R-module is injective if and only if every right (left) R-module is projective. Of course, if every right R-module is injective then every right R-module has a

$D3$-envelope ($D3$-cover). We raise naturally a question: "Is the converse true?" and we obtain the following theorem.

Theorem 2.1. *The following statements are equivalent for a ring R:*

(1) *R is a semisimple artinian ring.*
(2) *Every right R-module has a $D3$-cover.*
(3) *Every right R-module has a $D3$-envelope.*

(2) and (3) are also true for left R-modules.

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are obvious.

(2) $\Rightarrow$ (1) Let S be a simple right R-module. Call $\varphi : R_R \to S$ an epimorphism. By (2), $M = R_R \oplus S$ has a $D3$-cover, say $\alpha : C \to M$ where C is a $D3$-module. Let $\iota_1 : S \to M$ and $\iota_2 : R_R \to M$ be the inclusion maps for all $i = 1, 2$. Note that S and R_R are $D3$-modules, and there are homomorphisms $\beta_1 : S \to C; \beta_2 : R_R \to C$ such that $\alpha\beta_i = \iota_i$. Clearly, $\mathrm{id}_M = \iota_1 \oplus \iota_2 = \alpha(\beta_1 \oplus \beta_2)$. This shows that M is isomorphic to a direct summand of C, which implies that M is a $D3$-module. We deduce that $\mathrm{Ker}(\varphi)$ is a direct summand of R_R by [27, Proposition 4]. It follows that S is a projective module.

(3) $\Rightarrow$ (1) Let S be a simple right R-module. Call $\varphi : R_R \to S$ an epimorphism. By (3), $M = R_R \oplus S$ has a $D3$-envelope, named $\iota : M \to E$ where E is a $D3$-module. Since S and R are $D3$ right R-modules, there exist $f_1 : E \to S$; $f_2 : E \to R$ such that $f_i\iota = \pi_i$, where $\pi_1 : M \to S$ and $\pi_2 : M \to R$ are the projections. There exists $\phi : E \to M$ such that $\pi_i\phi = f_i$ for all $i = 1, 2$. It follows that $\phi\iota = \mathrm{id}_M$, and hence ι is a split monomorphism. Thus, M is isomorphic to a direct summand of E. This gives that $S \oplus R$ is also a $D3$-module. We deduce that $\mathrm{Ker}(\varphi)$ is a direct summand of R_R, and so S is a projective module. □

From this result, we can obtain the following characterizations of a right perfect ring via $D3$-covers ($D3$-envelopes).

Corollary 2.2. *The following statements are equivalent for a ring R:*

(1) *R is a right perfect ring.*
(2) *Every flat right R-module has a $D3$-cover.*
(3) *Every flat right R-module has a $D3$-envelope.*

Recall that R is a *right V-ring* if every simple right R-module is injective. Now, we can obtain the similar result for a right V-ring, but using $C3$-covers and $C3$-envelopes, as follows.

Theorem 2.3. *The following statements are equivalent for a ring R:*

(1) *R is a right V-ring.*
(2) *Every right finitely cogenerated R-module has a $C3$-cover.*
(3) *Every right finitely cogenerated R-module has a $C3$-envelope.*

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are obvious.

(2) $\Rightarrow$ (1) Assume that N is a simple right R-module and $f : C \to M$ is an $C3$-cover of the module $M = N \oplus E(N)$, where $E(N)$ is the injective envelope of N. Denote by π_1 and π_2 the projections of M on the first and second component, respectively. Since N is a $C3$-module, the identity map $1_N : N \to N$ factors through f, for some homomorphism $g : N \to C$, i.e. $\pi_1 f g = \mathrm{id}_N$. Then $C = X \oplus Y$, where $X = \mathrm{Im}(g) \cong N$ and $Y = \mathrm{Ker}(\pi_1 f)$. Similarly, $fh = i$ for some homomorphism $h : E(N) \to C$, where $i : E(N) \to M$. Clearly, $\mathrm{Im}(h) \leq Y$ and h is a monomorphism, so $\mathrm{Im}(h) \cong E(N)$. Now

$$C = X \oplus \mathrm{Im}(h) \oplus E,$$

for some submodule E of Y and $M \cong X \oplus \mathrm{Im}(h)$ is $C3$. We obtain that N is injective (see [2]). We deduce that R is a right V-ring.

(3) $\Rightarrow$ (1) Assume that N is a simple right R-module. Set $H := N \oplus E(N)$. Consider the inclusion map $\iota : H \to M$, where M is $C3$. Since N and $E(N)$ are $C3$, there exist homomorphisms $f_1 : M \to N$ and $f_2 : M \to E(N)$ such that $f_i \iota = \pi_i$, where $\pi_1 : H \to N$ and $\pi_2 : H \to N$ are the canonical projections. There exists a homomorphism $f : M \to H$ such that $\pi_i f = f_i$, for all $i = 1, 2$. It follows that $f\iota = \mathrm{id}_H$. Therefore, ι is a split monomorphism. It means that $N \oplus E(N)$ is a direct summand of M, and so $N \oplus E(N)$ is $C3$, and so N is isomorphic to a direct summand of $E(N)$. □

From this result, we can obtain the following characterizations of a right noetherian V-ring in terms of $C3$-covers ($C3$-envelopes).

Corollary 2.4. *The following statements are equivalent for a ring R:*

(1) *R is a right noetherian right V-ring.*
(2) *Every right R-module with essential socle has a $C3$-cover.*
(3) *Every right R-module with essential socle has a $C3$-envelope.*
(4) *Every right semisimple R-module has a $C3$-cover.*
(5) *Every right semisimple R-module has a $C3$-envelope.*

A nonzero right R-module M is called *isosimple* if any nonzero its submodule is isomorphic to M [8]. We call that a right R-module M is *isosimple-projective* if, for every epimorphism $f : N \to M$ and any homomorphism $g : C \to M$ with C an isosimple right R-module, there exists a homomorphism $h : C \to N$ such that $g = f \circ h$.

Now, we give some elementary properties of isosimple-projective modules. For an R-module M, the dual module $\mathrm{Hom}_R(M, R)$ is denoted by M^*, and $\delta_M : M \to M^{**}$ stands for the evaluation map. For two right R-modules A and B, there is a natural homomorphism

$$\sigma_{AB} : A \otimes_R B^* \to \mathrm{Hom}_R(B, A)$$

defined via $\sigma_{AB}(a \otimes f)b = af(b)$ for all $a \in A$, $f \in B^*$, $b \in B$.

Lemma 2.5. *The following are equivalent for a right R-module A:*

(1) *A is isosimple-projective.*
(2) *For any isosimple right R-module N and any homomorphism $f : N \to A$, f factors through a finitely generated free right R-module F, that is, there exist homomorphisms $g : N \to F$ and $h : F \to A$ such that $f = h \circ g$.*
(3) *For any isosimple right R-module B, σ_{AB} is an epimorphism.*

Proof. (1) $\Rightarrow$ (2) Let F_1 be a free module and $\pi : F_1 \to A$ be an epimorphism. Let N be an arbitrary isosimple module. By (1), A is isosimple-projective, then there exists a homomorphism $g : N \to F_1$ such that $f = \pi \circ g$. Note that N is cyclic, and so is $\mathrm{Im}(g)$. It follows that there is a finitely generated free module F such that $\mathrm{Im}(g) \subseteq F \subseteq F_1$. Take $\iota : F \to F_1$ the inclusion map and $h = \pi \circ \iota$. Then h is a homomorphism from F to A and $f = h \circ g$.

(2) $\Rightarrow$ (1) Let $\alpha : P \to A$ be an epimorphism and $f : N \to A$ be a homomorphism with N an isosimple right R-module. By (2), there exist a finitely generated free module F, homomorphisms $g : N \to F$ and $h : F \to A$ such that $f = h \circ g$. Since F is projective, there exists a homomorphism $\beta : F \to P$ with $\alpha \circ \beta = h$. Thus, $\alpha \circ (\beta \circ g) = h \circ g = f$.

(2) $\Rightarrow$ (3) Let B be an isosimple right R-module and $f \in \mathrm{Hom}_R(B, A)$. From (2), there exist an integer n and homomorphisms $g : B \to R^n$ and $h : R^n \to A$ such that $f = h \circ g$. Let $\pi_i : R^n \to R$ be the ith projection and $\iota_j : R \to R^n$ the jth injection, $i, j = 1, 2, \ldots, n$. Put $a_j = h \circ \iota_j$ and $g_j = \pi_j \circ g$. One can check that $f = \sigma_{AB}(\sum_{j=1}^n a_j \otimes g_j)$. We deduce that σ_{AB} is an epimorphism.

(3) $\Rightarrow$ (2) Let B be an isosimple right R-module and $f \in \mathrm{Hom}_R(B, A)$. By (3), there are $a_j \in A$ and $g_j \in B^*$, $j = 1, 2, \ldots, n$, such that $f = \sigma_{AB}(\sum_{j=1}^n a_j \otimes g_j)$. Define $g : B \to R^n$ via $g(b) = (g_1(b), g_2(b), \ldots, g_n(b))$ for all $b \in B$ and $h : R^n \to A$ via $h(r_1, r_2, \ldots, r_n) = \sum_{j=1}^n a_j r_j$ for all $r_j \in R$. It follows that $f = h \circ g$. Thus, A is isosimple-projective. □

Proposition 2.6. *Every pure submodule of an isosimple-projective module is isosimple-projective.*

Proof. Assume that M is an isosimple-projective module and N is a pure submodule of M. Take C an isosimple right R-module and f a homomorphism from C to N. Let $\iota : N \to M$ be the inclusion map. Since M is isosimple-projective, there exist homomorphisms $\alpha : C \to F$ and $\beta : F \to M$ such that $\iota \circ f = \beta \circ \alpha$ with F a finitely generated free right R-module. Then, we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} & & C & \xrightarrow{\alpha} & F & \xrightarrow{\pi_1} & F/\mathrm{Im}(\alpha) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \psi} & & \\ 0 & \longrightarrow & N & \xrightarrow{\iota} & M & \xrightarrow{\pi_2} & M/N & & \end{array}$$

where π_1, π_2 are the canonical projections and $\psi(x+\mathrm{Im}(\alpha)) = (\pi_2 \circ \beta)(x)$. We have that N is pure in M and $F/\mathrm{Im}(\alpha)$ is finitely presented, there exists a homomorphism $\gamma : F/\mathrm{Im}(\alpha) \to M$ such that $\psi = \pi_2 \circ \gamma$. By [25, Diagram Lemma], there exists a homomorphism $\omega : F \to N$ such that $\omega \circ \alpha = f$. We deduce that N is an isosimple-projective module. □

Corollary 2.7. *The class of isosimple-projective modules is closed under direct summands.*

Theorem 2.8. *The following are equivalent for a ring R:*

(1) *Every isosimple right R-module has a free preenvelope.*
(2) *Every isosimple right R-module has a projective preenvelope.*
(3) *Every isosimple right R-module has an isosimple-projective preenvelope.*
(4) *Every direct product of isosimple-projective right R-modules is isosimple-projective.*
(5) *Every right R-module has an isosimple-projective preenvelope.*
(6) *R_R^I is isosimple-projective for any index set I.*
(7) *The dual module of any isosimple right R-module is finitely generated.*

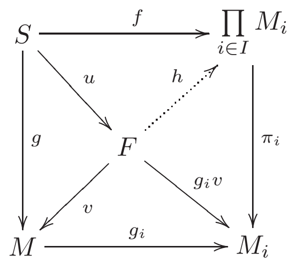
Proof. (1) $\Leftrightarrow$ (2) Let S be an isosimple module and $f : S \to P$ be a projective preenvelope. Since P is a direct summand of a free module F, there exist homomorphisms $\iota : P \to F$ and $\pi : F \to P$ such that $\pi\iota = 1_P$. We show that $\iota \circ f$ is a free preenvelope. Let $g : S \to F'$ be a homomorphism, where F' is a free module. Since $f : S \to P$ is a projective preenvelope, there exists a homomorphism $\alpha : P \to F'$ such that $\alpha \circ f = g$. It follows that

$$(\alpha \circ \pi) \circ (\iota \circ f) = \alpha \circ (\pi \circ \iota) \circ f = \alpha \circ f = g.$$

(2) $\Rightarrow$ (3) It is obvious.

(3) $\Rightarrow$ (4) Let $\{M_i\}_{i \in I}$ be a family of isosimple-projective right R-modules and $f : S \to \prod_{i \in I} M_i$ be a homomorphism, where S is an isosimple module. Let $\pi_i : \prod_{i \in I} M_i \to M_i$ be the projection for any $i \in I$. Call $g : S \to M$ an isosimple-projective preenvelope. By the definition of preenvelopes, for every $i \in I$, there exists homomorphisms $g_i : M \to M_i$ such that $g_i \circ g = \pi_i \circ f$. Inasmuch as M is isosimple-projective, there exist homomorphisms $u : S \to F$ and $v : F \to M$, where

F is a free module such that $g = v \circ u$.



By the universal property of direct product, there exists a (unique) homomorphism $h : F \to \prod_{i \in I} M_i$ such that $\pi_i \circ h = g_i \circ v$. Then, we have

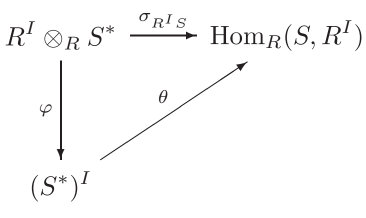
$$\pi_i \circ h \circ u = g_i \circ v \circ u = g_i \circ g = \pi_i \circ f.$$

This implies that $h \circ u = f$. We deduce that $\prod_{i \in I} M_i$ is isosimple-projective.

(4) $\Rightarrow$ (5) Let N be a right R-module. By [7, Lemma 5.3.12], there is a cardinal number $\aleph_\alpha$ such that for any homomorphism $g : N \to L$ with L isosimple-projective, there is a pure submodule Q of L such that $\mathrm{Card}(Q) \leq \aleph_\alpha$ and $g(N) \leq Q$. By Proposition 2.6, Q is isosimple-projective. Thus, by [7, Proposition 6.2.1], N has an isosimple-projective preenvelope.

(5) $\Rightarrow$ (6) By Corollary 2.7 and [5, Lemma 1].

(6) $\Rightarrow$ (7) Let S be an isosimple module and I be a set of indices. We have the following commutative diagram:



where θ and φ are the canonical homomorphisms. Since R^I is an isosimple-projective module, then $\sigma_{R^I S}$ is an epimorphism by Lemma 2.5, and so $\varphi = \theta^{-1} \circ \sigma_{R^I S}$ is also an epimorphism. The result follows from [21, Lemma 13.1].

(7) $\Rightarrow$ (2) Let S be an isosimple module and $f_1, f_2, \ldots, f_n$ be a set of generators of S^*. We consider the map $f : S \to R^n$ via $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$. We show that f is a projective preenvelope of S. Let $g : S \to P$ be a homomorphism with P a projective. As P is isosimple-projective, there exist homomorphisms $u : S \to R^m$ and $v : R^m \to P$ such that $g = v \circ u$. For every $i = 1, 2, \ldots, m$, we consider the projections $\pi_i : R^m \to R$. It follows that $\pi_i \circ u = r_{i1} f_1 + r_{i2} f_2 + \cdots + r_{in} f_n$.

Define a homomorphism $\varphi : R^n \to R^m$ by

$$\varphi = \begin{pmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ r_{21} & r_{22} & \dots & r_{2n} \\ \dots & \dots & \dots & \dots \\ r_{m1} & r_{m2} & \dots & r_{mn} \end{pmatrix}.$$

One can check that $\pi_i \circ \varphi \circ f = \pi_i \circ u$ for each $i \in I$. It follows that $\varphi \circ f = u$, and so $v \circ \varphi \circ f = v \circ u = g$. We deduce that f is a projective preenvelope of S. □

Example 2.9. If R is a right FGF (or left coherent) ring, then every right R-module has an isosimple-projective preenvelope, since the dual module of any isosimple right R-module is finitely generated, i.e. R satisfies the equivalences of Theorem 2.8.

Theorem 2.10. *If R is a semiperfect ring and satisfies Theorem* 2.8, *then every isosimple right R-module has a projective envelope.*

Proof. Let S be an isosimple right R-module. By Theorem 2.8, S^* is a finitely generated left R-module. Since R is a semiperfect ring, S^* has a projective cover $\pi : P \to S^*$. One can check that P is a finitely generated projective left R-module. Let $f = \pi^* \circ \delta_S : S \to P^*$. We show that f is a projective cover. In fact, let $\alpha : S \to F$ be a homomorphism with S projective. We have that F is projective and obtain that there exist a finitely generated free right R-module Q and homomorphisms $\beta : S \to Q$, $\gamma : Q \to F$ such that $\alpha = \gamma \circ \beta$. Note that Q^* is a finitely generated free left R-module, and so δ_Q is an isomorphism. There is a homomorphism $\theta : Q^* \to P$ such that $\pi \circ \theta = \beta^*$. Call $\omega = \gamma \circ \delta_Q^{-1} \circ \theta^* : P^* \to F$. It follows that

$$\begin{aligned} \omega \circ f &= \gamma \circ \delta_Q^{-1} \circ \theta^* \circ \pi^* \circ \delta_S = \gamma \circ \delta_Q^{-1} \circ (\pi \circ \theta)^* \circ \delta_S \\ &= \gamma \circ \delta_Q^{-1} \circ \beta^{**} \circ \delta_S = \gamma \circ \beta = \alpha. \end{aligned}$$

It means that f is a projective preenvelope of S.

Assume that v is an endomorphism of P^* such that $v \circ f = f$. Note that $\delta_{S^*} \circ \pi = \pi^{**} \circ \delta_P$ and $\delta_S^* \delta_{S^*} = 1_{S^*}$ by [3, Proposition 20.14], and so $\pi = \delta_S^* \circ \pi^{**} \circ \delta_P$. It follows that

$$\begin{aligned} \pi(\delta_P^{-1} v^* \delta_P) &= \delta_S^* \pi^{**} \delta_P (\delta_P^{-1} v^* \delta_P) = (v \pi^* \delta_S)^* \delta_P \\ &= (vf)^* \delta_P = f^* \delta_P = \delta_S^* \pi^{**} \delta_P = \pi. \end{aligned}$$

We deduce that $\delta_P^{-1} v^* \delta_P$ is an isomorphism, and so v is also an isomorphism. □

Corollary 2.11. *If R is a semiperfect right FGF ring* (*or left coherent ring*), *then every isosimple right R-module has a projective envelope.*

Corollary 2.12. *If R is a right perfect ring and satisfies Theorem* 2.8, *then every isosimple right R-module has a flat envelope.*

3. On Finitely Generated Flat Right *R*-Modules

A ring R is called *a right S-ring* if every finitely generated flat right R-module is projective [15].

Before proving the main results of this section, we need some lemmas.

Lemma 3.1 ([15, Example 3.7]). *Every semiperfect ring is a right and left S-ring.*

Lemma 3.2 ([15, Corollary 4.6]). *If R is a right S-ring, then for every n, the ring $M_n(R)$ is I-finite, where $M_n(R)$ is the ring of all $n \times n$ matrices with entries in R.*

Lemma 3.3. *The following conditions are equivalent for a ring R:*

(1) *R is a hereditary artinian serial ring.*
(2) *Every nonsingular right R-module is projective.*
(3) *R is right nonsingular and every projective right R-module is extending.*
(4) *R is Morita equivalent to a finite direct sum of full lower triangular matrix rings over division rings.*

Proof. By [9, Theorem 8.11; 14, Theorem 4.6; 10, Theorem 5.28]. □

Lemma 3.4 ([19, Corllary 4.7]). *The following conditions are equivalent:*

(1) *Every right R-module has an epic projective envelope.*
(2) *R is semiprimary, right hereditary and left coherent.*

Lemma 3.5 ([12, Theorem 3.7]). *R is a left semihereditary ring if and only if every right R-module has an epic flat envelope.*

Lemma 3.6. *Let R be a right S-ring (left S-ring) such that every right R-module has an epic flat envelope. Then R is semihereditary.*

Proof. Let R be a right S-ring such that every right R-module has an epic flat envelope. By Lemma 3.5, R is left semihereditary. Because R is a right S-ring so that the ring $M_n(R)$ is I-finite for all positive integers n. Therefore, R is right semihereditary by [20, Theorem 3].

In case R is a left S-ring, it is a similar proof. □

Corollary 3.7. *If R is a semiperfect ring such that every right R-module has an epic flat envelope, then R is a semihereditary ring.*

In this section, we characterize semihereditary serial rings (respectively, hereditary artinian serial rings) in terms of epi flat (respectively, projective) envelopes.

Theorem 3.8. *The following statements are equivalent for a ring R:*

(1) *R is a semihereditary serial ring.*

(2) *R is a right S-ring and*

 (a) *every right R-module has an epic flat envelope*;

 (b) *every 2-generated nonsingular right R-module can be embedded in a flat module.*

(3) *R is a right S-ring and*

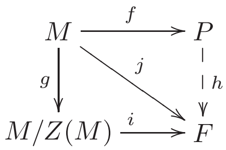
 (a) *every right R-module has an epic flat envelope*;

 (b) *every flat envelope of a 2-generated noncosmall right R-module is nonzero.*

 (2) *and* (3) *are also true for a left S-ring.*

Proof. (1) $\Rightarrow$ (2) Let R be a semihereditary serial ring. Then R is a right S-ring. Clearly, R is left semihereditary so that 2(a) satisfied by Lemma 3.5. Let M be a 2-generated nonsingular right R-module, then M is projective by [24, Theorem 4.6]. Hence, M can be embedded in a flat module. Condition 2(b) holds.

(2) $\Rightarrow$ (3) Suppose that R is a ring satisfying (2), $\mathcal{F}$ is the class of flat right R-modules. From (2) it follows that R is semihereditary by Lemma 3.6. Let M be a 2-generated noncosmall right R-module and let $f : M \to P$ be an epic flat of M. It is enough to show that $P \neq 0$. Since $M \neq Z(M)$ and R is right semihereditary it follows that $M/Z(M)$ is a 2-generated nonsingular right R-module by [10, Propositon 1.23(a)]. It follows from 2(b) that there is an embedding $i : M/Z(M) \to F$, where $F \in \mathcal{F}$. Let $g : M \to M/Z(M)$ be the canonical projection. Put $j = i \circ g$. Since $M \to P \to 0$ is an epic flat envelope of M, it implies that there is a homomorphism $h : P \to F$ such that $j = h \circ f$, i.e. we have a commutative diagram:



Because i is an embedding and $M/Z(M) \neq 0$ it implies that $j(M) = ig(M) = i(M/Z(M)) \neq 0$. Therefore, $hf(M) = j(M) \neq 0$, so that $f(M) = P \neq 0$.

(3) $\Rightarrow$ (1) Let R be a ring satisfying (3). Then R is a semihereditary right S-ring by Lemma 3.6. Suppose that M is a 2-generated noncosmall module right R-module. M has an epic flat envelope $f : M \to P$ with $P \neq 0$ by 3(b). Since R is a right S-ring, every finitely generated flat right R-module is projective. So that P is projective. Since f is epic, it follows that $M = P \oplus \mathrm{Ker}(f)$, i.e. M contains a nonzero projective direct summand. Next, we shall prove that $F = R_R^{(2)} = R_R \oplus R_R$ is an extending module. We have that R is a right S-ring and obtain that R is I-finite. Suppose that R has a decomposition

$$R_R = e_1 R \oplus e_2 R \oplus \cdots \oplus e_n R,$$

where $\{e_i R\}_{i=1}^n$ is a set of orthogonal primitive idempotents of R.

Suppose $W \neq 0$ is a proper submodule of F.

Put $\mathcal{P} = \{U \mid W \leq U \text{ and } U \leq^{\oplus} R_R\}$.

Consider a chain $U_1 \geq U_2 \geq \cdots \geq U_k \geq U_{k+1} \geq \cdots$ of elements in $\mathcal{P}$. Note that U_i are direct summands of R_R. Since R has DCC on direct summands of R_R, there exists a positive n such that $U_n = U_{n+k}$ for natural numbers k. Using Zorn's Lemma it follows that $\mathcal{P}$ has a minimal element, say P. We shall show that $W \leq^e P$. It is enough to prove in case that $W \neq P$. On the contrary suppose that there is a submodule $V \neq 0$ of P such that $W \cap V = 0$. Since $Z(R_R) = 0$, hence $Z(F) = 0$. Therefore, $Z(V) = 0$. It follows that P/W is a 2-generated noncosmall right R-module, since nonsingular module V can be embedded in P/W. So that there is an epic projective envelope $f : P/W \to B$, with $B \neq 0$ by 3(b). As above argument, B is projective. Therefore, the nonzero module P/W has a direct decomposition $P/W = P_1/W \oplus P_2/W$ with a projective module $P_1/W \cong B \neq 0$. Therefore, $0 \neq P_1/W \cong (P/W)/(P_2/W) \cong P/P_2$. So that we get $W \leq P_2 \leq^{\oplus} P$. Therefore, $P_2 \neq P$ and $P_2 \in \mathcal{P}$. This contradicts the minimality of P in $\mathcal{P}$. Hence, $W \leq^e P$. Thus, F is an extending module. It follows that R is an extending module and I-finite. Hence, R is a semiperfect ring. Since R is a semihereditary ring and $R_R^{(2)} = R_R \oplus R_R$ is an extending module, it implies that R is a semihereditary serial ring by [22, Theorem 2.5].

For a left S-ring, the results are obtained by applying [15, Proposition 4.10]. □

We obtain the following result immediately.

Corollary 3.9. *The following statements are equivalent for a ring R*:

(1) *R is a semihereditary serial ring.*
(2) *R is semiperfect and*
   (a) *every right R-module has an epic flat envelope*;
   (b) *every 2-generated nonsingular right R-module can be embedded in a flat module.*
(3) *R is semiperfect and*
   (a) *every right R-module has an epic flat envelope*;
   (b) *every flat envelope of a 2-generated noncosmall right R-module is nonzero.*

For more further details concerning semihereditary serial rings see, for example [22, 24]. The last result is obtained for characterizing a hereditary artinian serial ring in terms of epic projective envelopes.

Theorem 3.10. *The following statements are equivalent for a ring R*:

(1) *R is a hereditary artinian serial ring.*
(2) *Every right R-module has an epic projective envelope, and every 2-generated nonsingular right R-module can be embedded in a flat module.*
(3) *Every right R-module has an epic projective envelope, and every envelope of a 2-generated noncosmall right R-module is nonzero.*

Proof. (1) $\Rightarrow$ (2) Let R be a hereditary artinian serial ring. Then every right R-module has an epic projective envelope by Lemma 3.4. The last follows using the same argument as in the proof of Theorem 3.8 ((1) $\Rightarrow$ (2)).

(2) $\Rightarrow$ (3) As in the proof (2) $\Rightarrow$ (3) of Theorem 3.8.

(3) $\Rightarrow$ (1) Suppose R is a ring satisfying (3). Using the same as proof of (1) $\Rightarrow$ (2) in Theorem 3.8, it follows that $R_R^{(2)} = R_R \oplus R_R$ is an extending module. From (3) and Lemma 3.4, it implies that R is semiprimary right hereditary left coherent. Therefore, R is a hereditary artinian serial ring by [22, Theorem 3.6]. □

Remarks.

(1) Let $\mathbb{Z}$ be the ring of integers. Then $\mathbb{Z}$ satisfies all conditions 2(a), 2(b) and 3(a), 3(b) but $\mathbb{Z}$ is not a serial ring. This shows that the condition "R is a right S-ring" in Theorem 3.8 is needed.

(2) Let $\mathbb{R}$ and $\mathbb{C}$ be the fields of real and complex numbers, respectively, and set

$$S = \begin{bmatrix} \mathbb{R} & \mathbb{C} \\ 0 & \mathbb{C} \end{bmatrix}.$$

Then S is a two-sided artinian and right hereditary right serial ring. We have $S_S = e_{11}S \oplus e_{22}S$, where $e_{11}S$ has precisely one proper submodule and $e_{22}S$ is simple. Therefore, every right S-module has an epic projective envelope by Lemma 3.4. It is easy to check that S satisfies the property that every noncosmall cyclic module of S can be embedded in a free module (since S_S is extending). However, S is not a serial ring. Hence, in Theorems 3.8 and 3.10 if replace "2-generated" by "cyclic" then the results are no longer true.

Acknowledgments

Le Van Thuyet and Phan Dan were funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.04-2018.02. Le Van Thuyet and Truong Cong Quynh would like to thank Hue University for the received support. The authors would like to thank the referees for the very helpful comments and suggestions.

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