KÄHLER DIFFERENTIAL MODULES AND CONFIGURATIONS OF POINTS IN $\mathbb{P}^2$

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Abstract

Given a finite set of points in the projective plane, we use the module of Kähler differentials to investigate the configurations of these points. More precisely, depending on the values of the Hilbert function of the module of Kähler differential 3-forms we determine whether the set of points lies on a non-singular conic or on two different lines or on a line.

Keywords: Kähler differential module; finite set of points; configuration of points; Hilbert function.

Contents

1 Introduction

The theory of Kähler differentials has been used very extensively in the literature as a tool to study finitely generated algebras. Many structural properties of the algebra can be reflected in terms of the modules of Kähler differentials, for instance, the smoothness criterion, the regular criterion, the ramification criterion of the algebra (see Kunz, 1986; Scheja and Storch, 1988). In de Dominicis and Kreuzer, 1999, the authors introduced methods using this theory into the study of finite sets of points in the projective n-space $\mathbb{P}^n$ over a field K of characteristic zero. Explicitly, given a set of points $\mathbb{X} \subseteq \mathbb{P}^n$ with homogeneous coordinate ring $R = K[X_0,...,X_n]/I_{\mathbb{X}}$, the module of Kähler differential 1-forms of R/K is $\Omega^1_{R/K} = \mathscr{J}/\mathscr{J}^2$, where $\mathscr{J}$ is the kernel of the multiplication map $\mu : R \otimes_K R \to R, f \otimes g \mapsto fg$. One of main results of de Dominicis and Kreuzer, 1999 is the formula

$$\mathrm{HF}_{\Omega^1_{R/K}}(i) = (n+1)\mathrm{HF}_{\mathbb{X}}(i-1) - \sum_{j=1}^{n} \mathrm{HF}_{\mathbb{X}}(i-d_j)$$

when $\mathbb{X}$ is a complete intersection of hypersurfaces of degrees $d_1,...,d_n$. Also, they showed that the module $\Omega^1_{R/K}$ contains more information about $\mathbb{X}$. Later, in Kreuzer et al., 2015, 2019, these differential algebra techniques have been developed for arbitrary 0-dimensional schemes in $\mathbb{P}^n$.

In this paper we are interested in applying the theory of Kähler differentials to investigate the geometrical structure of finite sets of points in the projective spaces. More precisely, we first look at the Hilbert functions of the modules of Kähler differential m-forms for a finite set of points in the projective n-space $\mathbb{P}^n$ over a field K of characteristic zero and use these to determine the smallest projective space which contains the set of points. Then we restrict our attention to finite sets of points in $\mathbb{P}^2$ and show whether a set of points lies on a non-singular conic, on two different lines or on a line by detecting the the Hilbert functions of the modules of Kähler differential 3-forms.

Now let us describe the content of the paper in details. In Section 2, we first introduce the modules of Kähler differential m-forms and their basic properties for a finite set $\mathbb{X}$ of points in $\mathbb{P}^n$. In particular, we define the matrix of coordinates of points of $\mathbb{X}$ and show that the rank of the matrix is exactly the smallest positive integer m such that the module of Kähler differential m-forms is not zero (see Theorem 2.6). Also, we show that the Hilbert function of the module of Kähler differential m-forms for a subset of $\mathbb{X}$ is bounded by that one for $\mathbb{X}$.

Next, in Section 3, we use the modules of Kähler differential m-forms for $\mathbb{X}$ to investigate the configurations of $\mathbb{X}$ in case $\mathbb{X} \subseteq \mathbb{P}^2$. Basically, we apply the Hilbert function of the module of Kähler differential 3-forms to characterize the geometrical properties of $\mathbb{X}$. Several nice properties about the shapes of $\mathbb{X}$ are provided. For example, Proposition 3.2 shows that $\mathbb{X}$ lies on a conic, not a double line if and only if $\mathrm{HF}_{\Omega^3_{R/K}}(3) = 1$ and $\mathrm{HF}_{\Omega^3_{R/K}}(i) \leq 1$ for all $i \geq 0$. Specially, Theorem 3.8 shows us how to determine the set $\mathbb{X}$ to be on a line, on two different lines, or on a non-singular conic by checking the certain values of the Hilbert functions of $\Omega^3_{R/K}$.

All examples in this paper were calculated by using the computer algebraic system ApCo-CoA (see The ApCoCoA Team, 2021). Some of the results in this paper were improved and developed from part of Linh, 2015.

2   Kähler differential modules for sets of points

In the following let K be a field of characteristic zero, let $S = K[X_0, \ldots, X_n]$, and let $\mathbb{X} = \{p_1, \ldots, p_s\}$ be a set of s distinct K-rational points in the projective n-space $\mathbb{P}^n$. One may suppose that no point of $\mathbb{X}$ lies on the hyperplane at infinity $\mathscr{Z}^+(X_0)$. This enables us to write $p_j = (1 : p_{j1} : \cdots : p_{jn})$ with $p_{j1}, \ldots, p_{jn} \in K$ for $j = 1, \ldots, s$. Each point $p_j \in \mathbb{X}$ has its associated homogeneous prime ideal in S given by

$$\mathfrak{p}_j := \langle p_{j1}X_0 - X_1, \ldots, p_{jn}X_0 - X_n \rangle \subseteq S.$$

Then the homogeneous vanishing ideal of $\mathbb{X}$ is $I_{\mathbb{X}} = \bigcap_{j=1}^s \mathfrak{p}_j$ and the homogeneous coordinate of $\mathbb{X}$ is $R = S/I_{\mathbb{X}}$. The ring R is a Cohen-Macaulay ring of dimension 1 and is also a graded K-algebra (see e.g. Kreuzer and Robbiano, 2005, Section 6.3 or Linh, 2015, Section 2.4).

The ring $R^e := R \otimes_K R$ is known as the *enveloping algebra* of the algebra R/K. We have the *canonical multiplication map* $\mu : R^e \to R$ with $\mu(f \otimes g) = fg$ for all $f, g \in R$. It is clearly true that this map is homogeneous of degree zero and its kernel $\mathscr{J} := \mathrm{Ker}(\mu)$ is a homogeneous ideal of R^e. For a homogeneous system of generators of $\mathscr{J}$ see Kunz, 2005, Theorem G.7. These notions lead us to the following definition of Kähler differential modules of the algebra R/K.

Definition 2.1. Let m be a positive integer.

(a) The graded R-module $\Omega^1_{R/K} := \mathscr{J}/\mathscr{J}^2$ is called the *module of Kähler differentials* of R/K. The homogeneous K-linear map $d : R \to \Omega^1_{R/K}$ given by $f \mapsto f \otimes 1 - 1 \otimes f + \mathscr{J}^2$ is called the *universal derivation* of R/K.

(b) The exterior power $\Omega^m_{R/K} := \bigwedge^m_R \Omega^1_{R/K}$ is called the *module of Kähler differential m-forms* of R/K.

By x_i we denote the image of X_i in R for $i = 0, \ldots, n$. Then we have $\deg(dx_i) = \deg(x_i) = 1$ and $\Omega^1_{R/K} = Rdx_0 + \cdots + Rdx_n$. Hence $\Omega^m_{R/K} = 0$ if $m \geq n+2$, and for $m \geq 1$, $\Omega^m_{R/K}$ is a finitely generated graded R-module and its Hilbert function is defined by

$$\mathrm{HF}_{\Omega^m_{R/K}}(i) := \dim_K(\Omega^m_{R/K})_i, \quad \text{for all } i \in \mathbb{Z}.$$

Moreover, from Kunz, 1986, Proposition 4.12 we get the following presentation for $\Omega^m_{R/K}$.

Proposition 2.2. *For $1 \leq m \leq n+1$, the graded R-module $\Omega^m_{R/K}$ has a presentation*

$$\Omega^m_{R/K} \cong \Omega^m_{S/K}/(I_{\mathbb{X}}\Omega^m_{S/K} + dI_{\mathbb{X}}\Omega^{m-1}_{S/K})$$

where $dI_{\mathbb{X}} = \langle dF \mid F \in I_{\mathbb{X}} \rangle$.

In view of Kreuzer et al., 2019, Proposition 4.4, the Hilbert polynomial of the module of Kähler differential m-forms $\Omega^m_{R/K}$ can be described as follows.

Proposition 2.3. *For $1 \leq m \leq n+1$, the Hilbert polynomial of $\Omega^m_{R/K}$ satisfies*

$$\mathrm{HP}_{\Omega^m_{R/K}}(z) = \begin{cases} s & \text{if } m = 1, \\ 0 & \text{if } m \geq 2. \end{cases}$$

In addition, the regularity index of $\Omega^m_{R/K}$ is bounded by $\mathrm{ri}(\Omega^m_{R/K}) \leq 2\,\mathrm{ri}(R) + m$.

The next result found in Scheja and Storch, 1988, Proposition 85.12 is useful for the proof of Theorem 2.6.

Lemma 2.4. *For two R-modules M and N, there is a canonical isomorphism*

$$\textstyle\bigwedge_R^n(M \oplus N) \cong \sum_{i=0}^n \bigwedge_R^{n-i}(M) \otimes_R \bigwedge_R^i(N).$$

Lemma 2.5. *Let $1 \le m \le n+1$ and let $\mathcal{K}$ be the graded R-module*

$$\mathcal{K} = \langle (\tfrac{\partial F}{\partial x_0}, \ldots, \tfrac{\partial F}{\partial x_n}) \in R^{n+1} \mid F \in I_{\mathbb{X}} \rangle.$$

There is an exact sequence of graded R-modules

$$0 \longrightarrow \mathcal{K} \wedge_R \textstyle\bigwedge_R^m(R^{n+1}) \longrightarrow \bigwedge_R^{m+1}(R^{n+1}) \longrightarrow \Omega_{R/K}^{m+1}(m+1) \longrightarrow 0. \tag{1}$$

Proof. By de Dominicis and Kreuzer, 1999, Proposition 1.3, we have an epimorphism of graded R-modules $\varphi : R^{n+1} \to \Omega_{R/K}^1(1)$ with $\varphi((F_0, \ldots, F_n)) = F_0 dx_0 + \cdots + F_n dx_n$ for $F_0, \ldots, F_n \in R$ and the kernel of φ is $\mathcal{K}$. Thus the exact sequence (1) follows from Scheja and Storch, 1988, p. X. 83. $\qquad\square$

From the coordinates of points $p_1, \ldots, p_s$ of $\mathbb{X}$, we form the matrix

$$\mathscr{A}_{\mathbb{X}} = (p_{jk})_{\substack{i=1,\ldots,s \\ j=0,\ldots,n}} \in \mathrm{Mat}_{s \times (n+1)}(K),$$

and we let $\rho_{\mathbb{X}}$ denote the rank of the matrix $\mathscr{A}_{\mathbb{X}}$. The next theorem shows how the rank $\rho_{\mathbb{X}}$ can be detected via the Hilbert functions of the Kähler differential modules for $\mathbb{X}$.

Theorem 2.6. *The set $\mathbb{X}$ has $\rho_{\mathbb{X}} = m$ if and only if $\mathrm{HF}_{\Omega_{R/K}^m}(i) \ne 0$ for some i and $\mathrm{HF}_{\Omega_{R/K}^{m+1}}(i) = 0$ for all $i \in \mathbb{Z}$.*

In order to prove Theorem 2.6, we need only prove the following two lemmas.

Lemma 2.7. *If $m > \rho_{\mathbb{X}}$, then $\mathrm{HF}_{\Omega_{R/K}^m}(i) = 0$ for all $i \in \mathbb{Z}$.*

Proof. Set $M = \mathcal{K} \wedge_R \bigwedge_R^{\rho_{\mathbb{X}}}(R^{n+1})$ and $N = \bigwedge_R^{\rho_{\mathbb{X}}+1}(R^{n+1})$. According to Lemma 2.5, it suffices to show that $\mathrm{HF}_M(i) = \mathrm{HF}_N(i)$ for all $i \ge 0$. Let $\{e_1, \ldots, e_{n+1}\}$ be a standard basis of the graded-free R-module R^{n+1}, and let V be the space of solutions of the homogeneous system of linear equations with coefficient matrix $\mathscr{A}_{\mathbb{X}}$. Then we have $t := \dim_K(V) = n+1 - \rho_{\mathbb{X}}$. Let $\{v_1, \ldots, v_t\}$ be a K-basis of V. W.l.o.g. we may assume that $v_j = e_j + \sum_{k=t+1}^{n+1} a_{jk} e_k$ for $j = 1, \ldots, t$. For each $j \in \{1, \ldots, t\}$ we see that $X_{j-1} + \sum_{k=t+1}^{n+1} a_{jk} X_{k-1} \in I_{\mathbb{X}}$, and hence $v_j \in \mathcal{K}$. This implies $\langle v_1, \ldots, v_t \rangle_R \subseteq \mathcal{K} \subseteq R^{n+1}$.

Now we need only prove that $M' := \langle v_1, \ldots, v_t \rangle_R \wedge_R \bigwedge_R^{\rho_{\mathbb{X}}}(R^{n+1}) = N$. In M', we have

$$v_j \wedge e_{t+1} \wedge \cdots \wedge e_{n+1} = e_j \wedge e_{t+1} \wedge \cdots \wedge e_{n+1}.$$

If we can show that $e_{i_1} \wedge \cdots \wedge e_{i_{\rho_{\mathbb{X}}+1}} \in M'$ for all $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, t\}$, $\{i_{k+1}, \ldots, i_{\rho_{\mathbb{X}}+1}\} \subseteq \{t+1, \ldots, n+1\}$ and $1 \le k \le \rho_{\mathbb{X}}$ implies $e_{j_1} \wedge \cdots \wedge e_{j_{\rho_{\mathbb{X}}+1}} \in M'$ for all $\{j_1, \ldots, j_{k+1}\} \subseteq \{1, \ldots, t\}$ and $\{j_{k+2}, \ldots, j_{\rho_{\mathbb{X}}+1}\} \subseteq \{t+1, \ldots, n+1\}$. Then the set

$$\{e_{i_1} \wedge \cdots \wedge e_{i_{\rho_{\mathbb{X}}+1}} \mid \{i_1, \ldots, i_{\rho_{\mathbb{X}}+1}\} \subseteq \{1, \ldots, n+1\}, \exists i_k \in \{1, \ldots, t\}\}$$

is contained in M', and hence $M' = N$, since this set generates N as an R-module.

To see $e_{j_1} \wedge \cdots \wedge e_{j_{\rho_{\mathbb{X}}+1}} \in M'$, we represent

$$
\begin{aligned}
e_{j_1} \wedge \cdots \wedge e_{j_{\rho_{\mathbb{X}}+1}} &= (e_{j_1} + \sum_{\ell=t+1}^{n+1} a_{j_1\ell}e_\ell) \wedge e_{j_2} \wedge \cdots \wedge e_{j_{\rho_{\mathbb{X}}+1}} \\
&\quad - \sum_{\ell=t+1}^{n+1} (-1)^{\rho_{\mathbb{X}}} a_{j_1\ell}(e_{j_2} \wedge \cdots \wedge e_{j_{\rho_{\mathbb{X}}+1}} \wedge e_\ell) \\
&= v_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{\rho_{\mathbb{X}}+1}} \\
&\quad - \sum_{\ell=t+1}^{n+1} (-1)^{\rho_{\mathbb{X}}} a_{j_1\ell}(e_{j_2} \wedge \cdots \wedge e_{j_{k+1}}) \wedge (e_{j_{k+2}} \wedge \cdots \wedge e_{j_{\rho_{\mathbb{X}}+1}} \wedge e_\ell).
\end{aligned}
$$

It follows from the hypothesis that $e_{j_1} \wedge \cdots \wedge e_{j_{\rho_{\mathbb{X}}+1}} \in M'$, as desired. $\qquad\square$

Lemma 2.8. *We have* $\mathrm{HF}_{\Omega_{R/K}^{\rho_{\mathbb{X}}}}(i) \neq 0$ *for some* $i \in \mathbb{Z}$.

Proof. Using the notation given in the proof of Lemma 2.7, let $U = \langle e_{t+1}, \ldots, e_{n+1}\rangle_R$, $V = \langle v_1, \ldots, v_t\rangle_R$, and $N = \bigwedge_R^{\rho_{\mathbb{X}}}(R^{n+1})$. Then U, V are graded-free R-modules of rank $\rho_{\mathbb{X}}$ and t, respectively. We want to show that $V \wedge_R \bigwedge_R^{\rho_{\mathbb{X}}-1}(R^{n+1}) \subsetneq N$. We have $V \oplus U = R^{n+1}$ and

$$
\begin{aligned}
V \wedge_R \bigwedge_R^{\rho_{\mathbb{X}}-1}(R^{n+1}) &= V \wedge_R \bigwedge_R^{\rho_{\mathbb{X}}-1}(V \oplus U) \\
&= V \wedge_R \left(\sum_{i=0}^{\rho_{\mathbb{X}}-1} \bigwedge_R^{\rho_{\mathbb{X}}-1-i}(V) \otimes_R \bigwedge_R^i(U) \right) \\
&= \sum_{i=0}^{\rho_{\mathbb{X}}-1} \bigwedge_R^{\rho_{\mathbb{X}}-i}(V) \otimes_R \bigwedge_R^i(U),
\end{aligned}
$$

where the second equality follows from by Lemma 2.4. Since $U \cap V = \langle 0\rangle$, the graded-free R-module $V \wedge_R \bigwedge_R^{\rho_{\mathbb{X}}-1}(R^{n+1})$ has rank

$$
\mathrm{rank}(V \wedge_R \bigwedge_R^{\rho_{\mathbb{X}}-1}(R^{n+1})) = \sum_{i=0}^{\rho_{\mathbb{X}}-1} \binom{n+1-\rho_{\mathbb{X}}}{\rho_{\mathbb{X}}-i} \cdot \binom{\rho_{\mathbb{X}}}{i}.
$$

Moreover, N is a graded-free R-module of rank $\mathrm{rank}(N) = \sum_{i=0}^{\rho_{\mathbb{X}}} \binom{n+1-\rho_{\mathbb{X}}}{\rho_{\mathbb{X}}-i} \cdot \binom{\rho_{\mathbb{X}}}{i}$, and consequently $\mathrm{rank}(N) - \mathrm{rank}(V \wedge_R \bigwedge_R^{\rho_{\mathbb{X}}-1}(R^{n+1})) = 1$ or $V \wedge_R \bigwedge_R^{\rho_{\mathbb{X}}-1}(R^{n+1}) \subsetneq N$. An application of the the exact sequence (1) with $V \subseteq \mathcal{K}$ yields $\mathrm{HF}_{\Omega_{R/K}^{\rho_{\mathbb{X}}}}(i) \neq 0$ for some $i \in \mathbb{Z}$. $\qquad\square$

Due to Lemmas 2.7 and 2.8, $\mathrm{HF}_{\Omega_{R/K}^m}(i) \neq 0$ for some i and $\mathrm{HF}_{\Omega_{R/K}^{m+1}}(i) = 0$ for all $i \in \mathbb{Z}$ if and only if $\rho_{\mathbb{X}} = m$, and Proposition 2.6 is completely proved.

Now let us look at an explicit example.

Example 2.9. Let $\mathbb{X} = \{p_1, \ldots, p_6\} \subseteq \mathbb{P}^4$ be the set of six $\mathbb{Q}$-rational points, where $p_1 = (1 : 0 : 0 : 0 : 1)$, $p_2 = (1 : -1 : 1 : -1 : 2)$, $p_3 = (1 : 1 : 1 : 1 : 2)$, $p_4 = (1 : 2 : 4 : 8 : -1)$, $p_5 = (1 : -2 : 4 : -8 : 11)$, and $p_6 = (1 : 3 : 9 : 27 : -14)$ are on the intersection of a twisted cubic curve defined by $X_0X_2 - X_1^2, X_1X_3 - X_2^2, X_0X_3 - X_1X_2$ and an hyperplane defined by $F_1 = X_0 + X_1 + X_2 - X_3 - X_4$.

By forming $\mathscr{A}_{\mathbb{X}}$ and calculating its rank, we have $\rho_{\mathbb{X}} = 4$. The homogeneous vanishing ideal $I_{\mathbb{X}}$ of $\mathbb{X}$ is generated by $\{F_1, F_2, F_3, F_4, F_5\}$ with

$$F_2 = X_1^2 - 2X_2X_3 + X_3^2 - X_2X_4 + X_3X_4,$$
$$F_3 = X_1X_2 + \tfrac{23}{22}X_2X_3 - \tfrac{8}{11}X_3^2 - \tfrac{3}{11}X_1X_4 + \tfrac{4}{11}X_2X_4 - \tfrac{3}{4}X_3X_4,$$
$$F_4 = X_2^2 - \tfrac{1}{22}X_2X_3 - \tfrac{3}{11}X_3^2 + \tfrac{3}{11}X_1X_4 - \tfrac{4}{11}X_2X_4 - \tfrac{1}{4}X_3X_4,$$
$$F_5 = X_1X_3 - \tfrac{1}{22}X_2X_3 - \tfrac{3}{11}X_3^2 + \tfrac{3}{11}X_1X_4 - \tfrac{4}{11}X_2X_4 - \tfrac{1}{4}X_3X_4$$

and $\mathrm{HF}_{\mathbb{X}} = \mathrm{HF}_R : 1\,4\,6\,6\cdots$. In this case we also have $\mathrm{HF}_{\Omega^4_{R/K}}(4) = 1 \neq 0$ and $\mathrm{HF}_{\Omega^5_{R/K}}(i) = 0$ for all $i \in \mathbb{Z}$.

Remark 2.10. When $\mathbb{X} = \{p_1, \ldots, p_s\} \subset \mathbb{P}^n$ is a set of s distinct K-rational points that lie on a line, the Hilbert functions of $\mathbb{X}$ and of $\Omega^m_{R_{\mathbb{X}}/K}$ satisfy

$$\mathrm{HF}_{\mathbb{X}}: \qquad 1\,2\,3\,4\,\cdots\,s-1\,s\,s\,\cdots$$
$$\mathrm{HF}_{\Omega^1_{R_{\mathbb{X}}/K}}: \quad 0\,2\,4\,6\cdots\,(2s-2)\,(2s-1)\,(2s-2)\,(2s-3)\cdots(s+1)\,s\,s\cdots$$
$$\mathrm{HF}_{\Omega^2_{R_{\mathbb{X}}/K}}: \quad 0\,0\,1\,2\,\cdots\,(s-2)\,(s-1)\,(s-2)\,\cdots2\,1\,0\,0\cdots$$

and $\mathrm{HF}_{\Omega^m_{R_{\mathbb{X}}/K}}(i) = 0$ for $m \geq 3$ and for all $i \in \mathbb{Z}$.

We end this section with a relation between the Hilbert functions of the Kähler differential modules for $\mathbb{X}$ and its subsets. This property is also true when $\mathbb{X}$ is a 0-dimensional scheme in $\mathbb{P}^n$.

Lemma 2.11. *Let $\mathbb{Y}$ be a subset of $\mathbb{X}$ and let $R_{\mathbb{Y}}$ be the homogeneous coordinate of $\mathbb{Y}$. Then $1 \leq m \leq n+1$ and for all $i \in \mathbb{Z}$ we have*

$$\mathrm{HF}_{\Omega^m_{R_{\mathbb{Y}}/K}}(i) \leq \mathrm{HF}_{\Omega^m_{R/K}}(i).$$

Proof. We know that $I_{\mathbb{X}} \subseteq I_{\mathbb{Y}}$, and so there is a surjective homomorphism of rings $\pi : R \to R_{\mathbb{Y}} = S/I_{\mathbb{Y}}$. This map induces a surjective homomorphism of graded R-modules $\phi : \Omega^1_{R/K} \to \Omega^1_{R_{\mathbb{Y}}/K}$. For $1 \leq m \leq n+1$, the map ϕ induces a surjective homomorphism of graded R-modules $\phi^{(m)} : \Omega^m_{R/K} \to \Omega^m_{R_{\mathbb{Y}}/K}$. Therefore we get the inequality $\mathrm{HF}_{\Omega^m_{R_{\mathbb{Y}}/K}}(i) \leq \mathrm{HF}_{\Omega^m_{R/K}}(i)$ for all $i \in \mathbb{Z}$. $\qquad\square$

3 Several configurations of points in the projective plane

In this section, we restrict our attention to the sets of K-rational points $\mathbb{X} = \{p_1, \ldots, p_s\}$ in the projective plane $\mathbb{P}^2$. More precisely, we will discuss some geometrical configurations of points $\mathbb{X}$ in $\mathbb{P}^2$ which are reflected in terms of the Hilbert function of the module of Kähler differential 3-forms. Firstly, we look at a representation of this module. We denote

$$\partial I_{\mathbb{X}} := \langle \tfrac{\partial F}{\partial X_i} \mid F \in I_{\mathbb{X}}, 0 \leq i \leq n \rangle \subseteq S.$$

The ideal $\partial I_{\mathbb{X}}$ is also known as the *n-th Jacobian ideal* (or the *n-th Kähler different*) of R/K (see e.g. Kunz, 1986, Section 10). We can use this notation to give an explicit description for the module of Kähler differential 3-forms of R/K as follows.

Lemma 3.1. *There is an isomorphism of graded R-modules*

$$\Omega^3_{R/K} \cong (S/\partial I_{\mathbb{X}})(-n-1).$$

In particular, $\mathrm{HF}_{\Omega^3_{R/K}}(i) = \mathrm{HF}_{S/\partial I_{\mathbb{X}}}(i-n-1)$ *for all* $i \in \mathbb{Z}$.

Proof. See Kreuzer et al., 2019, Corollary 2.3. □

Now we may give a criterion for the set $\mathbb{X}$ to lie on a conic $\mathscr{C} = \mathscr{Z}^+(C)$, where

$$C = \sum_{0 \le i,j \le 2} a_{ij} X_i X_j \qquad (a_{ij} \in K). \tag{2}$$

Proposition 3.2. *A set of s distinct K-rational points* $\mathbb{X} \subseteq \mathbb{P}^2$ *lies on a conic* $\mathscr{C} = \mathscr{Z}^+(C)$ *which is not a double line if and only if we have* $\mathrm{HF}_{\Omega^3_{R/K}}(i) \le 1$ *for all* $i \in \mathbb{Z}$ *and* $\mathrm{HF}_{\Omega^3_{R/K}}(3) = 1$.

Proof. We may assume that $C \ne a\ell^2$ for any linear form $\ell \in S$ and $a \in K$ and that $a_{00} \ne 0$ (after a change of coordinates). We have $\frac{\partial C}{\partial X_i} = \sum_{j=0}^2 a_{ij} X_j \in \partial I_{\mathbb{X}}$ for $i = 0, 1, 2$, as $C \in I_{\mathbb{X}}$. Put $\mathscr{A} := (a_{ij})_{i,j=0,1,2} \in \mathrm{Mat}_3(K)$. Then $\mathrm{rank}(\mathscr{A}) = 1$ if and only if $\mathscr{A}$ is of the form

$$\mathscr{A} = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & \frac{a_{01}^2}{a_{00}} & \frac{a_{01}a_{02}}{a_{00}} \\ a_{02} & \frac{a_{01}a_{02}}{a_{00}} & \frac{a_{02}^2}{a_{00}} \end{pmatrix}.$$

In this case we get $C = \frac{1}{a_{00}}\ell^2$ with $\ell = \frac{\partial C}{\partial X_0} = \sum_{j=0}^2 a_{0j} X_j$, which is impossible. Hence $\mathrm{rank}(\mathscr{A}) \ge 2$ and suppose that two first rows of $\mathscr{A}$ are linearly independent. Then $\frac{\partial C}{\partial X_0}$, $\frac{\partial C}{\partial X_1}$ form a regular sequence for S. Notice that

$$J := \langle \tfrac{\partial C}{\partial X_0}, \tfrac{\partial C}{\partial X_1} \rangle_S \subseteq \langle \tfrac{\partial C}{\partial X_i} \mid 0 \le i \le 2 \rangle \subseteq \partial I_{\mathbb{X}}.$$

Hence, according to Lemma 3.1, for all $i \in \mathbb{Z}$ we have

$$\mathrm{HF}_{\Omega^3_{R/K}}(i) = \mathrm{HF}_{S/\partial I_{\mathbb{X}}}(i-3) \le \mathrm{HF}_{S/J}(i-3) \le 1.$$

On the other hand, we also have $\partial I_{\mathbb{X}} \subseteq \langle X_0, X_1, X_2 \rangle_S$, because $\mathbb{X}$ does not lie on a line. This implies $\mathrm{HF}_{\Omega^3_{R/K}}(3) = 1$.

Conversely, assume that $\mathrm{HF}_{\Omega^3_{R/K}}(i) \le 1$ for all $i \in \mathbb{Z}$ and $\mathrm{HF}_{\Omega^3_{R/K}}(3) = 1$. Theorem 2.6 guarantees that $\mathbb{X}$ lies neither on a line nor on a double line. If $\mathbb{X}$ does not lie on any conic, then the homogeneous vanishing ideal $I_{\mathbb{X}}$ is generated in degrees greater than 2, and hence $\mathrm{HF}_{\partial I_{\mathbb{X}}}(1) = 0$. By Lemma 3.1, we get $\mathrm{HF}_{\Omega^3_{R/K}}(4) = 3 > 1$, contradiction. Therefore $\mathbb{X}$ must lie on a conic. □

Due to the proof of Proposition 3.2, $\mathbb{X}$ lies on a line if and only if $\mathrm{HF}_{\Omega^3_{R/K}}(i) = 0$ for all $i \in \mathbb{N}$.

Lemma 3.3. *If the set* $\mathbb{X} = \{p_1, ..., p_s\} \subseteq \mathbb{P}^2$ *lies on a non-singular conic* $\mathscr{C} = \mathscr{Z}^+(C)$, *then* $\mathrm{HF}_{\Omega^3_{R/K}}(i) = 0$ *for* $i \ne 3$ *and* $\mathrm{HF}_{\Omega^3_{R/K}}(3) = 1$.

Proof. Let C be given in (2) and let $\mathscr{A} = (a_{ij})_{i,j=0,1,2} \in \mathrm{Mat}_3(K)$ be the coefficient matrix of C. It is well-known that $\mathscr{C}$ is a non-singular conic if and only if $\mathrm{rank}(\mathscr{A}) = 3$. The last condition yields $\langle X_0, X_1, X_2 \rangle_S = \langle \frac{\partial C}{\partial X_i} \mid 0 \le i \le 2 \rangle_S \subseteq \partial I_{\mathbb{X}}$. Hence $\langle X_0, X_1, X_2 \rangle_S = \partial I_{\mathbb{X}}$, and the claim follows by Lemma 3.1. $\qquad \square$

Next we consider the case that the set $\mathbb{X}$ lies on two different lines.

Proposition 3.4. *Let $\mathbb{X} \subseteq \mathbb{P}^2$ be a set of s distinct K-rational points which lie on on two different lines.*
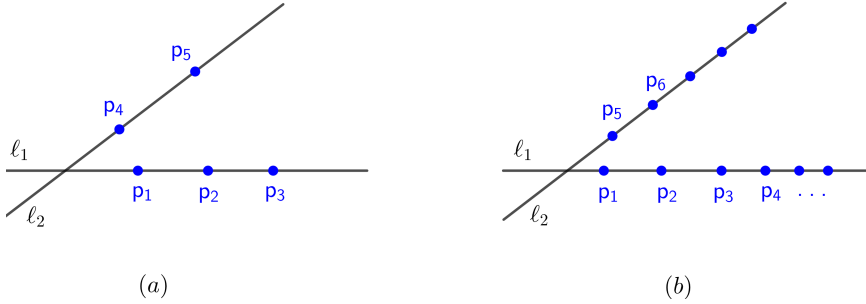
(a) *If $s = 5$ and none of four points of $\mathbb{X}$ lie on a line, then*

$$\mathrm{HF}_{\Omega^3_{R/K}} : 0\ 0\ 0\ 1\ 1\ 0\ \cdots.$$

(b) *If $s \ge 5$ and there exist five points such that no four of them lie on a line, then*

$$\mathrm{HF}_{\Omega^3_{R/K}}(3) = \mathrm{HF}_{\Omega^3_{R/K}}(4) = 1.$$

Proof. Suppose that $\mathbb{X}$ lies on two lines defined by $\ell_1 = \sum_{j=0}^2 a_j X_j$ and $\ell_2 = \sum_{j=0}^2 b_j X_j$ with $a_i, b_j \in K$. Note that $\gcd(\ell_1, \ell_2) = 1$ and w.l.o.g. assume that $a_0 b_1 - a_1 b_0 \ne 0$. We first treat the case of five points.

(a)   (b)

(a)   Assume that $\mathbb{X}_1 = \{p_1, p_2, p_3\} \subseteq \mathscr{L}^+(\ell_1)$ and $\mathbb{X}_2 = \{p_4, p_5\} \subseteq \mathscr{L}^+(\ell_2)$. Then we have $I_{\mathbb{X}_1} = \langle \ell'_1 \ell'_2 \ell'_3, \ell_1 \rangle_S$ and $I_{\mathbb{X}_2} = \langle \ell'_4 \ell'_5, \ell_2 \rangle_S$ for suitable linear forms $\ell'_i = \sum_{j=0}^2 c_{ij} X_j$ ($i = 1, ..., 5, c_{ij} \in K$). Clearly, the initial degree of the ideal $I_{\mathbb{X}}$ is 2 and $I_{\mathbb{X}} = I_{\mathbb{X}_1} \cap I_{\mathbb{X}_2}$. For a non-zero element $F \in (I_{\mathbb{X}})_2$, we may write

$$F = F_1 \ell_1 = F_2 \ell_2 + c \ell'_4 \ell'_5$$

for some $c \in K$ and $F_1, F_2 \in S_1$. Since $\ell_1(p_4) \ne 0$ and $\ell_1(p_5) \ne 0$, we have $F_1(p_4) = F_1(p_5) = 0$, and so $F_1 = c' \ell_2$ with $c' \in K \setminus \{0\}$. Hence $F \in (I_{\mathbb{X}})_2$ if and only if $F = \langle \ell_1 \ell_2 \rangle_K$. Observe that

$$\tfrac{\partial \ell_1 \ell_2}{\partial X_i} = \ell_1 \tfrac{\partial \ell_2}{\partial X_i} + \ell_2 \tfrac{\partial \ell_1}{\partial X_i} = b_i \ell_1 + a_i \ell_2 \in \partial I_{\mathbb{X}} \quad (i = 0, 1, 2)$$

and $\langle b_0 \ell_1 + a_0 \ell_2, b_1 \ell_1 + a_1 \ell_2 \rangle_K = \langle \ell_1, \ell_2 \rangle_K$, as $a_0 b_1 - a_1 b_0 \ne 0$. Since $b_2 \ell_1 + a_2 \ell_2 \in \langle \ell_1, \ell_2 \rangle_K$, we obtain $\langle \ell_1, \ell_2 \rangle_K = (\partial I_{\mathbb{X}})_1$. This implies

$$\mathrm{HF}_{\Omega^3_{R/K}}(4) = \mathrm{HF}_{S/\partial I_{\mathbb{X}}}(1) = 3 - \dim_K(\langle \ell_1, \ell_2 \rangle_K) = 1.$$

Furthermore, we have $\ell_1 \ell'_4 \ell'_5 \in I_{\mathbb{X}}$, and hence, for $a_i \ne 0$, we get

$$\ell'_4 \ell'_5 = \tfrac{1}{a_i}(\tfrac{\partial \ell_1 \ell'_4 \ell'_5}{\partial X_i} - \ell_1(\tfrac{\partial \ell'_4 \ell'_5}{\partial X_i})) \in \partial I_{\mathbb{X}}.$$

In particular, $\langle \ell_1, \ell_2, \ell_4' \ell_5' \rangle_S \subseteq \partial I_{\mathbb{X}}$. If $\ell_4' \ell_5' = H_1 \ell_1 + H_2 \ell_2$ for some $H_1, H_2 \in S_1$, then $H_1(p_4) = H_1(p_5) = 0$ or $H_1 = c'' \ell_2$, and thus $\ell_2 \mid \ell_4' \ell_5'$, this is impossible. Hence $\ell_4' \ell_5' \notin \langle \ell_1, \ell_2 \rangle_S$. It follows that

$$\dim_K(\langle \ell_1, \ell_2, \ell_4' \ell_5' \rangle_S)_2 = \dim_K(\partial I_{\mathbb{X}})_2 = \dim_K(S_2) = 6,$$

and consequently $\langle \ell_1, \ell_2, \ell_4' \ell_5' \rangle_S = \partial I_{\mathbb{X}}$ and $\mathrm{HF}_{\partial I_{\mathbb{X}}}(i) = \mathrm{HF}_S(i)$ for all $i \geq 2$. Therefore $\mathrm{HF}_{\Omega^3_{R/K}}(i) = 0$ for all $i > 4$.

(b)   Because the set $\mathbb{X}$ lies on two different lines, Proposition 3.2 implies that $\mathrm{HF}_{\Omega^3_{R/K}}(3) = 1$ and $\mathrm{HF}_{\Omega^3_{R/K}}(4) \leq 1$. Let $\mathbb{Y}$ be the set of five points of $\mathbb{X}$ such that no four points of $\mathbb{Y}$ lie on a line. By (a) and Lemma 2.11, we have $1 = \mathrm{HF}_{\Omega^3_{R_{\mathbb{Y}}/K}}(4) \leq \mathrm{HF}_{\Omega^3_{R/K}}(4) \leq 1$, and hence the last inequality becomes an equality. $\square$

When there is a line passing through all but one point of $\mathbb{X}$, we have the following property.

Lemma 3.5. *Let* $\mathbb{X} = \mathbb{Y} \cup \{p_s\} \subseteq \mathbb{P}^2$, *where* $\mathbb{Y} = \{p_1, \ldots, p_{s-1}\}$ *is a set of* $s - 1$ *distinct K-rational points on a line* $\mathscr{Z}^+(\ell)$ *and* $p_s \notin \mathscr{Z}^+(\ell)$. *Then*

$$\mathrm{HF}_{\Omega^3_{R/K}}(3) = 1 \quad and \quad \mathrm{HF}_{\Omega^3_{R/K}}(i) = 0, \quad \forall i \neq 3.$$

Proof. Since $I_{\mathbb{Y}} = \bigcap_{j=1}^{s-1} \mathfrak{p}_j$, if $\mathfrak{p}_j = \langle \ell, \ell_j \rangle_S$ with a linear form $\ell_j \in S$ for $j = 1, \ldots, s-1$, then $I_{\mathbb{Y}}$ is generated by ℓ and $F = \prod_{j=1}^{s-1} \ell_j$. Let us write $\mathfrak{p}_s = \langle L_1, L_2 \rangle_S$ with suitable $L_1 = \sum_{j=0}^2 a_i X_i, L_2 = \sum_{j=0}^2 b_i X_i \in S_1$ such that $p_1 \in \mathscr{Z}^+(L_1)$ and $p_2 \in \mathscr{Z}^+(L_2)$. Such $L_1, L_2 \in S_1$ exist, since $p_s \notin \mathscr{Z}^+(\ell)$. Then we have $I_{\mathbb{X}} = \bigcap_{j=1}^s \mathfrak{p}_j = \langle \ell, F \rangle_S \cap \langle L_1, L_2 \rangle_S$.

We claim that $\dim_K(\langle \ell, L_1, L_2 \rangle_K) = 3$. If there are $a, b, c \in K$ such that $a\ell + bL_1 + cL_2 = 0$, then $0 = a\ell(p_1) + bL_1(p_1) + cL_2(p_1) = cL_2(p_1)$ and $0 = a\ell(p_2) + bL_1(p_2) + cL_2(p_2) = bL_1(p_2)$, and so $b = c = 0$ and $a\ell = 0$. This means that $a = b = c = 0$ and the claim follows.

Consequently, we get $\langle \ell, L_1, L_2 \rangle_K = \langle X_0, X_1, X_2 \rangle_K$. Let $\ell = \sum_{j=0}^2 c_i X_i$ with $c_i \in K$ and suppose w.l.o.g. that $a_0 c_1 - a_1 c_0 \neq 0$ and $b_0 c_1 - b_1 c_0 \neq 0$ (as $\{\ell, L_1, L_2\}$ is linearly independent over K). Since $\ell L_1, \ell L_2 \in I_{\mathbb{X}}$, we have $a_i \ell + c_i L_1, b_i \ell + c_i L_2 \in \partial I_{\mathbb{X}}$ for every $i = 0, 1, 2$. We have

$$\langle a_i \ell + c_i L_1, b_i \ell + c_i L_2 \mid i = 0, 1, 2 \rangle_K = \langle \ell, L_1, L_2 \rangle_K = \langle X_0, X_1, X_2 \rangle_K,$$

and subsequently we get $\partial I_{\mathbb{X}} = \langle X_0, X_1, X_2 \rangle_S$. Therefore, Lemma 3.1 yields

$$\mathrm{HF}_{\Omega^3_{R/K}}(3) = \mathrm{HF}_{S/\partial I_{\mathbb{X}}}(0) = 1, \qquad \mathrm{HF}_{\Omega^3_{R/K}}(i) = \mathrm{HF}_{S/\partial I_{\mathbb{X}}}(i - 3) = 0$$

for all $i \neq 3$. $\square$

Note that the *Castelnuovo function* of $\mathbb{X}$ is defined by

$$\Delta \mathrm{HF}_{\mathbb{X}}(i) := \mathrm{HF}_{\mathbb{X}}(i) - \mathrm{HF}_{\mathbb{X}}(i - 1)$$

for all $i \in \mathbb{Z}$. In the proof of the lemma, we see that $I_{\mathbb{X}} = \langle \ell, F \rangle_S \cap \langle L_1, L_2 \rangle_S \supseteq \langle \ell L_1, \ell L_2, FL_1, FL_2 \rangle_S$. It follows that $\Delta \mathrm{HF}_{\mathbb{X}}(2) = 1$, and therefore $\Delta \mathrm{HF}_{\mathbb{X}}(i) \leq 1$ for all $i \geq 2$ by Kreuzer and Robbiano, 2005, Corollary 5.5.28. This proves the following corollary.

Corollary 3.6. *In the setting of Lemma 3.5, we have* $\Delta \mathrm{HF}_{\mathbb{X}}(i) \leq 1$ *for all* $i \geq 2$.

Using Bézout's Theorem (see e.g. Ueno, 1999, Theorem 1.32), we get the following consequence for a set of points on a non-singular conic in $\mathbb{P}^2$.

Corollary 3.7. *If $\mathbb{X}$ is a set of $s \geq 5$ distinct K-rational points in $\mathbb{P}^2$ which are on a non-singular conic $\mathscr{C} = \mathscr{L}^+(C)$, then $\Delta\mathrm{HF}_{\mathbb{X}}(2) = 2$.*

Proof. By Bézout's Theorem and $s \geq 5$, $I_{\mathbb{X}}$ has only one non-zero homogeneous polynomial F of degree 2. In this case we must have $F = aC$ with some $a \in K \setminus \{0\}$. Let $\{C, F_1, \ldots, F_t\}$, $\deg(F_i) > 2$, be a homogeneous system of generators of $I_{\mathbb{X}}$. Then the Hilbert function of $\mathbb{X}$ is given by

$$\mathrm{HF}_{\mathbb{X}} : \ 1 \ 3 \ 5 \ * \ * \cdots.$$

In particular, we obtain $\Delta\mathrm{HF}_{\mathbb{X}}(2) = 2$. □

Combining the above results we get the following classification of sets of points corresponding to values of the Hilbert function of the module of Kähler differentials.

Theorem 3.8. *Let $\mathbb{X}$ be a set of $s \geq 5$ distinct K-rational points in $\mathbb{P}^2$.*

    (a) $\mathbb{X}$ lies on two different lines and no $s-1$ points of $\mathbb{X}$ lie on a line, when $\mathrm{HF}_{\Omega^3_{R/K}}(3) = 1$ and $\mathrm{HF}_{\Omega^3_{R/K}}(4) = 1$.

    (b) If $\mathrm{HF}_{\Omega^3_{R/K}}(3) = 1$ and $\mathrm{HF}_{\Omega^3_{R/K}}(i) = 0$ for $i \neq 3$, then $\mathbb{X}$ either contains $s-1$ points on a line or lies on a non-singular conic depending on the value of $\Delta\mathrm{HF}_{\mathbb{X}}(2)$ equals 1 or not.

Proof. We see that $\mathbb{X}$ lies on a conic if and only if the Hilbert function of $\Omega^3_{R/K}$ satisfies $\mathrm{HF}_{\Omega^3_{R/K}}(4) \leq 1$ and $\mathrm{HF}_{\Omega^3_{R/K}}(3) = 1$ by Theorem 2.6 and Proposition 3.2. We consider the following three possibilities for the shapes of the points of $\mathbb{X}$.

(c_1) (c_2) (c_3)
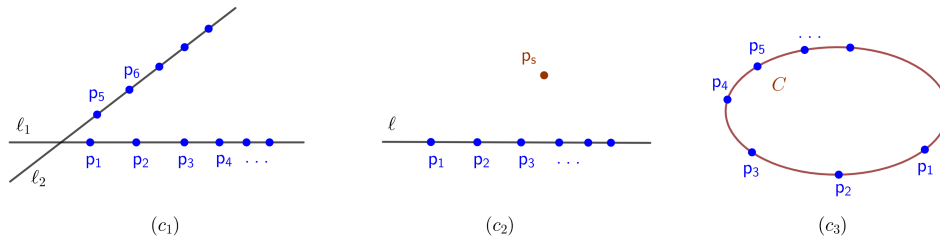
Case 1: If $\mathbb{X}$ lies on two different lines, such that no $s-1$ points lie on a line, then Proposition 3.4 shows that $\mathrm{HF}_{\Omega^3_{R/K}}(3) = \mathrm{HF}_{\Omega^3_{R/K}}(4) = 1$.

Case 2: In the case that there is a line passing through all but one point of $\mathbb{X}$, by Lemma 3.5 we have $\mathrm{HF}_{\Omega^3_{R/K}}(3) = 1$ and $\mathrm{HF}_{\Omega^3_{R/K}}(4) = 0$. Furthermore, Corollary 3.6 shows that $\Delta\mathrm{HF}_{\mathbb{X}}(2) = 1$.

Case 3: When $\mathbb{X}$ lies on a non-singular conic, Lemma 3.3 implies $\mathrm{HF}_{\Omega^3_{R/K}}(3) = 1$ and $\mathrm{HF}_{\Omega^3_{R/K}}(4) = 0$. In this case we have $\Delta\mathrm{HF}_{\mathbb{X}}(2) = 2$ by Corollary 3.7.

Altogether, the theorem is completely proved. □

The following example shows us how to distinguish two sets of K-rational points with the same cardinality using the modules of Kähler differentials.

Example 3.9. Let $\mathbb{X} \subseteq \mathbb{P}^2$ be a set of six K-rational points on the conic $X_0^2 + 2X_1^2 - X_2^2$, and let $\mathbb{Y} \subseteq \mathbb{P}^2$ be a set of six K-rational points on two lines defined by $\ell_1 = x_1$ and $\ell_2 = x_2$. Then the Hilbert functions of $\mathbb{X}$ and $\mathbb{Y}$ are the same and given by $\mathrm{HF}_{\mathbb{X}} = \mathrm{HF}_{\mathbb{Y}} : 1\ 3\ 5\ 6\ 6 \cdots$. Also, their homogeneous vanishing ideals have the same minimal graded free resolution (see Tohaneanu and van Tuyl, 2013, Example 4.1). However, their modules of Kähler differential m-forms have the following Hilbert functions:

$$\mathrm{HF}_{\Omega^1_{R/K}} : 0\ 3\ 8\ 11\ 10\ 7\ 6\ 6 \cdots, \quad \mathrm{HF}_{\Omega^1_{R_{\mathbb{Y}}/K}} : 0\ 3\ 8\ 11\ 10\ 7\ 6\ 6 \cdots,$$

$$\mathrm{HF}_{\Omega^2_{R/K}} : 0\ 0\ 3\ 6\ 4\ 1\ 0 \cdots, \qquad \mathrm{HF}_{\Omega^2_{R_{\mathbb{Y}}/K}} : 0\ 0\ 3\ 6\ 5\ 1\ 0\ 0 \cdots,$$

$$\mathrm{HF}_{\Omega^3_{R/K}} : 0\ 0\ 0\ 1\ 0\ 0 \cdots, \qquad \mathrm{HF}_{\Omega^3_{R_{\mathbb{Y}}/K}} : 0\ 0\ 0\ 1\ 1\ 0\ 0 \cdots.$$

Thus, by looking at the Hilbert functions of the modules of Kähler differential m-forms, one can distinguish two sets $\mathbb{X}$ and $\mathbb{Y}$.

References

de Dominicis, G., & Kreuzer, M. (1999). Kähler differentials for points in P^n. *Journal of Pure and Applied Algebra*, *141*, 153–173.

Kreuzer, M., Linh, T. N. K., & Long, L. N. (2015). Kähler differentials and Kähler differents for fat point schemes. *Journal of Pure and Applied Algebra*, *2015*, 4479–4509.

Kreuzer, M., Linh, T. N. K., & Long, L. N. (2019). Kähler differential algebras for 0-dimensional schemes. *Journal of Algebra*, *501*, 255–284.

Kreuzer, M., & Robbiano, L. (2005). *Computational commutative algebra 2*. Springer-Verlag.

Kunz, E. (1986). *Kähler differentials*. Vieweg Verlag, Braunschweig.

Kunz, E. (2005). *Introduction to plane algebraic curves*. Birkhäuser.

Linh, T. N. K. (2015). *Kähler differential modules for 0-dimensional schemes and applications*. PhD Thesis, University of Passau.

Scheja, G., & Storch, U. (1988). *Lehrbuch der algebra, teil 2*. B.G. Teubner, Stuttgart.

The ApCoCoA Team. (2021). Apcocoa: Applied computations in commutative algebra.

Tohaneanu, S. O., & van Tuyl, A. (2013). Bounding invariants of fat points using a coding theory construction. *Journal of Pure and Applied Algebra*, *217*, 269–279.

Ueno, K. (1999). *Algebraic geometry 1, from algebraic varieties to schemes*. Amer. Math. Soc., Providence.