THE LEFSCHETZ PROPERTIES OF ARTINIAN MONOMIAL ALGEBRAS ASSOCIATED TO SOME GRAPHS

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Abstract: In this paper, we study the weak Lefschetz property of an artinian monomial algebra A_G defined by the sum of the edge ideal of a simple graph G and the square of the variables. We classify some class of graphs G where A_G has or fails the weak Lefschetz properties.

Keywords: Artinian algebra, edge ideal, independence polynomial, weak Lefschetz property.

1 INTRODUCTION

The Lefschetz property is an algebrization of the Hard Lefschetz theorem, one of the important theorems in algebraic geometry. More precisely, we say that a graded artinian algebra A has the weak Lefschetz property (WLP) if there exists a linear form ℓ such that the multiplication map $\times\ell : A_i \longrightarrow A_{i+1}$ has maximal rank for all degree i, while A has the strong Lefschetz property (SLP) if the multiplication map $\times\ell^j : A_i \longrightarrow A_{i+j}$ has maximal rank for all i and all j.

The Lefschetz properties of graded algebras have connections to several areas of mathematics. Due to this ubiquity, many classes of algebras have been studied with respect to the WLP and the SLP. At first glance, checking the WLP or the SLP might seem to be a simple problem of linear algebra. However, determining which graded algebras have the WLP or the SLP is notoriously difficult, and a number of natural families of algebras still simply remain uncharacterized. We refer the reader to the monography *The Lefschetz Properties* [4].

In this paper, we study the SLP and/or WLP of artinian monimial algebras associated the edge ideals of graphs. More precisely, let G be a simple graph, i.e. $G = (V, E)$ is a pair where V is a set of elements called *vertices*, and E is a set of elements called *edges* which are unordered pairs of vertices from V. Suppose that $V = \{1, 2, \dots, n\}$ and let $R = k[x_1, \dots, x_n]$ be a standard graded polynomial ring

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over a field k. The *edge ideal* of G is the ideal

$$I_G = (x_i x_j \mid \{i,j\} \in E) \subset R.$$

Then, we say that

$$A_G = R/((x_1^2, \dots, x_n^2) + I_G)$$

is the *artinian monomial algebra associated to* G. We are interested in the following problem.

Problem 1.1. Classify the simple graph G that A_G has or fails the WLP or SLP.

Note that A_G is an artinian algebra generated by quadratic monomials. The WLP of these algebras is also studied by Michałek and Miró-Roig [7] and Migliore, Nagel and Schenck [9]. In this paper, we will study the WLP/SLP of the artinian monomial algebras associated to some classes of graphs such as the empty graphs, the complete graphs, the disjoint union of complete graphs, the star graphs, the Barbell graphs and the cone of graphs.

2 PRELIMINARIES

We consider standard graded algebra $A = \oplus_{i\geq 0}[A]_i = R/I$, where $R = k[x_1, \dots, x_n]$ is a polynomial ring over a field k with all x_i's have degree 1 and $I \subset R$ is an artinian homogeneous ideal. Let us define the weak and strong Lefschetz properties for artinian algebras.

Definition 2.1. We say that A has the *weak Lefschetz property* (WLP) if there is a linear form $\ell \in [A]_1$ such that, for all integers j, the multiplication map

$$\times\ell : [A]_j \longrightarrow [A]_{j+1}$$

has maximal rank, i.e. it is injective or surjective. In this case the linear form ℓ is called a *Lefschetz element* of A. If for the general form $\ell \in [A]_1$ and for an integer number j the map $\times\ell : [A]_j \longrightarrow [A]_{j+1}$ does not have the maximal rank we will say that A *fails the WLP in degree* j.
We say that A has the *strong Lefschetz property* (SLP) if there is a linear form $\ell \in [A]_1$ such that, for all integers j and s, the multiplication map

$$\times\ell^s : [A]_j \longrightarrow [A]_{j+s}$$

has maximal rank.

In the case of one variable, the WLP and SLP trivially hold since all ideals are principal. In the case of two variables there is a nice result in characteristic zero by Harima, Migliore, Nagel and Watanabe [5, Proposition 4.4].

Proposition 2.2. *Every artinian algebra $A = k[x, y]/I$, where k has characteristic zero, has the SLP (and consequently also the WLP).*

In a polynomial ring with more than two variables it is not true in general that every artinian monomial algebra has the SLP or WLP. The most general result in this case proved by Stanley in [10].

Theorem 2.3. *Let $R = k[x_1, \ldots, x_n]$, where k is of characteristic zero. Let I be an artinian monomial complete intersection, i.e. $I = (x_1^{d_1}, \ldots, x_n^{d_n})$. Then $A = R/I$ has the SLP.*

By using the action of a torus on monomial algebras, Migliore, Miró-Roig and Nagel proved the existence of the canonical Lefschetz element.

Proposition 2.4. [8, Proposition 2.2] *Let $I \subset R = k[x_1, \ldots, x_n]$ be an artinian monomial ideal. Then $A = R/I$ has the WLP if and only if $\ell = x_1 + x_2 + \cdots + x_n$ is a Lefschetz element for A.*

A necessary condition for the WLP and SLP of an artinian algebra A is the unimodality of the Hilbert series of A. To do it, we need some notations.

Definition 2.5. Let k be a field and $A = \oplus_{j \geq 0}[A]_j$ be a standard graded k-algebra. The *Hilbert series* of A is the power series $\sum \dim_k[A]_i t^i$ and is denoted by $HS(A, t)$. The *Hilbert function* of A is the function $h_A : \mathbb{N} \longrightarrow \mathbb{N}$ defined by $h_A(j) = \dim_k[A]_j$. If A is an artinian graded algebra, then $[A]_i = 0$ for $i \gg 0$. We denote

$$D = \max\{i \mid [A]_i \neq 0\}.$$

The integer D is called the *socle degree* of A. In this case, the Hilbert series of A is a polynomial

$$HS(A, t) = 1 + h_1 t + \cdots + h_D t^D,$$

where $h_i = H_A(i) = \dim_k[A]_i > 0$. By definition, the degree of the Hilbert series for an artinian graded algebra A is equal to its socle degree $D := \max\{i \mid [A]_i \neq 0\}$. Since A is artinian and non-zero, this number also agrees with the *Castelnuovo-Mumford regularity* of A, so

$$\operatorname{reg}(A) = D = \deg(HS(A, t)).$$

Definition 2.6. A polynomial $\sum_{k=0}^{n} a_k x^k$ with integer coefficients is called *unimodal* if there is an m, such that

$$a_0 \leq a_1 \leq \cdots \leq a_{m-1} \leq a_m \geq a_{m+1} \geq \cdots \geq a_n.$$

The *mode* of the unimodal polynomial $\sum_{k=0}^{n} a_k x^k$ defined by

$$\min\{k \mid a_{k-1} < a_k \geq a_{k+1} \geq \cdots \geq a_m\}.$$

Proposition 2.7. [4, Proposition 3.2] *If A has the WLP or SLP then the Hilbert series of A is unimodal.*

3 ARTINIAN MONOMIAL ALGEBRAS ASSOCIATED TO EDGE IDEALS OF GRAPHS

In this section, we study the WLP or SLP of an artinian monomial algebra A_G associated to the edge ideal of a simple graph G. In detail, let $G = (V, E)$ is a simple graph, with the set of vertices $V = \{1, 2, \ldots, n\}$ and the set of edges E. Denote by $R = k[x_1, \ldots, x_n]$ the standard graded polynomial ring over a field k. Then we consider the artinian monomial algebra

$$A_G = R/((x_1^2, \ldots, x_n^2) + I_G),$$

where $I_G = (x_i x_j \mid \{i, j\} \in E) \subset R$ is the edge ideal of G. We are interested in studying the following problem.

Problem 3.1. Classify the simple graphs G such that A_G has or fails the WLP/SLP.

The algebra A_G contains significant combinatorial information about G. In detail, a subset X of vertices V is called *independent* if for any $i, j \in X$, $\{i, j\} \notin E$, i.e., the vertices in X are pairwise non-adjacent. The *independence number* of a graph G is the largest cardinality among all independent sets of G. We denote this value by $\alpha(G)$. The *independence polynomial* of a graph G, denoted by $I(G; t)$, is the polynomial

$$I(G; t) = \sum_{k=0}^{\alpha(G)} s_k(G) t^k,$$

where $s_k(G)$ is the number of independent sets of order k in G. The independence polynomial of a graph was defined by I. Gutman and F. Harary in [3] as a generalization of the matching polynomial of a graph. Then we have the following.

Proposition 3.2. *The Hilbert series of A_G is equal to the independence polynomial of G, i.e.*

$$HS(A_G;t) = I(G;t) = \sum_{k=0}^{\alpha(G)} s_k(G)t^k.$$

Therefore, the WLP/SLP of A_G has strong consequences on the unimodality of the independence polynomial of G (see Proposition 2.7). In particular, if $I(G;t)$ is not unimodal, then A_G fails the WLP/SLP. Thus, to study the WLP/ SLP of A_G, it is enough to consider the graphs G such that their independence polynomial are unimodal.

Example 3.3. [1] Given positive integers m and $n > m$, let $G = (V, E)$ with $V = V_1 \cup V_2 \cup V_3$, where V_1, V_2, V_3 are disjoint; $|V_1| = n - m$ and $|V_2| = |V_3| = m$; E consists of a complete bipartite graph between V_1 and V_2 and a perfect matching between V_2 and V_3. Then G is a bipartite graph and for every $i \geq 0$, $s_i(G) = (2^i - 1)\binom{m}{i} + \binom{n}{i}$. Therefore, for $m \geq 95$ and $n = \lfloor m \log_2(3) \rfloor$. Then $I(G;t)$ is not unimodal. As a consequence, A_G fails the WLP.

Let $G = (V, E)$ be a simple graph. A graph G is said to be *well-covered* if every maximal independent set of G has the same size and is equal to $\alpha(G)$. The result is immediately implied from the above proposition.

Corollary 3.4. *Let A_G be the artinian monomial algebra associated to a simple graph G. Then* $\operatorname{reg}(A_G) = \alpha(G)$ *and A_G is level if and only if G is well-covered.*

It is known that the Lefschetz properties depend strongly on the characteristic of field. For simplicity, we always assume that k is a field of characteristic zero.

Example 3.5. An empty graph is simply a graph with no edges. We denote the empty graph on n vertices by E_n. Then

$$A_{E_n} = R/(x_1^2, \ldots, x_n^2) \quad \text{and } I(E_n;t) = (1+t)^n.$$

A result of Stanley says that A_{E_n} has the SLP (see Theorem 2.3).

If G has a small number of vertices, the we have the following result.

Proposition 3.6. *Let $G = (V, E)$ be a graph. If $|V| \leq 3$, then A_G has the SLP.*

Proof. Since all artinian algebras in the polynomial ring with one or two variables have the SLP (see Proposition 2.2), it is enough to consider the case where $|V| = 3$. In this case, G is a empty graph; or a complete graph; or a path and one isolated vertex. A simple computation with `Macaulay2` [2] shows that A_G has the SLP. □

A complete graph on n vertices, denoted by K_n, is the graph where every vertex is adjacent to every other vertex. It follows that

$$A_{K_n} = R/(x_1, \dots, x_n)^2 \quad \text{and } I(K_n; t) = 1 + nt.$$

It is easy to see that A_{K_n} has the SLP. Now, we consider the joint union of complete graphs $G = \cup_{i=1}^{n} K_{m_i}$. Recall that the *disjoint union* of the graphs G_1, G_2 is a graph $G = G_1 \cup G_2$ having as vertex set the disjoint union of $V(G_1), V(G_2)$, and as edge set the disjoint union of $E(G_1), E(G_2)$.

Proposition 3.7. *Let A be the artinian monomial algebra associated to $G = \cup_{i=1}^{n} K_{m_i}$. Assume $m_1 \geq m_2 \geq \cdots \geq m_n \geq 1$. Then A has the WLP if and only if one of the following holds:*

(1) *$m_2 = \cdots = m_n = 1$, i.e. G is the disjoint union of a complete graph K_{m_1} and an empty graph of order $n - 1$.*

(2) *$m_3 = \cdots = m_n = 1$ and n is odd.*

In particular for $n \geq 2$, the disjoint of n complete graphs at least two vertices does not have the WLP.

Proof. First, I. Gutman and F. Harary in [3] was given a formula to calculate the independence polynomial of disjoint graphs as follows

$$I(G_1 \cup G_2; t) = I(G_1; t).I(G_2; t).$$

By using this formula, it is easy to see that

$$A = \bigotimes_{i=1}^{n} k[x_{i,1}, \dots, x_{i,m_i}]/(x_{i,1}^2, \dots, x_{i,m_i}^2).$$

The proposition is implied from [9, Theorem 4.8]. □

Recall that a *star graph* of order n is a graph on $n+1$ vertices. This graph is formed by starting with a single vertex and adjoining n leaves. We denote this graph by S_n. Let A_{S_n} be the artinian monomial algebra associated to a star S_n. Then $A_{S_n} = R/I$, where $R = k[x_0, x_1, \dots, x_n]$ and

$$I = (x_0^2, x_1^2, \dots, x_n^2) + (x_0x_1, \dots, x_0x_n).$$

It is easy to see that the independence polynomial of S_n is $I(S_n;t) = (1+t)^n + t$.

Proposition 3.8. *The algebra A_{S_n} has the WLP if and only if $n = 1, 2$. Moreover, if $n \geq 3$, then A_{S_n} fails the WLP in only one degree, namely it fails the injectivity from degree 1 to degree 2.*

Proof. Write $A_{S_n} = R/I$ as above. By Proposition 2.4, set $\ell = x_0 + x_1 + \cdots + x_n$. Then the following exact sequence

$$0 \longrightarrow R/(I\colon x_0)(-1) \xrightarrow{\times x_0} R/I \longrightarrow R/(I, x_0) \longrightarrow 0$$

deduces the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & [R/(I\colon x_0)]_{j-1} & \xrightarrow{\times x_0} & [R/I]_j & \longrightarrow & [R/(I,x_0)]_j & \longrightarrow & 0\,, \\ & & \downarrow{\scriptstyle \times\ell} & & \downarrow{\scriptstyle \times\ell} & & \downarrow{\scriptstyle \times\ell} & & \\ 0 & \longrightarrow & [R/(I\colon x_0)]_j & \xrightarrow{\times x_0} & [R/I]_{j+1} & \longrightarrow & [R/(I,x_0)]_{j+1} & \longrightarrow & 0 \end{array} \tag{3.1}$$

with rows are exact, for all integer $j \geq 0$. Note that

$$\begin{aligned} (I, x_0) &= (x_1^2, \dots, x_n^2, x_0) \\ I\colon x_0 &= (x_0, x_1, \dots, x_n). \end{aligned}$$

It follows that $R/(I, x_0) \cong S/J := k[x_1, \dots, x_n]/(x_1^2, \dots, x_n^2)$ an artinian monomial complete intersection, and hence it has the WLP by Theorem 2.3. Clearly, $R/(I\colon x_0) \cong k$. It follows from (3.1) that the multiplication map

$$\times\ell : [R/I]_j \longrightarrow [R : I]_{j+1}$$

has maximal rank for all j, except $j = 1$. In the later case, $\times\ell : [R/I]_1 \longrightarrow [R : I]_2$ is not injective. Thus R/I fails the WLP in degree 1. $\square$

Definition 3.9. The Barbell graph of order n is a graph on $2n$ vertices which is formed by joining two copies of K_n by a single edge, known as a bridge. We denote this graph Bar_n.

It is known that the independence polynomial of the Barbell graph of order n is

$$I(\mathrm{Bar}_n;t) = (1+nx)^2 - x^2 = 1 + 2nx + (n^2-1)x^2.$$

Theorem 3.10. *The artinian monomial algebra associated to* Bar_n *has the WLP if and only if* $n = 1, 2$. *Furthermore, for any* $n \geq 3$, *this algebra fails the injectivity from degree 1 to degree 2.*

Proof. The artinian monomial algebra associated to Bar_n is $A = R/I$, where

$$R = k[x_1, \dots, x_n, y_1, \dots, y_n] \text{ and } I = (x_1, \dots, x_n)^2 + (y_1, \dots, y_n)^2 + (x_n y_n).$$

It is easy to see that Bar_n has the WLP for $n = 1, 2$. For $n \geq 3$, one has

$$\dim_k [A]_1 = 2n < n^2 - 1 = \dim_k [A]_2.$$

Let B be the artinian monomial algebra associated to disjoint union of two complete graph K_n, so $B = R/J$, where $J = (x_1, \dots, x_n)^2 + (y_1, \dots, y_n)^2$. Then

$$J \colon x_n y_n = (x_1, \dots, x_n, y_1, \dots, y_n).$$

It follows that $R/(J \colon x_n y_n) \cong k$. By Proposition 2.4, let $\ell = x_1 + \cdots + x_n + y_1 + \cdots + y_n$. Therefore, we have the following commutative diagram, with rows are exact

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \xrightarrow{\times x_n y_n} & [B]_1 & \longrightarrow & [A]_1 & \longrightarrow & 0\,, \\
 & & \downarrow{\scriptstyle \times\ell} & & \downarrow{\scriptstyle \times\ell} & & \downarrow{\scriptstyle \times\ell} & & \\
0 & \longrightarrow & k & \xrightarrow{\times x_n y_n} & [B]_2 & \longrightarrow & [A]_2 & \longrightarrow & 0
\end{array}$$

By [7, Proposition 2.8], the multiplication map $\times\ell : [B]_1 \longrightarrow [B]_2$ is not injective. Thus $\times\ell : [A]_1 \longrightarrow [A]_2$ is not injective, as desired. □

Finally, we consider the cone of graphs. Recall that the *cone* of a graph G is a graph formed by taking a copy of G and adding a vertex which is adjacent to every vertex in G, denoted this graph by $\mathrm{Cone}(G)$.

Theorem 3.11. *Let* G *be a graph on* n *vertices. If* $s_2(G) \geq n+1$, *then cone of* G *has the unimodal independence polynomial and* $A_{\mathrm{Cone}(G)}$ *fails the WLP.*

Proof. Assume that $A_G = \frac{k[x_1,\dots,x_n]}{(x_1^2,\dots,x_n^2)+I_G}$. A computation shows that the independence polynomial of $\mathrm{Cone}(G)$ is

$$I(\mathrm{Cone}(G);t) = I(G;t) + t.$$

Since $s_2(G) \geq n+1, s_2(\text{Cone}(G)) = s_2(G) \geq n+1 = s_1(\text{Cone}(G))$. We will show that the multiplication map

$$\times \ell : [A_{\text{Cone}(G)}]_1 \longrightarrow [A_{\text{Cone}(G)}]_2$$

is not injective. To do it, we write $A_{\text{Cone}(G)} = R/I$, where $R = k[x_0, \dots, x_n]$ and

$$I = (x_0^2, \dots, x_n^2) + I_G + (x_0x_1, \dots, x_0x_n).$$

Set $\ell = x_0 + x_1 + \cdots + x_n$ (see Proposition 2.4). Consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & [R/(I\colon x_0)]_0 & \xrightarrow{\times x_0} & [R/I]_1 & \longrightarrow & [R/(I, x_0)]_1 & \longrightarrow & 0\ , \\
 & & \downarrow{\scriptstyle \times\ell} & & \downarrow{\scriptstyle \times\ell} & & \downarrow{\scriptstyle \times\ell} & & \\
0 & \longrightarrow & [R/(I\colon x_0)]_1 & \xrightarrow{\times x_0} & [R/I]_2 & \longrightarrow & [R/(I, x_0)]_2 & \longrightarrow & 0
\end{array}$$

with rows are exact. Note that $R/(I, x_0) \cong A_G$ and $R/(I\colon x_0) \cong k$. It follows that the kernel of the first vertices map is equal to k, and we conclude that the multiplication map

$$\ell : [R/I]_1 \longrightarrow [R/I]_2$$

is not injective, as desired. □

Denote by P_n the path on n vertices ($n \geq 1$). In [6], G. Hopkins and W. Staton showed that the independence polynomial of P_n

$$I(P_n; t) = \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-i}{i} t^i.$$

Corollary 3.12. *The artinian monomial algebra of* Cone(P_n) *has the WLP if and only if* $n \leq 4$.

Proof. By using `Macaulay2`, it is easy to show that $A_{\text{Cone}(P_n)}$ has the WLP for $n \leq 4$. Now if $n \geq 5$ then $s_2(P_n) = \binom{n-1}{2} \geq n+1$. By Theorem 3.11, $A_{\text{Cone}(P_n)}$ fails the WLP. □

Denote by C_n the cycle on n vertices ($n \geq 3$). It is showed in [6] that the independence polynomial of C_n is

$$I(C_n; t) = 1 + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{i} \binom{n-i-1}{i-1} t^i.$$

Recall that the cone of C_n is called a *wheel graph* of order n, i.e., this graph is formed by taking a copy of C_n and adding a central vertex which is adjacent to every vertex in C_n. We denote the wheel graph of order n by W_n.

Corollary 3.13. *The artinian monomial algebra of* W_n *has the WLP if and only if* $n = 3, 4, 5$.

Proof. By using `Macaulay2`, it is easy to show that A_{W_n} has the WLP for $n = 3, 4, 5$. Now if $n \geq 6$ then $s_2(C_n) = \frac{n(n-3)}{2} \geq n + 1$. By Theorem 3.11, A_{W_n} fails the WLP. □

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