Math. Proc. Camb. Phil. Soc. (2007), **142**, 429 © 2007 *Cambridge Philosophical Society*
doi:10.1017/S0305004106009819 *Printed in the United Kingdom*

Castelnuovo–Mumford regularity and Degree of nilpotency

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(*Received* 30 *September* 2005; *revised* 19 *April* 2006)

Abstract

In this paper we show that the Castelnuovo–Mumford regularity of the associated graded module with respect to an $\mathfrak{m}$-primary ideal I is effectively bounded by the degree of nilpotency of I. From this it follows that there are only a finite number of Hilbert-Samuel functions for ideals with fixed degree of nilpotency.

1. *Introduction*

Let $R = \oplus_{n\geqslant 0} R_n$ be a finitely generated standard graded ring over a noetherian commutative ring R_0. Let R_+ be the ideal of R generated by the elements of positive degrees of R. If E is a finitely generated graded R-module, we set

$$a_i(E) = \begin{cases} \max\{n \mid H^i_{R_+}(E)_n \neq 0\} & \text{if } H^i_{R_+}(E) \neq 0, \\ -\infty & \text{if } H^i_{R_+}(E) = 0. \end{cases}$$

The Castelnuovo–Mumford regularity of E is the number

$$\operatorname{reg}(E) := \max\{a_i(E) + i \mid i \geqslant 0\}.$$

Let $(A, \mathfrak{m})$ be a Noetherian local ring, I an $\mathfrak{m}$-primary ideal of A and M a finitely generated A-module. We denote by $G_I(M)$ the associated graded ring $\bigoplus_{n\geqslant 0} I^n M/I^{n+1}M$ of M with respect to I. It is known that the Castelnuovo–Mumford regularity $\operatorname{reg}(G_I(M))$ provides upper bounds for several invariants of M with respect to I such as the postulation number, the relation type and the reduction number [**16**]. It is of great interest to find upper bounds for $\operatorname{reg}(G_I(M))$ by means of simpler invariants.

Our first main result gives a bound for the Castelnuovo–Mumford regularity of $G_I(M)$ in terms of the degree of nilpotency $n(I)$ of I. Recall that the *degree of nilpotency* is the least integer n such that $\mathfrak{m}^n \subseteq I$.

THEOREM 2.4. *Let $(A, \mathfrak{m})$ be a Noetherian local ring and I be an $\mathfrak{m}$-primary ideal of A. Let M be a finitely generated A-module with $d = \dim A \geqslant 1$. Then:*

(i) $\operatorname{reg}(G_I(M)) \leqslant n(I)\operatorname{hdeg}(M) - 1$ if $d = 1$;
(ii) $\operatorname{reg}(G_I(M)) \leqslant 2^{(d-1)!}\operatorname{hdeg}(M)^{3(d-1)!-1} n(I)^{3d!-d} - 1$ if $d \geqslant 2$.

Here hdeg (M) denotes the homological degree of M, which was introduced by Vasconcelos [**17**] in order to control the complexity of M. If M is a Cohen–Macaulay module, hdeg (M) is the multiplicity (degree) of M. Theorem 2·4 implies effective bounds for the postulation number, the relation type and the reduction number of I in terms of the degree of nilpotency of I. It should be noted that bounds for the postulation number were already established by Schwartz [**11**] for $M = A$ in the characteristic zero case.

Our proof is based on a bound for $\operatorname{reg}(G_I(M))$ by means of an extended degree, a generalization of the degree, which was introduced by Doering, Gunston and Vasconcelos [**3**]. Using this bound we are also able to give bounds for the coefficients of the Hilbert–Samuel polynomials.

If we denote by $P_M(n)$ the Hilbert–Samuel polynomial associated with the Hilbert-Samuel function $\ell(M/I^{n+1}M)$ and if we write

$$P_M(n) = \sum_{i=0}^{d} (-1)^i e_i(I, M) \binom{m+d-i}{d-i},$$

then $e_i(I, M)$ are called the Hilbert coefficients of M with respect to I. We will set $e(I, M) := e_0(I, M)$, the multiplicity of M with respect to I, and $e(M) := e(\mathfrak{m}, M)$.

The Hilbert coefficients have become an interesting subject in recent years [**4**, **6**, **20**]. In particular, Srinivas and Trivedi [**12–14**] gave bounds for the Hilbert coefficients in terms of the dimension, multiplicity, and lengths of local cohomologies for Cohen–Macaulay rings and generalized Cohen–Macaulay modules. Recently, Rossi, Trung and Valla [**8**] gave bounds for the Hilbert coefficients of A with respect to $\mathfrak{m}$ in terms of an extended degree. We will extend this result to the module case for an arbitrary $\mathfrak{m}$-primary ideal I. As an application we obtain bounds for the Hilbert coefficients of the Hilbert function in terms of the degree of nilpotency of I and the homological degree of M.

THEOREM 3.3. *Let $(A, \mathfrak{m})$ be a Noetherian local ring and I an $\mathfrak{m}$-primary ideal of A. Let M be a finitely generated A-module with* $\dim(M) \geqslant 1$. *Then*:

(i) $e(I, M) \leqslant e(M)n(I)^d$;

(ii) $|e_1(I, M)| \leqslant \operatorname{hdeg}(M)n(I)^d[\operatorname{hdeg}(M)n(I)^d - 1]$;

(iii) $|e_i(I, M)| \leqslant (i+1)2^{2i!+2} \operatorname{hdeg}(M)^{3i!-i+1} n(I)^{3di!-di+d} - 1 \quad \text{if } i \geqslant 2.$

It follows from Theorem 2·4 and Theorem 3·3 that there are only a finite number of Hilbert-Samuel functions of M for $\mathfrak{m}$-primary ideals with fixed degree of nilpotency. This extends another result of Schwartz in the characteristic zero case [**11**].

2. *Bounds for the regularity of the associated graded module*

Let $\mathcal{M}(A)$ denote the class of finitely generated A-modules. An *extended degree on $\mathcal{M}(A)$ with respect to I* (see [**5**]) is a numerical function $D(I, \cdot)$ on $\mathcal{M}(A)$ such that the following properties hold for every module $M \in \mathcal{M}(A)$.

(i) $D(I, M) = D(I, M/L) + \ell(L)$, where L is the maximal submodule of M having finite length,

(ii) $D(I, M) \geqslant D(I, M/xM)$ for a generic x on M with respect to I,

(iii) $D(I, M) = e(I, M)$ if M is a Cohen–Macaulay A-module, where $e(I, M)$ denotes the multiplicity of M with respect to I.

Remark. Any extended degree $D(I, M)$ will satisfy $D(I, M) \geqslant e(I, M)$, where equality holds if and only if M is a Cohen–Macaulay module.

The extended degree $D(I, M)$ is a generalization of the notion $D(M) := D(\mathfrak{m}, M)$ introduced in [**3**] and [**18**].

We have the following bound for $\operatorname{reg}(G_I(M))$ in terms of an extended degree of M with respect to I.

THEOREM 2·1 ([**5**, theorem 4·4]). *Let M be a finitely generated A-module with $d = \dim M \geqslant 1$. Let $D(I, M)$ be an arbitrary extended degree of M with respect to I. Then*:

(i) $\operatorname{reg}(G_I(M)) \leqslant D(I, M) - 1$ if $d = 1$;

(ii) $\operatorname{reg}(G_I(M)) \leqslant 2^{(d-1)!} D(I, M)^{3(d-1)!-1} - 1$ if $d \geqslant 2$.

In this paper we are interested only in the following special case of extended degrees.

If A is a homomorphic image of a Gorenstein ring S with $\dim S = n$ and $M \in \mathcal{M}(A)$ with $\dim M = d$, we define the *homological degree* with respect to I as the number

$$\operatorname{hdeg}(I, M) := e(I, M) + \sum_{i=0}^{d-1} \binom{d-1}{i} \operatorname{hdeg}(I, \operatorname{Ext}_S^{n-i}(M, S)),$$

if $d > 0$, and $\operatorname{hdeg}(I, M) = \ell(M)$ if $d = 0$. This is a recursive definition on the dimension since $\dim \operatorname{Ext}_S^{n-i}(M, S) \leqslant i$ for $i = 0, \ldots, d-1$. If A is not a homomorphic image of a Gorenstein ring, we define

$$\operatorname{hdeg}(I, M) := \operatorname{hdeg}(I, M \otimes_A \hat{A}),$$

where $\hat{A}$ denotes the $\mathfrak{m}$-adic completion of A. We can easily verify that $\operatorname{hdeg}(I, M)$ is an extended degree of M with respect to I (see [**17**] for the case $I = \mathfrak{m}$).

Remark. If M is a generalized Cohen–Macaulay module, then

$$\operatorname{hdeg}(I, M) = e(I, M) + \sum_{i=0}^{d-1} \binom{d-1}{i} \ell\big(H_{\mathfrak{m}}^i(M)\big).$$

For simplicity we put

$$\operatorname{hdeg}(M) := \operatorname{hdeg}(\mathfrak{m}, M).$$

To study the relationship between $\operatorname{hdeg}(I, M)$ and $\operatorname{hdeg}(M)$ we shall need the following observation.

LEMMA 2·2. *Assume that $d = \dim M \geqslant 1$. Then*

$$e(M) \leqslant e(I, M) \leqslant n(I)^d e(M).$$

Proof. Since

$$\ell(M/\mathfrak{m}^{m+1}M) \leqslant \ell(M/I^{m+1}M) \leqslant \ell(M/\mathfrak{m}^{n(I)(m+1)}M),$$

we have

$$\begin{aligned} \frac{e(M)}{d!} m^d + (\text{terms of lower degree}) &\leqslant \frac{e(I, M) n(I)^d}{d!} m^d + (\text{terms of lower degree}) \\ &\leqslant \frac{e(M) n(I)^d}{d!} m^d + (\text{terms of lower degree}) \end{aligned}$$

for $m \gg 0$. Hence $e(M) \leqslant e(I, M) \leqslant n(I)^d e(M)$.

Similarly, the relationship between $\operatorname{hdeg}(M)$ and $\operatorname{hdeg}(I, M)$ is given by the following inequalities.

LEMMA 2·3. *Let $(A, \mathfrak{m})$ be a Noetherian local ring and I an $\mathfrak{m}$-primary ideal of A. Let M be a finite generated A-module with* $\dim M = d$. *Then*

$$\operatorname{hdeg}(M) \leqslant \operatorname{hdeg}(I, M) \leqslant n(I)^d \operatorname{hdeg}(M).$$

Proof. It suffices to prove the case A is a homomorphic image of a Gorenstein ring S.

If $d = 0$ then $\operatorname{hdeg}(M) = \operatorname{hdeg}(I, M) = \ell(M)$.

If $d \geqslant 1$, we put $M_i := \operatorname{Ext}_S^{n-i}(M, S)$, where $n = \dim S$. It is well known that $\dim M_i \leqslant i$. By induction on d we may assume that

$$\operatorname{hdeg}(M_i) \leqslant \operatorname{hdeg}(I, M_i) \leqslant n(I)^i \operatorname{hdeg}(M_i)$$

for $i = 0, ..., d-1$. Then

$$\begin{aligned}\operatorname{hdeg}(M) &= e(M) + \sum_{i=0}^{d-1} \binom{d-1}{i} \operatorname{hdeg}(M_i)\\ &\leqslant e(I, M) + \sum_{i=0}^{d-1} \binom{d-1}{i} \operatorname{hdeg}(I, M_i)\\ &= \operatorname{hdeg}(I, M).\end{aligned}$$

On the other hand, by Lemma 2·2 we have $e(I, M) \leqslant n(I)^d e(M)$. Therefore

$$\begin{aligned}\operatorname{hdeg}(I, M) &= e(I, M) + \sum_{i=0}^{d-1} \binom{d-1}{i} \operatorname{hdeg}(I, M_i)\\ &\leqslant n(I)^d e(M) + \sum_{i=0}^{d-1} \binom{d-1}{i} n(I)^i \operatorname{hdeg}(M_i)\\ &\leqslant n(I)^d [e(M) + \sum_{i=0}^{d-1} \binom{d-1}{i} \operatorname{hdeg}(M_i)]\\ &= n(I)^d \operatorname{hdeg}(M).\end{aligned}$$

Now we are able to give an explicit bound for the Castelnuovo–Mumford regularity of $G_I(M)$ in terms of $n(I)$.

THEOREM 2·4. *Let $(A, \mathfrak{m})$ be a Noetherian local ring, I an $\mathfrak{m}$-primary ideal of A and M a finitely generated A-module with $d =$* $\dim M \geqslant 1$. *Then*:

(i) $\operatorname{reg}(G_I(M)) \leqslant n(I) \operatorname{hdeg}(M) - 1$, if $d = 1$;

(ii) $\operatorname{reg}(G_I(M)) \leqslant 2^{(d-1)!} \operatorname{hdeg}(M)^{3(d-1)!-1} n(I)^{3d!-d} - 1$, if $d \geqslant 2$.

Proof. Applying Theorem 2·1 for $D(I, M) = \operatorname{hdeg}(I, M)$ we have

(i) $\operatorname{reg}(G_I(M)) \leqslant \operatorname{hdeg}(I, M) - 1$, if $d = 1$,

(ii) $\operatorname{reg}(G_I(M)) \leqslant 2^{(d-1)!} \operatorname{hdeg}(I, M)^{3(d-1)!-1} - 1$, if $d \geqslant 2$.

By Lemma 2·3, this implies

(i) $\operatorname{reg}(G_I(M)) \leqslant n(I) \operatorname{hdeg}(M) - 1$, if $d = 1$,

(ii) $\operatorname{reg}(G_I(M)) \leqslant 2^{(d-1)!} \operatorname{hdeg}(M)^{3(d-1)!-1} n(I)^{3d!-d} - 1$, if $d \geqslant 2$.

Remark. The bound is the best possible in the case $d = 1$. For example, if $M = A$ is a regular local ring and $I = \mathfrak{m}$, then we always have $\operatorname{reg}(G_I(M)) = 0$ ($n(\mathfrak{m}) = \operatorname{hdeg}(A) = 1$).

Recall that the postulation number $\rho_M(I)$ of M with respect to I is the least integer m such that $H_M(n) = P_M(n)$ for $n \geqslant m$. We denote by $h_{G_I(M)}(n)$ the Hilbert function and by $p_{G_I(M)}(n)$ the Hilbert polynomial of $G_I(M)$. Put $r := \operatorname{reg}(G_I(M))$. By [**2**, theorem 17·1·6]),

$$h_{G_I(M)}(n) - p_{G_I(M)}(n) = \sum_{i=0}^{d} \ell(H^i_{G_I(A)_+}(G_I(M))).$$

Hence $h_{G_I(M)}(n) = p_{G_I(M)}(n)$ for all $n \geqslant r+1$. We have

$$H_M(n) = \ell(M/I^{n+1}M) = \sum_{i=0}^{r} h_{G_I(M)}(i) + \sum_{i=r+1}^{n} h_{G_I(M)}(i)$$

for $n \geqslant r+1$. From this it follows that $H_M(n) = P_M(n)$ for all $n \geqslant r$. Thus $\rho_M(I) \leqslant \operatorname{reg}(G_I(M))$. Therefore, we obtain the following consequence.

COROLLARY 2·5. *Let M be an arbitrary finitely generated A-module with $d = \dim(M) \geqslant 1$. Then*

(i) $\rho_M(I) \leqslant n(I)\operatorname{hdeg}(M) - 1$, *if* $d = 1$,
(ii) $\rho_M(I) \leqslant 2^{(d-1)!}\operatorname{hdeg}(M)^{3(d-1)!-1} n(I)^{3d!-d} - 1$, *if* $d \geqslant 2$.

Remark. In [**11**, corollary 3], Schwartz proved the existence of a bound for the postulation number of A with respect to I in terms of $n(I)$ under the assumption that the characteristic of the residue field is 0. He used the method of GrRbner bases which can not be applied to study the general case. Let $R_I(A)$ be the Rees algebra of A with respect to Ooishi [**7**, lemma 4·8] proved that $\operatorname{reg}(G_I(A)) = \operatorname{reg}(R_I(A))$. Represent $R_I(A) = A[T]/J$, where $A[T]$ is a polynomial ring and J is a homogeneous ideal of $A[T]$. The relation type $\operatorname{retype}(I)$ of I is defined as the largest degree of the minimal generators of J. It is known [**15**, corollary 1·3 and proposition 4·1] that $\operatorname{retype}(I) \leqslant \operatorname{reg}(R_I(A)) + 1$. Therefore, we obtain the following bounds for the relation type of I in terms of $n(I)$.

COROLLARY 2·6. *Let* $(A, \mathfrak{m})$ *be a noetherian local ring with* $d = \dim(A) \geqslant 1$ *and I an* $\mathfrak{m}$*-primary ideal. Then*

(i) $\operatorname{retype}(I) \leqslant n(I)\operatorname{hdeg}(A)$, *if* $d = 1$,
(ii) $\operatorname{retype}(I) \leqslant 2^{(d-1)!}\operatorname{hdeg}(A)^{3(d-1)!-1} n(I)^{3d!-d}$, *if* $d \geqslant 2$.

Recall that an ideal $J \subseteq I$ is called a reduction of I if $I^{n+1} = JI^n$ for $n \gg 0$. If J is a reduction of I and no other reduction of I is contained in J, then J is said to be a minimal reduction of I. If J is a reduction of I, then the reduction number of I with respect to J, $r_J(I)$, is given by

$$r_J(I) := \min\{\, n \mid I^{n+1} = JI^n\}.$$

The reduction number of I, denoted $r(I)$, is given by

$$r(I) := \min\{r_J(I) \mid J \text{ is a minimal reduction of } I\}.$$

By [**15**, proposition 3·2], $r(I) \leqslant \operatorname{reg}(G_I(A))$. This gives the following consequence.

COROLLARY 2·7. *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring with* $d = \dim A \geqslant 1$ *and I an* $\mathfrak{m}$*-primary ideal of A. Then*:

(i) $r(I) \leqslant n(I)\operatorname{hdeg}(A) - 1$, *if* $d = 1$;
(ii) $r(I) \leqslant 2^{(d-1)!}\operatorname{hdeg}(A)^{3(d-1)!-1} n(I)^{3d!-d} - 1$, *if* $d \geqslant 2$.

Remark. In [**11**, corollary 4], Schwartz could establish the existence of a bound for the reduction number of I in terms of $n(I)$ only for a Cohen–Macaulay ring A in the characteristic zero case. Vasconcelos [**19**, theorem 2·45] also gave a bound for the reduction number of I in terms of $e(I, A)$ and $n(I)$ for a Cohen–Macaulay ring A.

3. *Bounds for the Hilbert coefficients*

Throughout this section let $(A, \mathfrak{m})$ be a Noetherian local ring and I an $\mathfrak{m}$-primary ideal of A. Let M be a finitely generated A-module. Once we have a bound for the postulation number of M with respect to I, we can derive a bound for the Hilbert coefficients of I following a method proposed by Vasconcelos [**10**] (see [**8**, **9**] for the case $M = A$ and $I = \mathfrak{m}$).

THEOREM 3·1. *Let $(A, \mathfrak{m})$ be a Noetherian local ring and I an $\mathfrak{m}$-primary ideal of A. Let M be a finite generated A-module with* $\dim(M) \geqslant 1$ *and $D(I, M)$ an arbitrary extended degree of M with respect to I. Then*:

(i) $|e_1(I, M)| \leqslant D(I, M)[D(I, M) - 1]$;

(ii) $|e_i(I, M)| \leqslant (i + 1)2^{i!+2} D(I, M)^{3i!-i+1} - 1$ *if* $i \geqslant 2$.

Proof.

If $d = 1$ then $\operatorname{reg}(G_I(M)) \leqslant D(I, M) - 1$, by Theorem 2·1. This implies

$$\ell(M/I^{r+1}M) = (r + 1)e(I, M) - e_1(I, M),$$

where $r = \operatorname{reg}(G_I(M))$. Therefore,

$$\begin{aligned} |e_1(I, M)| &= |(r + 1)e(I, M) - \ell(M/I^{r+1}M)| \\ &\leqslant |(r + 1)e(I, M) - (r + 1)| \\ &\leqslant (r + 1)[e(I, M) - 1] \\ &\leqslant D(I, M)[D(I, M) - 1] \\ &= D(I, M)[D(I, M) - 1]. \end{aligned}$$

If $d \geqslant 2$, without loss of generality we may further assume that the residue field of A is infinite. Then we may choose $x \in I \backslash \mathfrak{m}I$ such that its initial form x^* is a $G_I(M)$-filter-regular element. We have $\dim M/xM = d - 1$. By induction we may assume that

(i′) $|e_1(I, M/xM)| \leqslant D(I, M/xM)[D(I, M/xM) - 1]$,

(ii′) $|e_i(I, M/xM)| \leqslant i2^{(i-1)!+2} D(I, M/xM)^{3(i-1)!-i+2} - 1$ if $i = 2, ..., d - 1$.

Since $e_i(I, M) = e_i(I, M/xM)$ for $i = 0, ..., d - 1$ and $D(I, M/xM) \leqslant D(I, M)$, this implies

$|e_1(I, M)| \leqslant D(I, M)[D(I, M) - 1]$,

$|e_i(I, M)| \leqslant (i + 1)2^{i!+2} D(I, M)^{3i!-i+1} - 1$, $i = 2, ..., d - 1$.

It remains to prove the bound for $e_d(I, M)$. We have

$$(-1)^d e_d(I, M) = P_M(m) - \sum_{i=0}^{d-1} (-1)^i e_i(I, M) \binom{m + d - i}{d - i}$$

for all $m \geqslant 0$. Put

$$m := 2^{(d-1)!} D(I, M)^{3(d-1)!-1} - 1.$$

Then $m \geqslant \operatorname{reg} G_I(M)$, by Theorem 2·1. Thus $H_M(m) = P_M(m)$. By [**8**, theorem 2·1] or [**5**, theorem 3·6],

$$H_M(m) = \ell(M/I^{m+1}M) \leqslant D(I, M)\left[\binom{m+d-1}{d} + \binom{m+d-1}{d-1}\right]$$
$$= D(I, M)\binom{m+d}{d}.$$

Therefore

$$P_M(m) \leqslant D(I, M)\binom{m+d}{d}.$$

Then

$$|e_d(I, M)| = \left|P_M(m) - \sum_{i=0}^{d-1}(-1)^i e_i(I, M)\binom{m+d-i}{d-i}\right|$$
$$\leqslant D(I, M)\binom{m+d}{d} + \sum_{i=0}^{d-1}|e_i(I, M)|\binom{m+d-i}{d-i}$$
$$= [D(I, M) + e(I, M)]\binom{m+d}{d} + \sum_{i=1}^{d-1}|e_i(I, M)|\binom{m+d-i}{d-i}$$
$$\leqslant 2D(I, M)\binom{m+d}{d} + \sum_{i=1}^{d-1}|e_i(I, M)|\binom{m+d-i}{d-i}.$$

It is easily seen that

$$\binom{m+d-i}{d-i} \leqslant (d-i+1)m^{d-i} - 1$$

for $m \geqslant 1$. Then we obtain

$$|e_d(I, M)| \leqslant 2(d+1)D(I, M)m^d + d|e_1(I, M)|m^{d-1} + \sum_{i=2}^{d-1}(d-i+1)|e_i(I, M)|m^{d-i}.$$

Since $2 \leqslant i \leqslant d-1$, we have

$$e_1(I, M) \leqslant D(I, M)[D(I, M) - 1] \leqslant m,$$

$$e_i(I, M) \leqslant (i+1)2^{i!+2}D(I, M)^{3i!-i+1} - 1 \leqslant 4(i+1)D(I, M)m.$$

We thus get

$$|e_d(I, M)| \leqslant 4(d+1)D(I, M)[(m+1)^d - 1]$$
$$= 4(d+1)D(I, M)[2^{d!}D(I, M)^{3d!-d} - 1]$$
$$\leqslant (d+1)2^{d!+2}D(I, M)^{3d!-d+1} - 1.$$

COROLLARY 3·2. *Given two positive integers d and q. There exist only a finite number of Hilbert-Samuel functions for a A-module M with respect to an $\mathfrak{m}$-primary ideal I of the local ring A such that* $\dim M = d$ *and* $D(I, M) \leqslant q$.

Proof. We have $P_M(n) = \ell(M/I^{n+1}M)$ for $n > \operatorname{reg}(G_I(M))$. By Theorem 3·1, there are only a finite number of polynomials $P_M(n)$. On the other hand, by Theorem [**8**, theorem 2·1] or [**5**, theorem 3·6], there are only a finite number of possibilities for $\ell(M/I^{n+1}M)$ for a fixed n. Hence finiteness of the number of the possibilities for the function $\ell(M/I^{n+1}M)$ follows from the finiteness of possibilities for reg $(G_I(M))$ which have been proved in Theorem 2·1.

Now we are able to give a bound for the Hilbert coefficients of M with respect to I in terms of the degree of nilpotency of I.

THEOREM 3·3. *Let $(A, \mathfrak{m})$ be a Noetherian local ring and I an $\mathfrak{m}$-primary ideal of A. Let M be a finitely generated A-module with* $\dim(M) \geqslant 1$. *Then*:

(i) $e(I, M) \leqslant e(M)n(I)^d$;

(ii) $|e_1(I, M)| \leqslant \operatorname{hdeg}(M)n(I)^d[\operatorname{hdeg}(M)n(I)^d - 1]$;

(iii) $|e_i(I, M)| \leqslant (i+1)2^{2i!+2} \operatorname{hdeg}(M)^{3i!-i+1}n(I)^{3di!-di+d} - 1, \quad \text{if } i \geqslant 2.$

Proof. We only need to prove (ii) and (iii). Applying Theorem 3·1 we get

$$|e_1(I, M)| \leqslant \operatorname{hdeg}(I, M)[\operatorname{hdeg}(I, M) - 1],$$
$$|e_i(I, M)| \leqslant (i+1)2^{i!+2} \operatorname{hdeg}(I, M)^{3i!-i+1} - 1 \text{ if } i \geqslant 2.$$

By Lemma 2·3, this implies

$$|e_1(I, M)| \leqslant \operatorname{hdeg}(M)n(I)^d[\operatorname{hdeg}(M)n(I)^d - 1],$$
$$|e_i(I, M)| \leqslant (i+1)2^{2i!+2} \operatorname{hdeg}(M)^{3i!-i+1}n(I)^{3di!-di+d} - 1 \text{ if } i \geqslant 2,$$

COROLLARY 3·4. *Let M be a finitely generated A-module with* $\dim(M) = d \geqslant 1$ *and r a positive integer. There exist only a finite number of Hilbert–Samuel functions for a module M with respect to an $\mathfrak{m}$-primary ideal I such that $n(I) \leqslant r$.*

Proof. By definition, the homological degree hdeg (M) of M is determinate. By an argument analogous to that for the proof of Corollary 3·2 we obtain the conclusion.

Remark. In [**11**, theorem 1], Schwartz proved that there are only a finite number of Hilbert-Samuel functions $\ell(A/I^n)$ with fixed degree of nilpotency $n(I)$ in the characteristic zero case.

Acknowledgements. The author is grateful to Prof. N. V. Trung for his advice and encouragement.

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