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MODULES COINVARIANT UNDER THE IDEMPOTENT ENDOMORPHISMS OF THEIR COVERS

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Abstract: We study the modules that are coinvariant under the idempotent endomorphisms of their covers. Some generalizations of discrete and continuous modules are introduced and inspected on using the theory of covers and envelopes of modules. By way of application, we consider the cases of flat covers, injective envelopes, and pure injective envelopes.

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1. Introduction

Many important classes of the modules close to the injective (projective) may be defined by injective envelopes (projective covers). The quasi-injective modules were introduced in [1] as those invariant under the endomorphisms of their injective envelopes. In the same paper, it was shown that a module M is quasi-injective if and only if each homomorphism from a submodule of M to M is extended to an endomorphism of M. A module is *automorphism-invariant* provided that it is invariant under the automorphisms of its injective envelope. The automorphism-invariant modules over finite-dimensional algebras were first studied by Dickson and Fuller in [2]. The notion of pseudoinjective module was introduced in [3]; i.e., such a module in which every monomorphism from a submodule of M to M is extended to an endomorphism of M. In [4], it was shown that M is pseudoinjective if and only if M is an automorphism-invariant module. The dual notion to an automorphism-invariant module is that of an automorphism-coinvariant (or dual automorphism-invariant) module. This notion was recently studied in [5–7].

The continuous modules and their extensions, the so-called quasicontinuous modules, were introduced and studied in [8–11] as the module analogs of continuous and quasicontinuous rings which were considered by Utumi in [12]. It was shown in [13] that a module M is quasicontinuous if and only if M is invariant under the idempotent endomorphisms of the injective envelope of M. Many important properties of continuous and quasicontinuous modules and their duals are reflected in [14–19].

The general theory of modules invariant or coinvariant under the automorphisms of their envelopes or covers, respectively, was recently developed in [20–22]. The theory of modules invariant under the idempotent endomorphisms of their envelopes was studied in [23].

In Section 2, we consider the modules coinvariant under the idempotent endomorphisms of their covers. Given an arbitrary class $\mathscr{X}$ of right R-modules closed under the isomorphic images, we introduce and study the notion of lifting $\mathscr{X}$-module. In the case when R is a perfect right ring and $\mathscr{X}$ is the class of projective right R-modules, the class of lifting $\mathscr{X}$-modules coincides with the class of right lifting R-modules. In Section 3, we show that the continuous (discrete) modules may be defined by the injective (projective) envelopes (covers). This fact allows us to give the definitions of $\mathscr{X}$-continuous module and

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$\mathscr{X}$-discrete module which are the natural and wide generalizations of the notions of continuous module and discrete module. We study the general properties of the endomorphism rings of $\mathscr{X}$-continuous and $\mathscr{X}$-discrete modules. We show that every $\mathscr{X}$-continuous ($\mathscr{X}$-discrete) module has the finite exchange property. By way of application, we address the cases of projective covers, injective envelopes, and pure injective envelopes.

The Jacobson radical of a ring R is denoted by $J(R)$. The fact that N is a submodule of M (a small submodule and an essential submodule) is denoted by $N \leq M$ (respectively, by $N \ll M$ and $N \leq_e M$). The Jacobson radical of a right R-module M is denoted by $J(M)$.

We use the standard notions and facts of ring and module theories (for example, see [18, 24–26]).

2. $\mathscr{X}$-Idempotent Coinvariant Modules

We assume that $\mathscr{X}$ is a class of right R-modules which is closed under the isomorphic images and direct summands. A homomorphism $g : X \to M$ of right R-modules is an *$\mathscr{X}$-cover* of a module M provided that

(1) $X \in \mathscr{X}$; and, for every homomorphism $g' : X' \to M$ with $X' \in \mathscr{X}$, there exists a homomorphism $h : X' \to X$ such that $g' = gh$;

(2) $g = gh$ implies that h is an automorphism for every endomorphism $h : X \to X$.

A module M is a *lifting module* provided that for every submodule N of M there are submodules M_1 and M_2 of M satisfying $M = M_1 \oplus M_2$, $M_1 \leq N$, and $M_2 \cap N \ll M_2$.

A module M is a *$D3$-module* if $X \cap Y$ is a direct summand of M for all direct summands X and Y of M such that $X + Y = M$. A module M is *quasidiscrete* provided that M is a lifting module and a $D3$-module simultaneously.

Proposition 1 [16, Proposition 4.45]. *Let $u : P \to M$ be a projective cover of a module M. The following are equivalent*:

(1) *M is a quasidiscrete module*;

(2) *M is an idempotent coinvariant module; i.e., $\alpha(\mathrm{Ker}(u)) \subseteq \mathrm{Ker}(u)$ for every idempotent endomorphism α of P.*

Let M be a right R-module. A module M is *$\mathscr{X}$-idempotent coinvariant* provided that there exists an $\mathscr{X}$-cover $u : X \to M$ such that for every idempotent $g \in \mathrm{End}(X)$ there is an endomorphism $f : M \to M$ such that the diagram commutes:

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "g"'] & M \arrow[d, dotted, "f"] \\ X \arrow[r, "p"'] & M . \end{tikzcd}$$

Lemma 2. *Let $p : X \to M$ be an epimorphic $\mathscr{X}$-cover of a right R-module M. If M is an $\mathscr{X}$-idempotent coinvariant module then for every idempotent $g^2 = g \in \mathrm{End}(X)$ there is a unique homomorphism $f \in \mathrm{End}(M)$ such that $fp = pg$ and $f^2 = f$.*

Proof. There are $f, f' \in \mathrm{End}(M)$ satisfying $fp = pg$ and $f'p = p(1-g)$. Then $f'fp = f'pg = 0$. Since p is an epimorphism, $f'f = 0$. As $p = pg + p(1-g) = f'p + fp = (f' + f)p$, we have $\mathrm{id} = f' + f$. Thus, $f = f^2 \in \mathrm{End}(M)$. Since p is an epimorphism, f is unique. $\square$

Lemma 3. *Let $p : X \to M$ be an epimorphic $\mathscr{X}$-cover of M. The following are equivalent*:

(1) *M is an $\mathscr{X}$-idempotent coinvariant module*;

(2) *$g(\mathrm{Ker}(p)) \leq \mathrm{Ker}(p)$ for every idempotent endomorphism of X.*

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1): Assume that $g = g^2 \in \mathrm{End}(X)$. Then $g(\mathrm{Ker}(p)) \leq \mathrm{Ker}(p)$. Consider the homomorphism $\psi : X/g(\mathrm{Ker}(p)) \to M$ defined as $\psi(x + g(\mathrm{Ker}(p)) = p(x)$ for all $x \in X$. Since p is an epimorphism, for

every $m \in M$ there is $x \in X$ such that $m = p(x)$. Consider the mapping

$$\phi : M \to X/g(\mathrm{Ker}(p)), \quad m \mapsto g(x) + g(\mathrm{Ker}(p)).$$

It is easy to see that ϕ is a homomorphism. Put $f = \psi\phi : M \to M$. Then

$$fp(x) = \psi\phi(p(x)) = \psi(g(x) + g(\mathrm{Ker}(p)) = pg(x)$$

for every $x \in X$. Hence, $fp = pg$. $\square$

Corollary 4. *Let $p : X \to M$ be an epimorphic $\mathscr{X}$-cover of M. The following are equivalent*:
(1) *M is an $\mathscr{X}$-idempotent coinvariant module*;
(2) *if $X = \bigoplus_I X_i$ then* $\mathrm{Ker}(p) = \bigoplus_I (X_i \cap \mathrm{Ker}(p))$;
(3) *if $X = X_1 \oplus X_2$ then* $\mathrm{Ker}(p) = (X_1 \cap \mathrm{Ker}(p)) \oplus (X_2 \cap \mathrm{Ker}(p))$;
(4) *if $e \in \mathrm{End}(X)$ is an idempotent then* $\mathrm{Ker}(p) = e(\mathrm{Ker}(p)) \oplus (1 - e)(\mathrm{Ker}(p))$.

Lemma 5. *Let M be a module, and let N be a direct summand of M. If M is an $\mathscr{X}$-idempotent coinvariant module and N possesses an $\mathscr{X}$-cover then N is an $\mathscr{X}$-idempotent coinvariant module.*

Proof. Let $p : X \to M$ and $p_1 : X_1 \to N$ be some $\mathscr{X}$-covers, let $\pi : M \to N$ be a projection, and let $\iota : N \to M$ be an embedding. Consider an arbitrary idempotent endomorphism g_1 of X_1. There are homomorphisms $h_1 : X_1 \to X$ and $h_2 : X \to X_1$ satisfying $ph_1 = \iota p_1$ and $p_1 h_2 = \pi p$. Hence, $p_1 h_2 h_1 = p_1$, and $h_2 h_1$ is an isomorphism. Then $h(h_2 h_1) = \mathrm{id}_{X_1}$ for some homomorphism $h : X_1 \to X_1$. Let $g = h_1(g_1 h)h_2 : X \to X$. Then g is an idempotent endomorphism of X. Since M is an $\mathscr{X}$-idempotent coinvariant module, there is a homomorphism $f : M \to M$ such that $fp = pg$. Let $f_1 = \pi f \iota$. Then

$$f_1 p_1 = \pi f \iota p_1 = \pi f p h_1 = \pi p g h_1 = \pi p h_1 (g_1 h) h_2 h_1 = \pi p h_1 g_1 = p_1 h_2 h_1 g_1 = p_1 g_1.$$

Thus, N is an $\mathscr{X}$-idempotent coinvariant module. $\square$

Let M_1 and M_2 be some right R-modules. A module M_1 is *$\mathscr{X}$-M_2-projective* provided that there exist $\mathscr{X}$-covers $p_1 : X_1 \to M_1$ and $p_2 : X_2 \to M_2$ such that for every homomorphism $g : X_1 \to X_2$ there is a homomorphism $f : M_1 \to M_2$ such that the diagram commutes:

$$\begin{array}{ccc} X_1 & \xrightarrow{p_1} & M_1 \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ X_2 & \xrightarrow[p_2]{} & M_2 \end{array} .$$

If M is $\mathscr{X}$-M-projective then M is *$\mathscr{X}$-endomorphism coinvariant*. The two right R-modules M_1 and M_2 are *mutually $\mathscr{X}$-projective* provided that M_1 is $\mathscr{X}$-M_2-projective and M_2 is $\mathscr{X}$-M_1-projective.

Lemma 6. *Let M_1 and M_2 be mutually $\mathscr{X}$-projective right R-modules, and let $p_1 : X_1 \to M_1$ and $p_2 : X_2 \to M_2$ be epimorphic $\mathscr{X}$-covers. If $X_1 \simeq X_2$ then $M_1 \simeq M_2$.*

Proof. Let $g : X_1 \to X_2$ be an isomorphism. By hypothesis, there are homomorphisms $f_1 : M_1 \to M_2$ and $f_2 : M_2 \to M_1$ such that $f_1 p_1 = p_2 g$ and $f_2 p_2 = p_1 g^{-1}$. Then $f_1 f_2 p_2 = p_2$ and $f_2 f_1 p_1 = p_1$. Hence, $f_1 f_2 = \mathrm{id}_{M_2}$ and $f_2 f_1 = \mathrm{id}_{M_1}$. $\square$

Proposition 7. *Assume that a module M_2 possesses an epimorphic $\mathscr{X}$-cover $p_2 : X_2 \to M_2$ and each quotient module M_1/A of M_1 possesses an $\mathscr{X}$-cover $p_A : X_A \to M_1/A$ such that for every natural homomorphism $f : M_1 \to M_1/A$ there exists a split epimorphism $\psi : X_1 \to X_A$ satisfying $p_A\psi = fp_1$, where $p_1 : X_1 \to M_1$ is an $\mathscr{X}$-cover. Then if M_2 is M_1-$\mathscr{X}$-projective then M_2 is M_1-projective.*

Proof. Let A be a submodule of $M = M_1 \oplus M_2$ such that $M = A + M_1$. It is easy to notice that there is a homomorphism $g : M_2 \to M_1/(A \cap M_1)$ such that $g(m_2) = m_1 + A \cap M_1$ if $a = m_1 + m_2$, where

$m_1 \in M_1$, $m_2 \in M_1$, and $a \in A$. By hypothesis, we have the commutative diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\psi} & X_1' \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_1'} \\ M_1 & \xrightarrow[\pi]{} & M_1/(A \cap M_1) \end{array}$$

for an epimorphism $\psi : X_1 \to X_1'$, where $p_1' : X_1' \to M_1/(A \cap M_1)$ is an $\mathscr{X}$-envelope, π is the natural homomorphism, and $\psi\iota = 1_{X_1'}$ for some homomorphism $\iota : X_1' \to X_1$.

By the definition of $\mathscr{X}$-cover, there exists a homomorphism $f : X_2 \to X_1'$ such that the diagram commutes:

$$\begin{array}{ccc} X_2 & \xrightarrow{f} & X_1' \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_1'} \\ M_2 & \xrightarrow[g]{} & M_1/(A \cap M_1). \end{array}$$

Since M_2 is $\mathscr{C}$-M_1-projective, there is a homomorphism $\phi : M_2 \to M_1$ such that the diagram commutes:

$$\begin{array}{ccc} X_2 & \xrightarrow{\iota f} & X_1 \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_1} \\ M_2 & \xrightarrow[\phi]{} & M_1 . \end{array}$$

Given $b \in M_2$, there is $x \in X_2$ such that $b = p_2(x)$. Then

$$\phi(b) = \phi p_2(x) = p_1 \iota f(x),$$

$$g(b) = g p_2(x) = p_1' f(x) = p_1' \psi \iota f(x) = \pi p_1 \iota f(x) = \pi\phi(b).$$

Put $C = \{\phi(m_2) + m_2 \mid m_2 \in M_2\} \leq M$. Then $M = M_1 \oplus C$, and $C \leq A$. Hence, M_2 is M_1-projective by [15, 4.12]. $\square$

Proposition 8. *Let $p_1 : X_1 \to M_1$ and $p_2 : X_2 \to M_2$ be some epimorphic $\mathscr{X}$-covers, and* $\operatorname{Ker}(p_1) \ll X_1$. *If M_1 is M_2-projective then M_1 is M_2-$\mathscr{X}$-projective.*

Proof. Let $f : X_1 \to X_2$ be a module homomorphism. Without loss of generality, we may assume that $M_2 = X_2/\operatorname{Ker}(p_2)$, and $p_2 : X_2 \to X_2/\operatorname{Ker}(p_2)$ is the natural homomorphism. Put $N = \operatorname{Ker}(p_2) + f(\operatorname{Ker}(p_1))$. Since $f(\operatorname{Ker}(p_1)) \subseteq N$; therefore, $\pi p_2 f = f_1 p_1$ for some homomorphism $f_1 : M_1 \to X_2/N$, where $\pi : X_2/\operatorname{Ker}(p_2) \to X_2/N$ is the natural homomorphism. By hypothesis, $\pi f_2 = f_1$ for a homomorphism $f_2 : M_1 \to M_2$. By the definition of $\mathscr{X}$-cover, there is a homomorphism $g : X_1 \to X_2$ satisfying $p_2 g = f_2 p_1$. Then, for an arbitrary $x \in X_1$ there are $x_1 \in \operatorname{Ker}(p_1)$ and $x_2 \in \operatorname{Ker}(p_2)$ such that $(f-g)(x) = x_2 + f(x_1)$. Since $p_2(f-g)(x - x_1) = p_2(x_2 + f(x_1)) - p_2 f(x_1) = 0$; therefore, $x \in \operatorname{Ker}(p_1) + \operatorname{Ker}(p_2(f-g))$. Thus, $X_1 = \operatorname{Ker}(p_1) + \operatorname{Ker}(p_2(f-g)) = \operatorname{Ker}(p_2(f-g))$. Hence, $(f-g)(X_1) \subseteq \operatorname{Ker}(p_2)$. Since $g(\operatorname{Ker}(p_1)) \leq \operatorname{Ker}(p_2)$; therefore, $f(\operatorname{Ker}(p_1)) \leq \operatorname{Ker}(p_2)$. Then $p_2 f = f' p_1$ for a homomorphism $f' : M_1 \to M_2$. $\square$

The following corollary of Propositions 7 and 8 is important:

Corollary 9 [27]. *Let M and N be some right R-modules, and let $\pi_1 : P \to M$ and $\pi_2 : P' \to N$ be some projective covers of M and N, respectively. Then the following are equivalent*:

(1) *M is N-projective*;

(2) *for every homomorphism* $f : P \to P'$ *there is a homomorphism* $g : M \to N$ *such that the diagram commutes:*

$$\begin{array}{ccc} P & \xrightarrow{\pi_1} & M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ P' & \xrightarrow[\pi_2]{} & N. \end{array}$$

In particular, if $\pi : P \to M$ *is a projective cover of* M *then* M *is quasiprojective if and only if* $\mathrm{Ker}(\phi)$ *is a completely invariant submodule of* P.

Theorem 10. *Let* $M = M_1 \oplus M_2$ *be a module, and let* $p_1 : X_1 \to M_1$, $p_2 : X_2 \to M_2$, *and* $p_1 \oplus p_2 : X_1 \oplus X_2 \to M$ *be some* $\mathscr{X}$*-covers of right* R*-modules. If* M *is* $\mathscr{X}$*-idempotent coinvariant then* M_i *is* $\mathscr{X}$*-*M_j*-projective for every* $i \neq j$.

Proof. Let $g : X_1 \to X_2$ be a homomorphism. Define the homomorphism $g' : X_1 \oplus X_2 \to X_1 \oplus X_2$ by the matrix $g' = \begin{pmatrix} \mathrm{id}_{X_1} & 0 \\ g & 0 \end{pmatrix}$. Then ${g'}^2 = g'$. Since M is an $\mathscr{X}$-idempotent coinvariant module, $f'p = pg'$ for some $f' \in \mathrm{End}(M)$. Let $f = \pi_2 f' \iota_1$, where $\iota_1 : M_1 \to M$ is the natural embedding, and $\pi_2 : M \to M_2$ is the canonical projection. Then $fp_1 = p_2 g$. Thus, M_1 is $\mathscr{X}$-M_2-projective. □

Theorem 11. *Let* $M = \bigoplus_{i=1}^n M_i$ *be a module, and let* $p_i : X_i \to M_i$ *be some* $\mathscr{X}$*-covers. Then the following are equivalent*:

(1) $M_1 \oplus M_2 \oplus \cdots \oplus M_n$ *is an* $\mathscr{X}$*-endomorphism coinvariant module*;

(2) M_i *and* M_j *are mutually* $\mathscr{X}$*-projective for all* $i, j \in \{1, 2, \dots, n\}$.

Proof. It suffices to consider the case $n = 2$.

(1) ⇒ (2): Since every $\mathscr{X}$-endomorphism coinvariant module is $\mathscr{X}$-idempotent coinvariant, item (2) follows from Lemma 5 and Theorem 10.

(2) ⇒ (1): Assume that M_i is an $\mathscr{X}$-M_j-projective module for all $i, j \in \{1, 2\}$. By [24, Proposition 5.5.4], $p_1 \oplus p_2 : X_1 \oplus X_2 \to M_1 \oplus M_2$ is an $\mathscr{X}$-cover. Let g be an endomorphism of $X_1 \oplus X_2$, let $\iota_1 : X_1 \to X_1 \oplus X_2$ and $\iota_2 : X_2 \to X_1 \oplus X_2$ be some embeddings, and let $\pi_1 : X_1 \oplus X_2 \to X_1$ and $\pi_2 : X_1 \oplus X_2 \to X_2$ be the canonical projections. Since M_i and M_j are mutually $\mathscr{X}$-projective for all $i, j \in \{1, 2\}$; there is a homomorphism $f_{ji} : M_i \to M_j$ such that $p_j(\pi_j g \iota_i) = f_{ji} p_i$. Let $f : M_1 \oplus M_2 \to M_1 \oplus M_2$ be an endomorphism whose matrix is $\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$. Then $(p_1 \oplus p_2)g = f(p_1 \oplus p_2)$. Thus, $M = M_1 \oplus M_2$ is an $\mathscr{X}$-endomorphism coinvariant module. □

Corollary 12. *A module* M *is an* $\mathscr{X}$*-endomorphism coinvariant if and only if* $M \oplus M$ *is an* $\mathscr{X}$*-endomorphism coinvariant module.*

Corollary 13. *Let* $M = \bigoplus_{i=1}^n M_i$ *be a right module over a perfect right ring. The following are equivalent*:

(1) $M_1 \oplus M_2 \oplus \cdots \oplus M_n$ *is a quasiprojective module*;

(2) M_i *and* M_j *are mutually projective for all* $i, j \in \{1, 2, \dots, n\}$.

Corollary 14. *If* M *is a right module over a perfect right ring then* M *is a quasiprojective module if and only if* $M \oplus M$ *is a quasiprojective module.*

A module M is *pure infinite* provided that $M = M \oplus M$. If M is not isomorphic to a proper direct summand then M is *directly finite.*

Theorem 15. *Let* M *be an* $\mathscr{X}$*-idempotent coinvariant module, and let* $p : X \to M$ *be an epimorphic* $\mathscr{X}$*-cover of* M*. The following are equivalent*:

(1) M *is pure infinite if and only if* X *is pure infinite module*;

(2) *if* X *is a directly finite module then* M *is directly finite*;

(3) *if $\mathscr{X}$ is the class of projective modules and M is not directly finite then $M = M_1 \oplus M_2 \oplus M_3$, where $M_1 \cong M_2 \neq 0$.*

Proof. (1): ($\Rightarrow$) Assume that M is infinite. Then $M = M_1 \oplus M_2$, where $M_1 \simeq M_2 \simeq M$. Let $p_1 : X_1 \to M_1$ and $p_2 : X_2 \to M_2$ be $\mathscr{X}$-envelopes. Clearly, $X \simeq X_1 \oplus X_2$ and $X \simeq X_1 \simeq X_2$. Thus, X is pure infinite.

($\Leftarrow$) Assume that $X = X_1 \oplus X_2$ and $X_1 \simeq X_2 \simeq X$. Then $X_1 = e(X)$ and $X_2 = (1-e)(X)$ for some $e^2 = e \in \mathrm{End}(X)$. By Corollary 4 $\mathrm{Ker}(p) = e(\mathrm{Ker}(p)) \oplus (1-e)(\mathrm{Ker}(p))$. Then

$$X/\mathrm{Ker}(p) \simeq X_1/e(\mathrm{Ker}(p)) \oplus X_2/(1-e)(\mathrm{Ker}(p)).$$

Denote $M_1 = X_1/e(\mathrm{Ker}(p))$ and $M_2 = X_2/(1-e)(\mathrm{Ker}(p))$. Show that the natural homomorphisms $p_1 : X_1 \to M_1$ and $p_2 : X_2 \to M_2$ are $\mathscr{X}$-envelopes. Let $\iota : M_1 \to M_1 \oplus M_2$ and $\iota' : X_1 \to X_1 \oplus X_2$ be the natural embeddings, and let $\pi : M_1 \oplus M_2 \to M_1$ and $\pi' : X_1 \oplus X_2 \to X_1$ be the projections. Consider an arbitrary homomorphism $f : U \to M_1$, where $U \in \mathscr{X}$. Since $p_1 \oplus p_2 : X_1 \oplus X_2 \to M_1 \oplus M_2$ is an $\mathscr{X}$-envelope, $(p_1 \oplus p_2)g = \iota f$ for a homomorphism $g : U \to X_1 \oplus X_2$, and so $f = \pi\iota f = \pi(p_1 \oplus p_2)\iota'\pi' g = p_1\pi' g$. Assume that $p_1\alpha = p_1$ for a homomorphism $\alpha : X_1 \to X_1$. Then $(p_1 \oplus p_2)(\alpha \oplus 1_{X_2}) = p_1 \oplus p_2$. By the definition of $\mathscr{X}$-envelope, $\alpha \oplus 1_{X_2}$ is an isomorphism, and so α is an isomorphism as well. Thus, $p_1 : X_1 \to M_1$ is an $\mathscr{X}$-envelope. Analogously, $p_2 : X_2 \to M_2$ is an $\mathscr{X}$-envelope. By Theorem 10, M_1 and M_2 are mutually $\mathscr{X}$-projective. Then Lemma 6 implies $M_1 \simeq M_2 \simeq M$. Thus, M is pure infinite.

(2): Assume that M is not directly finite. Then $M = M_1 \oplus M_2$, where $M_1 \simeq M$ and $M_2 \neq 0$. It is easy to see that $X \simeq X_1 \oplus X_2$ and $X_1 \simeq X$. Thus, X is not directly finite.

(3): By (2), X is not directly finite. Then there are some submodules X_1, X_2, and X_3 of X such that $X = X_1 \oplus X_2 \oplus X_3$, where $X_1 \neq 0$ and $X_1 \simeq X_2$. The equalities $X_1 = e_1(X)$, $X_2 = e_2(X)$, and $X_3 = e_3(X)$ hold for some orthogonal idempotents $\{e_1, e_2, e_3\}$ in $\mathrm{End}(X)$. By Corollary 4

$$\mathrm{Ker}(p) = e_1(\mathrm{Ker}(p)) \oplus e_2(\mathrm{Ker}(p)) \oplus e_3(\mathrm{Ker}(p)).$$

Then

$$X/\mathrm{Ker}(p) \simeq X_1/e_1(\mathrm{Ker}(p)) \oplus X_2/e_2(\mathrm{Ker}(p)) \oplus X_3/e_3(\mathrm{Ker}(p)).$$

Denote $M_1 = X_1/e(\mathrm{Ker}(p))$, $M_2 = X_2/(1-e)(\mathrm{Ker}(p))$, and $M_3 = X_3/e_3(\mathrm{Ker}(p))$. Then $M \simeq M_1 \oplus M_2 \oplus M_3$. Since $\pi_i : X_i \to X_i/e_i(\mathrm{Ker}(p))$ is an $\mathscr{X}$-envelope for every $1 \leq i \leq 3$; therefore, $M_1 \simeq M_2$. $\square$

Let M be a right R-module. A module M is a *lifting $\mathscr{X}$-module* provided that there is an $\mathscr{X}$-cover $p : X \to M$ of M such that for every idempotent $g \in \mathrm{End}(X)$ there is an idempotent $f : M \to M$ such that $g(X) + \mathrm{Ker}(p) = p^{-1}(f(M))$.

The following is immediate:

Proposition 16. *Let $p : X \to M$ be an epimorphic $\mathscr{X}$-cover. If M is an $\mathscr{X}$-idempotent coinvariant module then M is a lifting $\mathscr{X}$-module.*

Proposition 17. *Let N be a direct summand of M. If M is a lifting $\mathscr{X}$-module that possesses an epimorphic $\mathscr{X}$-cover, and N has an $\mathscr{X}$-cover; then N is a lifting $\mathscr{X}$-module.*

Proof. Let $p_1 : X_1 \to N$ be an $\mathscr{X}$-cover. It is easy to see that X_1 is isomorphic to a direct summand K of X such that $p|_K : K \to N$ is an $\mathscr{X}$-cover of N. Thus, we may assume that $p_1 = p|_{X_1} : X_1 \to N$ is an $\mathscr{X}$-cover of N and X_1 is a direct summand of X. Let $g : X_1 \to X_1$ be an idempotent endomorphism of X_1. Consider the homomorphism $g' = \iota g \pi : X \to X$, where $\iota : X_1 \to X$ and $\pi : X \to X_1$ are embeddings. Then ${g'}^2 = g'$. Since M is a lifting $\mathscr{X}$-module, there is a homomorphism ${f'}^2 = f' : M \to M$ such that $g'(X) + \mathrm{Ker}(p) = p^{-1}(f'(M))$. Then $f'(M) = p(g(X_1)) = p_1(g(X_1)) \leq N$, and so $f'(M)$ is a direct summand of N. There is an idempotent homomorphism $f : N \to N$ such that $p_1(g(X_1)) = f'(M) = f(N)$. Then $g(X_1) + \mathrm{Ker}(p_1) = p_1^{-1}(f(N))$. Thus, N is a lifting $\mathscr{X}$-module. $\square$

Let $p : X \to M$ be an $\mathscr{X}$-cover of M, and let A be a submodule of M. A submodule A is *$\mathscr{X}$-coclosed* in M provided that $A = p(g(X))$ for an idempotent endomorphism $g \in \mathrm{End}(X)$.

Theorem 18. *Let $p : X \to M$ be an epimorphic $\mathscr{X}$-cover. The following are equivalent*:
(1) *M is a lifting $\mathscr{X}$-module*;
(2) *each $\mathscr{X}$-coclosed submodule of M is a direct summand of M.*

Proof. (1) $\Rightarrow$ (2): Let $U = p(g(X))$, where $g^2 = g \in \mathrm{End}(X)$. There exists an endomorphism $f^2 = f \in \mathrm{End}(M)$ such that $g(X) + \mathrm{Ker}(p) = p^{-1}(f(M))$. Hence, $U = p(g(X)) = f(M)$ is a direct summand of M.

(2) $\Rightarrow$ (1): Let g be an idempotent in $\mathrm{End}(X)$. By hypothesis, $U = p(g(X))$ is a direct summand of M. There exists a homomorphism $f^2 = f \in \mathrm{End}(M)$ such that $p(g(X)) = f(M)$. Then $g(X) + \mathrm{Ker}(p) = p^{-1}(f(M))$. Thus, M is a lifting $\mathscr{X}$-module. $\square$

If $\mathscr{X}$ is the class of projective right R-modules then the standard argument shows that a right module M over a perfect right ring R is a lifting $\mathscr{X}$-module if and only if M is a lifting module [28, Theorem 2.6].

By the end of this section we assume that all modules M under consideration possess $\mathscr{C}$-covers $p : X \to M$, where $\mathscr{C}$ is the class of modules which satisfies the conditions

(1) $\mathscr{C}$ is closed under isomorphisms;

(2) each quotient module M/A for M possesses an epimorphic $\mathscr{C}$-cover $p_{M/A} : X_{M/A} \to M/A$ such that $\mathrm{Ker}(p_{M/A}) \ll X_{M/A}$;

(3) for every direct summand N of M and every natural homomorphism $\pi : N \to N/A$, there exists a split epimorphism $\psi : X_N \to X_{N/A}$ such that the diagram commutes:

$$\begin{array}{ccc} X_N & \xrightarrow{\psi} & X_{N/A} \\ {\scriptstyle p_N}\downarrow & & \downarrow{\scriptstyle p_{N/A}} \\ N & \xrightarrow[\pi]{} & N/A\,. \end{array}$$

A module M is an SSP-*module* provided that the sum of two direct summands of M is a direct summand of M.

Proposition 19. *Let $M = M_1 \oplus M_2$ be a module, and let $p_i : X_i \to M_i$, $i = 1, 2$, and $p = p_1 \oplus p_2 : X_1 \oplus X_2 \to M$ be $\mathscr{C}$-covers. If X is an* SSP*-module and each $\mathscr{C}$-coclosed submodule $N \leq M$ satisfying either $N + M_1 = M$ or $N + M_2 = M$ is a direct summand of M; then M is a lifting $\mathscr{C}$-module.*

Proof. Let X be an SSP-module, and let $N = p(g(X))$ be a $\mathscr{C}$-coclosed submodule of M, where g is an idempotent of $\mathrm{End}(X)$. Then the submodule $H = g(X) + X_2$ is a direct summand of X. Hence, $p(H)$ is a $\mathscr{C}$-coclosed submodule of M. On the other hand, $X = H + X_1$, and so $M = p(H) + M_1$. By hypothesis, $p(H)$ is a direct summand of M. Then $M = p(H) \oplus H'$ for a submodule H' of M. It is easy to notice that $H' = p(X')$ for a direct summand X' of X such that $p|_{X'} : X' \to H'$ is a $\mathscr{C}$-cover. Since $X' + g(X)$ is a direct summand of X; therefore, $pg(X) \oplus H' = p(g(X) + X')$ is a $\mathscr{C}$-coclosed submodule of M. Since $M = p(g(X)) + p(X_2) + H'$, $M = [pg(X) \oplus H'] + M_2$. So, $pg(X) \oplus H'$ is a direct summand of M. Thus, $N = pg(X)$ is a direct summand of M. $\square$

Theorem 20. *The following are equivalent*:
(1) *M is a $\mathscr{C}$-idempotent coinvariant module*;
(2) *M is a lifting $\mathscr{C}$-module, and M_1 and M_2 are mutually $\mathscr{C}$-projective for every decomposition $M = M_1 \oplus M_2$*;
(3) *M is a lifting $\mathscr{C}$-module, and M_1 and M_2 are mutually projective for every decomposition $M = M_1 \oplus M_2$.*

Proof. (1) $\Rightarrow$ (2) follows from Theorem 10 and Proposition 16.

(2) $\Rightarrow$ (3) follows from Proposition 7.

(3) $\Rightarrow$ (1): Assume that $p : X \to M$ is an epimorphic $\mathscr{C}$-cover of M. Let g be an idempotent in $\mathrm{End}(X)$. Since M is a lifting $\mathscr{C}$-module, $A = p(g(X))$ and $B = p((1-g)(X)$ are direct summands

of M. Then $A+B=M$ and $M=B\oplus B'$ for some $B'\le M$. Since A is B-projective, there is a submodule $C\le A$ such that $M=B\oplus C$. Let $\pi: B\oplus C\to C$ be the canonical projection. For every $x\in X$ there are $x_1,y_1,y_2\in X$ such that $pg(y_1)+p(1-g)(y_2)\in C$ and

$$p(x)=p(1-g)(x_1)+pg(y_1)+p(1-g)(y_2)=pg(x)+p(1-g)(x).$$

Then

$$\begin{aligned}0&=p(1-g)(x_1)+p(1-g)(y_2)-p(1-g)(x)+pg(y_1)-pg(x)\\&=p[(1-g)(x_1)+(1-g)(y_2)-(1-g)(x)+g(y_1)-g(x)].\end{aligned}$$

Thus,

$$(1-g)(x_1)+(1-g)(y_2)-(1-g)(x)+g(y_1)-g(x)\in \operatorname{Ker}(p).$$

Hence, $g(y_1)-g(x)\in g(\operatorname{Ker}(p))$. Let $a\in\operatorname{Ker}(p)$ be such that $g(y_1)-g(x)=g(a)$. Since $pg(y_1)+p(1-g)(y_2)\in C\le A=pg(X)$; therefore, $(1-g)(y_2)\in(1-g)(\operatorname{Ker}(p))$, and $(1-g)(y_2)=(1-g)(b)$ for some $b\in\operatorname{Ker}(p)$. Then

$$\begin{aligned}(\pi p-pg)(x)&=pg(y_1)+p(1-g)(y_2)-pg(x)=p[g(y_1)-g(x)]+p(1-g)(y_2)\\&=p(g(a))+p(1-g)(b)=(\pi p-pg)(-a)+(\pi p-pg)(b).\end{aligned}$$

Hence, $X=\operatorname{Ker}(p)+\operatorname{Ker}(\pi p-pg)$. Since $\operatorname{Ker}(p)\ll X$, $\pi p-pg=0$. Thus, M is a $\mathscr{C}$-idempotent coinvariant module. $\square$

Corollary 21. *Let R be a perfect right ring. If M is a right R-module then the following are equivalent*:

(1) *M is a quasidiscrete module*;

(2) *M is a lifting module, and M_1 and M_2 are mutually projective for every decomposition $M=M_1\oplus M_2$.*

3. $\mathscr{X}$-Discrete and $\mathscr{X}$-Continuous Modules

Let M be a right R-module, let $p: X\to M$ be an $\mathscr{X}$-cover of M, and let $S=\operatorname{End}(X)$. If $pg_1=fp=pg_2$ for some homomorphisms $g_1,g_2\in S$ and $f\in\operatorname{End}(M)$ then $p(g_1-g_2)h=0$ for every $h\in S$, and so $p=p(1-(g_1-g_2)h)$. Then $1-(g_1-g_2)h$ is an automorphism. Thus, $g_1-g_2\in J(S)$, and the ring homomorphism $\Phi:\operatorname{End}(M)\to S/J(S)$ is defined, which acts by the rule $\Phi(f)=f'+J(S)$, where $f': X\to X$ is a homomorphism such that $pf'=fp$. Denote the kernel of Φ by $\nabla(M)=\operatorname{Ker}(\Phi)$. Then we have an embedding $\overline{\Phi}: M/\nabla(M)\to S/J(S)$. It is easy to notice that if $\mathscr{X}$ is the class of projective right R-modules then $\nabla(M)=\{f\in\operatorname{End}(M)\mid f(M)\ll M\}$.

Lemma 22. *Let R be a perfect right ring, let M be a quasidiscrete right R-module, and let $p: P\to M$ be a projective cover of M. The following are equivalent*:

(1) *M is a discrete module*;

(2) *if $pe_i=e_i'p$, $i=1,2$, for some idempotents $e_1,e_2\in\operatorname{End}(P)$ and $e_1',e_2'\in\operatorname{End}(M)$, the diagram commutes:*

$$\begin{tikzcd} e_1(P) \arrow[r, "\alpha"] \arrow[d, "p"'] & e_2(P) \arrow[d, "p"] \\ e_1'(M) \arrow[r, "\alpha'"'] & e_2'(M) \end{tikzcd}$$

for some homomorphisms α, α', and α is an isomorphism then α' is an isomorphism.

Proof. (1) $\Rightarrow$ (2): Let M be a discrete module, let $e_1,e_2\in\operatorname{End}(P)$ and $e_1',e_2'\in\operatorname{End}(M)$ be some idempotents, let $\alpha: e_1(P)\to e_2(P)$ be an isomorphism, let $\alpha: e_1'(M)\to e_2'(M)$ be a homomorphism, and $pe_i=e_i'p$, $i=1,2$, $p\alpha=\alpha'p$. Clearly, α' is an epimorphism. Since M is discrete, $e_1'(M)=\operatorname{Ker}(\alpha')\oplus N$ with $N\le M$. Since $p^{-1}_{|e_1(P)}(\operatorname{Ker}(\alpha'))=\operatorname{Ker}(p\alpha)$ is a small submodule of $e_1(P)$, we have $\operatorname{Ker}(\alpha')=0$.

(2) $\Rightarrow$ (1): Assume that M satisfies the hypotheses of (2). Let $e \in \mathrm{End}(M)$, and let $f : M \to eM$ be an epimorphism. Consider a projective envelope $p' : P' \to eM$ of eM. We have $p'\alpha = fp$ for some homomorphism $\alpha : P \to P'$. It is easy to notice that α is an epimorphism. Then $P = P_0 \oplus \mathrm{Ker}(\alpha)$, where $P_0 \leq P$. Since M is a quasidiscrete module, $M = p(P_0) \oplus p(\mathrm{Ker}(\alpha))$. By (2), $f_{|p(P_0)}$ is an isomorphism. Thus, f is a split epimorphism. $\square$

In this section we assume unless otherwise stated that $\mathscr{X}$ is the class of right R-modules which is closed under the isomorphisms, $p : X \to M$ is an epimorphic $\mathscr{X}$-cover of a right R-module M, and $S = \mathrm{End}(X)$ is a semiregular ring. By Lemma 2, for every idempotent $e^2 = e \in \mathrm{End}(X)$ there is a unique idempotent $f \in \mathrm{End}(M)$ such that $pe = fp$. In what follows, we denote this idempotent by $\hat{e}$.

Let $p : X \to M$ be an $\mathscr{X}$-cover of M. A module M is *$\mathscr{X}$-discrete* provided that

(1) M is an $\mathscr{X}$-idempotent coinvariant module;

(2) if for some idempotents $e_1, e_2 \in \mathrm{End}(X)$ and homomorphisms α and α' the diagram commutes:

$$\begin{array}{ccc} e_1(X) & \xrightarrow{\alpha} & e_2(X) \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ \hat{e}_1(M) & \xrightarrow[\alpha']{} & \hat{e}_2(M), \end{array}$$

while α is an isomorphism; then α' is an isomorphism.

Lemma 23. *If M is an $\mathscr{X}$-idempotent coinvariant module then all idempotents in* $\mathrm{End}(M)/\nabla(M)$ *are lifted modulo the ideal* $\nabla(M)$.

Proof. Let $s + \nabla(M)$ be an idempotent in $\mathrm{End}(M)/\nabla(M)$. Then $\overline{\Phi}(s + \nabla(M)) = s' + J(S)$ is an idempotent in $S/J(S)$, where $sp = ps'$. Since S is a semiregular ring, there is an idempotent ε in S such that $s' + J(S) = \varepsilon + J(S)$. Since M is $\mathscr{X}$-idempotent coinvariant, by Lemma 2 there is an idempotent e in $\mathrm{End}(M)$ satisfying $\overline{\Phi}(e + \nabla(M)) = \varepsilon + J(S)$. Hence, $s + \nabla(M) = e + \nabla(M)$. $\square$

Theorem 24. *If M is an $\mathscr{X}$-discrete module then* $\mathrm{End}(M)$ *is semiregular and* $J(\mathrm{End}(M)) = \nabla(M)$.

Proof. Let $T = \mathrm{End}(M)$ and $\nabla = \nabla(M)$. Consider an arbitrary endomorphism $\alpha : M \to M$. Then $p\beta = \alpha p$ for an endomorphism $\beta : X \to X$. Let $\overline{S} = S/J(S)$. Given $y \in S$, denote the coset $y + J(S)$ by $\bar{y}$. We use an analogous notation for the ring T/∇. Since $\overline{S}$ is a regular ring, there is $x \in S$ such that $\bar{\beta} = \bar{\beta}\bar{x}\bar{\beta}$. Let $\bar{e} = \bar{x}\bar{\beta}$ and $\bar{f} = \bar{\beta}\bar{x}$. Then $\bar{e}^2 = \bar{e}$, $\bar{f}^2 = \bar{f}$ and $\bar{\beta} = \bar{\beta}\bar{e}$, $\bar{f}\bar{\beta}\bar{e} = \bar{\beta}$. Since the idempotents in S are lifted modulo $J(S)$, we may assume without loss of generality that $e, f \in S$ are some idempotents. Since

$$(\bar{e}\bar{x}\bar{f})(\bar{f}\bar{\beta}\bar{e}) = \bar{e}\bar{x}(\bar{f}\bar{\beta}\bar{e}) = \bar{e}\bar{x}\bar{\beta} = \bar{e},$$

we have $(exf)(f\beta e) = e + j$ for some $j \in J(S)$. Thus, $(exf)(f\beta e) = e + eje$. Since $e + eje \in U(eSe)$, there is $x' \in S$ such that $(ex'f)(f\beta e) = e$. Hence, $f\beta e : e(X) \to f(X)$ is a monomorphism.

On the other hand, $(\bar{f}\bar{\beta}\bar{e})(\bar{e}\bar{x}\bar{f}) = \bar{\beta}(\bar{e}\bar{x}\bar{f}) = (\bar{\beta}\bar{e}\bar{x})\bar{f} = \bar{f}$. So there is $j' \in J(S)$ such that $(f\beta e)(exf) = f + j'$. Therefore, $(f\beta e)(ex''f) = f$ for some $x'' \in S$. Thus, $f\beta e : e(X) \to f(X)$ is an isomorphism. Since $\hat{e}p = pe$ and $\hat{f}p = pf$, the diagram commutes:

$$\begin{array}{ccc} e(X) & \xrightarrow{f\beta e} & f(X) \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ \hat{e}(M) & \xrightarrow[\hat{f}\alpha\hat{e}]{} & \hat{f}(M). \end{array}$$

Since M is an $\mathscr{X}$-discrete module, $\hat{f}\alpha\hat{e} : \hat{e}(M) \to \hat{f}(M)$ is an isomorphism. Let $\hat{e}\alpha'\hat{f} : \hat{f}(M) \to \hat{e}(M)$ be an inverse homomorphism to $\hat{f}\alpha\hat{e}$. Then $(\hat{f}\alpha\hat{e})(\hat{e}\alpha'\hat{f}) = \hat{f}$ and $(\hat{e}\alpha'\hat{f})(\hat{f}\alpha\hat{e}) = \hat{e}$.

Let $\gamma = \hat{e}\alpha'\hat{f} \in T$. Then

$$\overline{\Phi}(\overline{\alpha\gamma\alpha}) = \bar{\beta}\overline{\Phi}(\bar{\gamma})\bar{\beta} = \bar{\beta}\Phi(\gamma)\bar{f}\bar{\beta}\bar{e} = \bar{\beta}\Phi(\gamma)\Phi(\hat{f}\alpha\hat{e}) = \bar{\beta}\Phi(\gamma(\hat{f}\alpha\hat{e})) = \bar{\beta}\Phi(\hat{e}) = \bar{\beta}\bar{e} = \bar{\beta} = \overline{\Phi}(\bar{\alpha}).$$

Since $\overline{\Phi}$ is injective, $\overline{\alpha\gamma\alpha} = \bar{\alpha}$. Thus, T/∇ is a regular ring.

Show that $J(T) = \nabla$. Since T/∇ is regular, $J(T) \leq \nabla$. Prove an inverse inclusion. Let $\alpha \in \nabla$, and let $\beta : X \to X$ be a homomorphism satisfying $\alpha p = p\beta$. Then $0 = \overline{\Phi}(\alpha + \nabla) = \beta + J(S)$, and so $\beta \in J(S)$. Given an arbitrary $\gamma : M \to M$, let γ' be an endomorphism of X such that $\gamma p = p\gamma'$. Since $\overline{\Phi}(\alpha\gamma + \nabla) = \beta\gamma' + J(S) = 0$; therefore, $1_X - \beta\gamma'$ is an isomorphism. Since M is $\mathscr{X}$-discrete, $1_M - \alpha\gamma$ is an isomorphism for every $\gamma \in T$. Thus, $\alpha \in J(T)$. $\square$

Theorem 25. *If M is an $\mathscr{X}$-idempotent coinvariant module then M is $\mathscr{X}$-discrete if and only if $\nabla(M) = J(\mathrm{End}(M))$ and $\mathrm{End}(M)/\nabla(M)$ is a regular ring.*

Proof. Necessity follows from Theorem 24. Assume that $\nabla(M) = J(\mathrm{End}(M))$ and $\mathrm{End}(M)/\nabla(M)$ is regular. Show that M is $\mathscr{X}$-discrete. Let $T = \mathrm{End}(M)$ and $S = \mathrm{End}(X)$. Consider the commutative diagram

$$\begin{array}{ccc} e_1(X) & \xrightarrow{\alpha} & e_2(X) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \hat{e}_1(M) & \xrightarrow[\alpha']{} & \hat{e}_2(M)\,, \end{array}$$

where α is an isomorphism, while e_1 and e_2 are some idempotents in S. Show that α' is an isomorphism. Since α is an isomorphism, $\alpha\alpha^{-1} = 1_{e_2(X)}$ and $\alpha^{-1}\alpha = 1_{e_1(X)}$ for some homomorphism $\alpha^{-1} : e_2(X) \to e_1(X)$. Let $\gamma \in S$ be an endomorphism acting by the rule $\gamma(e_1 m + (1 - e_1)m) = \alpha e_1 m$, and let $\gamma' \in S$ be an endomorphism defined by $\gamma(e_2 m + (1 - e_2)m) = \alpha^{-1} e_2 m$, where $m \in P$. Then $\gamma\gamma' = e_2$ and $\gamma'\gamma = e_1$. Consider the endomorphism $\omega \in T$ that acts by the rule $\omega(\hat{e}_1 m + (1 - \hat{e}_1)m) = \alpha'\hat{e}_1 m$, where $m \in M$. Clearly, $\omega = \omega\hat{e}_1 = \hat{e}_2\omega$. Then for every $m \in P$ we have

$$p\gamma(m) = p\alpha(e_1(m)) = \alpha' p(e_1(m)) = \alpha'\hat{e}_1 p(x) = \omega p(m).$$

Since $T/J(T)$ is a regular ring, $\omega - \omega\beta_1\omega \in J(T)$ for some $\beta_1 \in T$. We have $p\beta = \beta_1 p$ for some $\beta \in S$. Hence, $\gamma - \gamma\beta\gamma \in J(S)$.

The containment $\gamma'(\gamma - \gamma\beta\gamma) \in J(S)$ yields $e_1 - e_1\beta\gamma \in J(S)$. Since $\overline{\Phi}$ is injective, $\hat{e}_1 - \hat{e}_1\beta_1\omega \in J(T)$. Hence, $\hat{e}_1 - \hat{e}_1\beta_1\omega\hat{e}_1 \in \hat{e}_1 J(T)\hat{e}_1 = J(\hat{e}_1 T\hat{e}_1)$. Then $\hat{e}_1\beta_1\omega\hat{e}_1 \in U(\hat{e}_1 T\hat{e}_1)$, and there is $t \in T$ such that $e_1' t\omega\hat{e}_1 = \hat{e}_1$. Thus, $\alpha' : \hat{e}_1(M) \to \hat{e}_2(M)$ is a monomorphism.

The inclusion $(\gamma - \gamma\beta\gamma)\gamma' \in J(S)$ implies $e_2 - \gamma\beta e_2 \in J(S)$. Then $\hat{e}_2\omega\beta_1\hat{e}_2 \in U(\hat{e}_2 T\hat{e}_2)$. Hence, $\hat{e}_2\omega t'\hat{e}_2 = \hat{e}_2$ for some $t' \in T$. Thus, α' is an isomorphism. $\square$

Corollary 26. *The endomorphism ring of every indecomposable $\mathscr{X}$-discrete module is local.*

Proof. This is immediate from Theorem 24. $\square$

Theorem 27. *If M is an $\mathscr{X}$-discrete module then M has the finite exchange property.*

Proof. This is immediate from Theorems 24 and [29, Proposition 1.6]. $\square$

A ring R is *clean* provided that every r in R may be represented as $r = e + u$, where $e^2 = e \in R$ and u is invertible in R. A module M is *clean* if $\mathrm{End}(M)$ is a clean ring.

Theorem 28. *If M is an $\mathscr{X}$-discrete module and $\mathrm{End}(X)$ is a clean ring then $\mathrm{End}(M)$ is clean.*

Proof. Let α be an arbitrary element in $\mathrm{End}(M)$. There is an endomorphism $\beta \in X$ such that $p\beta = \alpha p$. Since $\mathrm{End}(X)$ is a clean ring, $\beta = e + \gamma$ for an automorphism γ of X and an idempotent $e \in \mathrm{End}(X)$. Since M is $\mathscr{X}$-idempotent coinvariant, $pe = e_1 p$ for an idempotent $e_1 \in \mathrm{End}(M)$. Let $\gamma' = \alpha - e_1 \in \mathrm{End}(M)$. Then $p\gamma = p(\beta - e) = p\beta - pe = \alpha p - e_1 p = \gamma' p$. Since M is $\mathscr{X}$-discrete, γ' is an automorphism of M. Thus, M is a clean module. $\square$

A module M is *quasicontinuous* provided that M is invariant under the idempotent endomorphism ring of the injective envelope of M. A quasicontinuous module M is *continuous* if each submodule of M isomorphic to a direct summand of M is a direct summand of M. The following may be proved by the standard argument.

Lemma 29. *Let M be a quasicontinuous module, and let $u : M \to E(M)$ be an injective envelope of M. The following are equivalent*:

(1) *M is a continuous module*;

(2) *if $e_i u = ue'_i$ for $i = 1, 2$ and some idempotents $e_1, e_2 \in \mathrm{End}(E)$ and $e'_1, e'_2 \in \mathrm{End}(M)$, the diagram commutes:*

$$\begin{array}{ccc} e'_1(M) & \xrightarrow{\alpha'} & e'_2(M) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u} \\ e_1(E) & \xrightarrow[\alpha]{} & e_2(E) \end{array}$$

for some homomorphisms α and α', while α is an isomorphism; then α' is an isomorphism.

Let $u : M \to X$ be an $\mathscr{X}$-envelope of M. A module M is *$\mathscr{X}$-continuous* if the following hold:

(1) M is an $\mathscr{X}$-idempotent invariant module;

(2) if $e_i u = ue'_i$ for $i = 1, 2$ and some idempotents $e_1, e_2 \in \mathrm{End}(E)$ and $e'_1, e'_2 \in \mathrm{End}(M)$, the diagram commutes:

$$\begin{array}{ccc} e'_1(M) & \xrightarrow{\alpha'} & e'_2(M) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u} \\ e_1(E) & \xrightarrow[\alpha]{} & e_2(E) \end{array}$$

for some homomorphisms α and α', while α is an isomorphism; then α' is an isomorphism.

In what follows, we assume that $u : M \to X$ is a monomorphic $\mathscr{X}$-envelope of a right R-module M and $\mathrm{End}_R(X)$ is a semiregular ring. Then the ring homomorphism $\Phi : \mathrm{End}(M) \to S/J(S)$ holds, which is defined by the rule $\Phi(f) = \overline{f} + J(S)$, where $\overline{f} : X \to X$ is a homomorphism such that $\overline{f}u = uf$. Let $\Delta(M) = \mathrm{Ker}(\Phi)$. Then we have the monomorphism $\overline{\Phi} : M/\Delta(M) \to S/J(S)$. It is easy to notice that if $\mathscr{X}$ is the class of injective right R-modules then $\Delta(M) = \{f \in \mathrm{End}(M) \mid \mathrm{Ker}(f) \leq_e M\}$.

We list the dual analogs to Theorems 24, 25, 27, and 28.

Theorem 30. *If M is an $\mathscr{X}$-continuous module then $\mathrm{End}(M)$ is semiregular and $J(\mathrm{End}(M)) = \Delta(M)$.*

Theorem 31. *If M is an $\mathscr{X}$-idempotent invariant module then M is $\mathscr{X}$-continuous if and only if $\Delta(M) = J(\mathrm{End}(M))$ and $\mathrm{End}(M)/\Delta(M)$ is regular.*

Theorem 32. *If M is an $\mathscr{X}$-continuous module then M has the finite exchange property.*

Theorem 33. *If M is an $\mathscr{X}$-continuous module then $\mathrm{End}(M)$ is clean.*

4. Some Applications

As some applications of the above results, we consider the cases of flat covers as well as injective and pure injective envelopes.

Let R be a ring, and let $\mathscr{F}$ be the class of flat right R-modules. Each right R-module possesses a flat envelope by [30, Theorem 3]. If R is a perfect right ring then every flat envelope of an arbitrary right R-module M coincides with the projective envelope of M by [25, Proposition 1.3.1]. Thus, a right R-module M over a perfect right ring R is discrete if and only if M is an $\mathscr{F}$-discrete module.

Theorem 34. *Let M be an $\mathscr{F}$-discrete right R-module. The following hold*:

(1) $\mathrm{End}\,M$ *is a semiregular ring*;

(2) *M has the finite exchange property*;

(3) $\mathrm{End}\,M$ *is a clean ring*;

(4) *if M is an indecomposable module then* $\mathrm{End}\,M$ *is a local ring.*

Proof. This follows from [21, Lemma 5.1], Theorems 24 and 20, and Corollary 21. □

Corollary 35. *Let R be a perfect right ring, and let M be a discrete right R-module. Then*
(1) $\operatorname{End} M$ *is a semiregular ring, and* $J(\operatorname{End}(M)) = \{f \in \operatorname{End}(M) \mid f(M) \ll M\}$;
(2) *M has the finite exchange property*;
(3) $\operatorname{End} M$ *is a clean ring*;
(4) *if M is an indecomposable module then* $\operatorname{End} M$ *is a local ring.*

Corollary 36. *Let R be a semiperfect ring, and let M be a finitely generated discrete right R-module. Then*
(1) $\operatorname{End} M$ *is a semiregular ring, and* $J(\operatorname{End}(M)) = \{f \in \operatorname{End}(M) \mid f(M) \ll M\}$;
(2) *M has the finite exchange property*;
(3) $\operatorname{End} M$ *is a clean ring*;
(4) *if M is an indecomposable module then* $\operatorname{End} M$ *is a local ring.*

Theorem 37. *Let M be a continuous right R-module. Then*
(1) $\operatorname{End} M$ *is a semiregular ring, and* $J(\operatorname{End}(M)) = \{f \in \operatorname{End}(M) \mid \operatorname{Ker}(f) \leq_e M\}$;
(2) *M has the finite exchange property*;
(3) $\operatorname{End} M$ *is a clean ring*;
(4) *if M is an indecomposable module then* $\operatorname{End} M$ *is a local ring.*

Proof. This follows from Theorems 31, 33, and 34, if $\mathscr{X}$ is the class of all injective right R-modules in these theorems. □

A module M is *pure continuous* provided that M is an $\mathscr{X}$-continuous module, where $\mathscr{X}$ is the class of pure injective right R-modules. By [31, Proposition 6], each module possesses a monomorphic pure injective envelope, and [32, Theorem 9] implies that the endomorphism ring of every pure injective right module is semiregular and right self-injective. Then Theorems 24, 27, 28 and Corollary 26 imply the assertion that generalizes the results from [32] on the endomorphism rings of pure injective modules:

Theorem 38. *Let M be a pure continuous right R-module. Then*
(1) $\operatorname{End} M$ *is a semiregular ring*;
(2) *M has the finite exchange property*;
(3) $\operatorname{End} M$ *is a clean ring*;
(4) *if M is an indecomposable module then* $\operatorname{End} M$ *is a local ring.*

Remark. It is known that the Schröder–Bernstein problem is solved affirmatively for the discrete modules over perfect rings and for the continuous modules [16, Theorem 3.17]. Therefore, it is of interest to study this problem in the general case for the $\mathscr{X}$-continuous and $\mathscr{X}$-discrete modules. The flat, injective, discrete, and continuous modules play an essential role in the homological characterization of rings. It stands to reason to understand the structure of the rings over which every module is $\mathscr{F}$-discrete. Obviously, the regular rings are some examples of these rings. A module M over a Prüfer ring is flat if and only if M is a torsion-free module by [30, Theorem 3]. The description of pure injective $\mathbb{Z}$-modules [33, Theorem 3.2] and continuous $\mathbb{Z}$-modules [16, p. 19] is well known. Therefore, the problem is natural of describing $\mathscr{F}$-discrete modules and pure continuous modules over Prüfer rings (in particular, over the ring of integers).

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