On the distance from the origin to an entire graphic m-shrinker

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Abstract. Let Σ be an entire graphic m-shrinker in $\mathbb{R}^n$ and X be the position vector field of Σ. By using the generalized divergence theorem, we obtain a formula for the weighted volume of Σ that is related to X, and give a simple proof for it. Thanks to this formula, an upper bound for the distance from the origin to the m-shrinker is given.

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1 Introduction

A minimal surface is a surface that locally minimizes its area. This is equivalent to that the surface has zero mean curvature. The classical Bernstein's theorem asserts that an entire minimal graph over $\mathbb{R}^2$ is a plane in $\mathbb{R}^3$ (see [10]). Many mathematicians tried to generalize the Bernstein's theorem to higher dimensions as well as higher codimensions. In 1965, De Giorgi proved the Bernstein's theorem for entire minimal graphs over $\mathbb{R}^3$ in $\mathbb{R}^4$ (see [7]). In 1966, Almgren proved the theorem in $\mathbb{R}^5$ (see [1]). In 1968, Simons extended the theorem to $\mathbb{R}^8$. He proved that an entire minimal graph of dimension n has to be planar for $n \leq 7$ (see [11]). In 1969, Bombieri, De Giorgi, and Giusti produced a counterexample in dimension 8 and higher (see [3]). In the theory of minimal hypersurfaces, Bernstein's theorem is one of the most fundamental theorems. Thus, it is natural to ask whether there is a Bernstein type theorem in an ambient space other than $\mathbb{R}^n$, such as Riemannian manifolds, Lorentz-Minkowski spaces, warped product spaces, manifolds with density,... In particular, a theme widely approached in recent years is Bernstein type theorem in a manifold with density, a Riemannian manifold with a positive function e^{-f} used to weight both the volume and the perimeter area. A hypersurface Σ is said to be f-minimal (or weighted minimal or minimal with density e^{-f}) if the f-mean curvature (or weighted mean curvature) of Σ,

$$H_f(\Sigma) = H(\Sigma) + \langle \nabla f, N \rangle = 0,$$

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where $H(\Sigma)$ $(= -\operatorname{div} N)$ and N are the classical mean curvature and the unit normal vector field of Σ, respectively. If $H_f(\Sigma) = \lambda$, a constant, then Σ is called a λ-hypersurface.

Shrinkers are minimal hypersurfaces in $\mathbb{R}^n$ under the conformally changed metric $g_{ij} = e^{-\frac{|x|^2}{2(n-1)}}\delta_{ij}$ (see [6]). They are special examples of weighted minimal hypersurfaces in $\mathbb{R}^n$ with density $e^{-\frac{|x|^2}{4}}$, a modified version of Gauss space $\mathbb{G}^n$, $\mathbb{R}^n$ with density $e^{-f} = (2\pi)^{-\frac{n}{2}}e^{-\frac{|x|^2}{2}}$. To answer the question of whether there is a Bernstein type theorem for shrinkers, Ecker and Huisken (see [8]) showed that a smooth shrinker is a hyperplane if it is an entire graphic shrinker with polynomial area growth. Recently, Lu Wang (see [12]) removed the assumption of polynomial area growth. She proved that smooth shrinkers in $\mathbb{R}^n$, that are entire graphs are hyperplanes. Cheng and Wei (see [4]) proved that an entire graphic λ-hypersurface in Euclidean space is a hyperplane. In [9], D. T. Hieu established a weighted area estimate for entire graphs with bounded weighted mean curvature in Gauss space. Thanks to this, the Bernstein type theorems for graphic shrinkers as well as for graphic λ-hypersurfaces are immediately obtained as sequences. It should be noted that all mentioned above are for the case of codimension one, i.e., for hypersurfaces.

For m-surface (i.e., an m-dimensional surface) Σ^m in $\mathbb{R}^n$, the f-mean curvature vector of Σ^m is $\vec{H}_f(\Sigma^m) = \vec{H}(\Sigma^m) + X^N$, where $\vec{H}(\Sigma^m)$ and X^N are the mean curvature vector and the projection of the position vector field X into the normal bundle of Σ^m, respectively. An m-shrinker is an m-surface satisfies the equation $\vec{H} + \frac{X^N}{2} = 0$. Up to now, some Bernstein type theorems for entire graphic shrinkers with higher codimensions have been proved with some additional conditions. H. Zhou (see [14]) proved a Bernstein type theorem for entire graphic shrinkers in $\mathbb{R}^4$ of a smooth map f from $\mathbb{R}^2$ to $\mathbb{R}^2$ with its Jacobian, J_f, satisfies that $J_f + 1 > 0$ or $1 - J_f > 0$ for all $x \in \mathbb{R}^2$. After that, he proved such a theorem in $\mathbb{R}^{n+l}$ for a smooth map f from $\mathbb{R}^n$ to $\mathbb{R}^l$ with its eigenvalues $\{\lambda_i\}_{i=1}^r$, satisfy that $|\lambda_i\lambda_j| \leq 1$ for $i \neq j$, where r is the rank of the differential of f.

It should be noted that the Bernstein type theorem for entire graphic shrinkers of codimension one is a stronger version than the classical one. An entire graphic shrinker is not only a hyperplane but also passing through the origin, i.e., the distance from the origin to the shrinker is zero. It is natural to consider the weaker question whether the entire graphic m-shrinker (i.e., an m-dimensional entire graphic shrinker) in $\mathbb{R}^n$ $(n - m > 1)$ passes through the origin. In this paper, we prove a partial result for this question: an entire graphic m-shrinker is not too far from the origin.

2 Distance from the origin to an entire graphic m-shrinker

In $\mathbb{R}$, by integration by parts, it is not hard to check that

$$2\int_{-\infty}^{\infty} e^{-\frac{x^2}{4}}\,dx = \int_{-\infty}^{\infty} e^{-\frac{x^2}{4}}x^2\,dx.$$

Observe that the left-hand side of this equation represents twice the weighted length of a straight line passing through the origin in $\overline{\mathbb{G}}^n$, $\mathbb{R}^n$ with density $e^{-\frac{|x|^2}{4}}$.

In [2], Claudio Arezzo and Jun Sun considered that Σ^m is an m-dimensional complete submanifold of $\mathbb{R}^n$ without boundary, with polynomial volume growth, and $\vec{H} + \frac{X^N}{2} = 0$. They gave a similar equation for Σ^m without proof (see [2], Lemma 2.4):

$$\int_{\Sigma^m} (|X|^2 - 2m) e^{-\frac{|X|^2}{4}} = 0.$$

Now, we consider that Σ is an entire graph m-shrinker in $\mathbb{R}^n$. By using the generalized divergence theorem, a simple proof for the same equation for m-planes passing through the origin in $\overline{\mathbb{G}}^n$ as well as entire graphic m-shrinkers in $\mathbb{R}^n$ will be given in the next propositions. Thanks to them, the main theorem, Theorem 2.5, is obtained.

We first prove the following lemma.

Lemma 2.1. *Let Σ be a graphic m-shrinker in $\mathbb{R}^n$ and X be the position vector field of Σ. We have*

$$\operatorname{div}_\Sigma X = m;$$

and

$$\operatorname{div}_\Sigma (e^{-f} X) = m e^{-f} - \frac{1}{2} e^{-f} |X^T|^2,$$

where $f = \sum_{i=1}^n \frac{x_i^2}{4} = \frac{1}{4}\sum_{i=1}^n x_i^2$ and X^T is the projection of the position vector field X into the tangent bundle of Σ.

Proof. A direct computation shows that for every $v \in \operatorname{Tan}_p \Sigma$,

$$\nabla_v X = v.$$

Thus, for any orthonormal basis of $\operatorname{Tan}_p\Sigma$, $\{e_1(p), e_2(p), ..., e_m(p)\}$, we have

$$\operatorname{div}_\Sigma X = \sum_{i=1}^m e_i \nabla_{e_i} X = \sum_{i=1}^m e_i e_i = m;$$

and

$$\begin{aligned}
\operatorname{div}_\Sigma (e^{-f} X) &= \sum_{i=1}^m e_i \nabla_{e_i} (e^{-f} X) \\
&= \sum_{i=1}^m e_i \left(e^{-f} \nabla_{e_i} X + e_i(e^{-f}) X \right) \\
&= \sum_{i=1}^m e_i \left(e^{-f} \nabla_{e_i} X - e^{-f} \langle \nabla f, e_i \rangle X \right) \\
&= \sum_{i=1}^m e_i \left(e^{-f} \nabla_{e_i} X - \frac{1}{2} e^{-f} \langle X, e_i \rangle X \right) \\
&= e^{-f} \left(\sum_{i=1}^m e_i \nabla_{e_i} X - \frac{1}{2} \sum_{i=1}^m \langle X, e_i \rangle \langle X, e_i \rangle \right) \\
&= e^{-f} \left(\operatorname{div}_\Sigma X - \frac{1}{2} |X^T|^2 \right) \\
&= m e^{-f} - \frac{1}{2} e^{-f} |X^T|^2. \qquad (2.1)
\end{aligned}$$

This completes the proof of Lemma 2.1. □

Proposition 2.2. *Let Σ_0 be an m-plane passing through the origin in $\overline{\mathbb{G}}^n$, and X be a position vector field of Σ_0. Then we have*

$$2m \operatorname{Vol}_f \Sigma_0 = \int_{\Sigma_0} e^{-f}|X|^2 \, dV,$$

where $f = \sum_{i=1}^n \frac{x_i^2}{4} = \frac{1}{4}\sum_{i=1}^n x_i^2$.

Proof. Since Σ_0 is an m-plane passing through the origin in $\mathbb{G}^n$ and X is in Σ_0, $|X^T|^2 = |X|^2$ and it follows from (2.1) that

$$\begin{aligned} \int_{\Sigma_0} \operatorname{div}_{\Sigma_0}(e^{-f}X)\, dV &= m\int_{\Sigma_0} e^{-f}\, dV - \frac{1}{2}\int_{\Sigma_0} e^{-f}|X^T|^2\, dV \\ &= m \operatorname{Vol}_f \Sigma_0 - \frac{1}{2}\int_{\Sigma_0} e^{-f}|X|^2\, dV. \end{aligned} \tag{2.2}$$

Let B_R be an n-ball with center O and radius R in $\mathbb{R}^n$, $\nu(p)$ be the unit vector in the tangent plane to $B_R \cap \Sigma_0$ at p that is normal to $\partial(B_R \cap \Sigma_0)$ and that points away from $B_R \cap \Sigma_0$. Since X and ν have the same direction, $\langle X, \nu\rangle = R$ on $\partial(B_R \cap \Sigma_0)$.

By using the generalize divergence theorem (see [13]), we get

$$\begin{aligned} \int_{B_R\cap\Sigma_0} \operatorname{div}_{\Sigma_0}(e^{-f}X)\, dV &= \int_{\partial(B_R\cap\Sigma_0)} e^{-f}\langle X,\nu\rangle\, dV - \int_{B_R\cap\Sigma_0} e^{-f}\langle \vec{H}, X\rangle\, dV \\ &= R.e^{-\frac{R^2}{4}} \operatorname{Vol}(\partial(B_R\cap\Sigma_0)), \end{aligned} \tag{2.3}$$

because $\vec{H} = 0$ on Σ_0.

Moreover, it is clear that $\partial(B_R \cap \Sigma_0)$ is an $(m-1)$-sphere with center O and radius R. Therefore, we have

$$\operatorname{Vol}(\partial(B_R \cap \Sigma_0)) = \frac{2\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})} R^{m-1},$$

where Γ is the gamma function, which satisfies $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(1) = 1$, and $\Gamma(x+1) = x\Gamma(x)$ for any x.

Thus, taking the limit of both sides of (2.3) as R tends to infinity, we obtain

$$\int_{\Sigma_0} \operatorname{div}_{\Sigma_0}(e^{-f}X) = 0.$$

Combining with (2.2), we get

$$2m \operatorname{Vol}_f \Sigma_0 = \int_{\Sigma_0} e^{-f}|X|^2\, dV$$

as desired. □

Remark 2.1. Since an entire graphic m-shrinker Σ in $\mathbb{R}^n$ is proper, it has Euclidean volume growth (see [5]), that is, there exist constants C so that for all $R \geq 1$

$$\operatorname{Vol}(B_R \cap \Sigma) \leq CR^m.$$

Lemma 2.3. *Let Σ be an entire graphic m-shrinker in $\mathbb{R}^n$, $\nu(p)$ be the unit vector in the tangent plane to $B_R \cap \Sigma$ at p that is normal to $\partial(B_R \cap \Sigma)$ and that points away from $B_R \cap \Sigma$. We have*

$$\lim_{R\to\infty} e^{-\frac{R^2}{4}} \int_{\partial(B_R\cap\Sigma)} \langle X, \nu\rangle\, dV = 0.$$

Proof. It is not hard to check that

$$\Delta_\Sigma X = \vec{H} = -\frac{X^N}{2};$$
$$\Delta_\Sigma |X|^2 = 2\langle X, \Delta_\Sigma X\rangle + 2|\nabla_\Sigma X|^2 = 2\langle X, \vec{H}\rangle + 2m = 2m - |X^N|^2;$$
$$\nabla_\Sigma |X|^2 = 2X^T.$$

Hence,

$$\int_{B_R\cap\Sigma} \operatorname{div}_\Sigma(X^T)\, dV = \int_{B_R\cap\Sigma} \frac{\Delta_\Sigma |X|^2}{2}\, dV = \int_{B_R\cap\Sigma} (m - \frac{1}{2}|X^N|^2)\, dV.$$

Moreover, by using the divergence theorem, we have

$$\int_{B_R\cap\Sigma} \operatorname{div}_\Sigma(X^T)\, dV = \int_{\partial(B_R\cap\Sigma)} \langle X^T, \nu\rangle\, dV = \int_{\partial(B_R\cap\Sigma)} \langle X, \nu\rangle\, dV.$$

It follows that

$$0 \le \int_{\partial(B_R\cap\Sigma)} \langle X, \nu\rangle\, dV = \int_{B_R\cap\Sigma} (m - \frac{1}{2}|X^N|^2)\, dV \le m \int_{B_R\cap\Sigma} dV \le mCR^m,$$

since Σ has Euclidean volume growth.
Therefore,

$$0 \le e^{-\frac{R^2}{4}} \int_{\partial(B_R\cap\Sigma)} \langle X, \nu\rangle\, dV \le e^{-\frac{R^2}{4}} mCR^m. \tag{2.4}$$

Taking the limit of both sides of (2.4) as R tends to infinity, we get

$$\lim_{R\to\infty} e^{-\frac{R^2}{4}} \int_{\partial(B_R\cap\Sigma)} \langle X, \nu\rangle\, dV = 0$$

as desired. □

Proposition 2.4. *Proposition* 2.2 *also holds true for Σ being an entire graphic m-shrinker in $\mathbb{R}^n$.*

Proof. From (2.1),

$$\int_{B_R\cap\Sigma} \operatorname{div}_\Sigma(e^{-f}X)\, dV = m \int_{B_R\cap\Sigma} e^{-f}\, dV - \frac{1}{2}\int_{B_R\cap\Sigma} e^{-f}|X^T|^2\, dV.$$

By using the generalized divergence theorem, we have

$$\int_{B_R\cap\Sigma} \operatorname{div}_\Sigma(e^{-f}X)\, dV = \int_{\partial(B_R\cap\Sigma)} e^{-f}\langle X, \nu\rangle\, dV - \int_{B_R\cap\Sigma} e^{-f}\langle \vec{H}, X\rangle\, dV.$$

Therefore,

$$m\int_{B_R\cap\Sigma} e^{-f}\,dV = \int_{\partial(B_R\cap\Sigma)} e^{-f}\langle X,\nu\rangle\,dV + \frac{1}{2}\int_{B_R\cap\Sigma} e^{-f}|X^T|^2\,dV - \int_{B_R\cap\Sigma} e^{-f}\langle \vec{H},X\rangle\,dV.$$

Since Σ is an m-shrinker in $\mathbb{R}^n$, i.e., $\vec{H}+\frac{1}{2}X^N=0$,

$$\langle \vec{H},X\rangle = -\frac{1}{2}\langle X^N,X\rangle = -\frac{1}{2}|X^N|^2.$$

It follows that

$$\begin{aligned} m\int_{B_R\cap\Sigma} e^{-f}\,dV &= \int_{\partial(B_R\cap\Sigma)} e^{-f}\langle X,\nu\rangle\,dV + \frac{1}{2}\int_{B_R\cap\Sigma} e^{-f}|X^T|^2\,dV \\ &\quad + \frac{1}{2}\int_{B_R\cap\Sigma} e^{-f}|X^N|^2\,dV \\ &= e^{-\frac{R^2}{4}}\int_{\partial(B_R\cap\Sigma)} \langle X,\nu\rangle\,dV + \frac{1}{2}\int_{B_R\cap\Sigma} e^{-f}|X|^2\,dV. \end{aligned} \tag{2.5}$$

According to Lemma 2.3, we have

$$\lim_{R\to\infty} e^{-\frac{R^2}{4}}\int_{\partial(B_R\cap\Sigma)} \langle X,\nu\rangle\,dV = 0.$$

Therefore, taking the limit of both sides of (2.5) as R tends to infinity, we get

$$2m\,\mathrm{Vol}_f\,\Sigma = \int_\Sigma e^{-f}|X|^2\,dV.$$

Thus, Proposition 2.2 also holds true for Σ being an entire graphic m-shrinker in $\mathbb{R}^n$. □

The following theorem gives an upper bound for the distance from O to an entire graphic m-shrinker in $\mathbb{R}^n$. It shows that an entire graphic m-shrinker in $\mathbb{R}^n$ is not too far from the origin.

Theorem 2.5. *Let Σ be an entire graphic m-shrinker in $\mathbb{R}^n$. Then the distance from the origin to Σ, $d(O,\Sigma)$, satisfies*

$$d(O,\Sigma) < \sqrt{2m}.$$

Therefore, an entire graphic m-shrinker in $\mathbb{R}^n$ is not too far from the origin.

Proof. Let $k=\sqrt{2m}$ and B_k be the n-ball with center O and radius k. It is clear that

$$2m = k^2 \le |X|^2,\ \forall X\in\Sigma - B_k \ \text{ and } \ \exists X\in\Sigma-B_k:\ 2m<|X|^2 \tag{2.6}$$

From Proposition 2.4, we have

$$\int_\Sigma e^{-f}(2m-|X|^2)\,dV = \int_{\Sigma\cap B_k} e^{-f}(2m-|X|^2)\,dV + \int_{\Sigma-B_k} e^{-f}(2m-|X|^2)\,dV = 0.$$

It follows from (2.6) that the last integral is negative, therefore the first one is positive. Hence, $\Sigma\cap B_k \neq \varnothing$, and $d(O,\Sigma)<\sqrt{2m}$. Thus, an entire graphic m-shrinker in $\mathbb{R}^n$ is not too far from the origin. □

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