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On Ahlfors currents

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ABSTRACT

We answer a basic question in Nevanlinna theory that Ahlfors currents associated to the same entire curve may be *nonunique*. Indeed, we will construct one exotic entire curve $f : \mathbb{C} \to X$ which produces infinitely many cohomologically different Ahlfors currents. Moreover, concerning Siu's decomposition, for an arbitrary $k \in \mathbb{Z}_+ \cup \{\infty\}$, some of the obtained Ahlfors currents have singular parts supported on k irreducible curves. In addition, they can have *nonzero* diffuse parts as well. Lastly, we provide new examples of diffuse Ahlfors currents on the product of two elliptic curves and on $\mathbb{P}^2(\mathbb{C})$, and we show cohomologically elaborate Ahlfors currents on blow-ups of X.

RÉSUMÉ

On répond à une question fondamentale dans la théorie de Nevanlinna que les courants d'Ahlfors associés à la même courbe entière peuvent être *nonuniques*. En effet, on construira une courbe entière exotique $f : \mathbb{C} \to X$ qui produit infinité beaucoup des cohomologiquement différents courants d'Ahlfors. De plus, concernant la décomposition de Siu, pour un arbitraire $k \in \mathbb{Z}_+ \cup \{\infty\}$, certains des courants d'Ahlfors obtenus ont des parties singulières supportées sur k courbes irréductibles. En outre, ils peuvent également avoir des parties diffuses *nonnulles*. Enfin, on fournit nouveaux exemples des courants d'Ahlfors diffuses sur le produit de deux courbes elliptiques et sur $\mathbb{P}^2(\mathbb{C})$, et on montre des cohomologiquement élaborés courants d'Ahlfors sur des éclatements de X.

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1. Introduction

Let X be a compact complex manifold equipped with an area form ω. Let $f : \mathbb{C} \longrightarrow X$ be a nonconstant entire holomorphic curve. An associated *Ahlfors current* of f is a positive closed current of bidimension $(1, 1)$ obtained as the weak limit of a certain sequence of positive currents of bounded masses

$$\left\{ \frac{[f(\mathbb{D}_{r_n})]}{\mathrm{Area}_\omega f(\mathbb{D}_{r_n})} \right\}_{n \geqslant 1},$$

where $\mathbb{D}_{r_n}$ are discs of increasing radii $r_n \nearrow \infty$ centered at the origin. Here, to ensure that such a limit current is closed, the sequence $\{r_n\}$ is chosen in such a way that the lengths of boundaries of the discs are asymptotically negligible compared with their areas, namely

$$\lim_{n\to\infty} \frac{\mathrm{Length}_\omega(f(\partial \mathbb{D}_{r_n}))}{\mathrm{Area}_\omega(f(\mathbb{D}_{r_n}))} = 0.$$

By Ahlfors' lemma (cf. [1,2]), for each positive number $\epsilon > 0$, the set

$$\left\{ r > 0 : \frac{\mathrm{Length}_\omega(f(\partial \mathbb{D}_r))}{\mathrm{Area}_\omega(f(\mathbb{D}_r))} \geqslant \epsilon \right\}$$

is of finite measure with respect to $\frac{\mathrm{d}\, r}{r}$. Hence the above *"length-area" condition* is satisfied for most choices of increasing radii. Moreover, given a sequence of radii $\{r_n\}_{n \geqslant 1}$ with $r_n \nearrow \infty$, after some small perturbation by scaling and extracting a subsequence, one can always obtain an Ahlfors current for f.

Ahlfors currents and their analogs obtained by taking the logarithmic average $\int \frac{\mathrm{d}\, t}{t}(\cdot)$, called *Nevanlinna currents*, are fundamental tools in studying complex hyperbolicity, value distribution theory and complex dynamical systems. Notably, they played a crucial role in the work McQuillan [3] on Green-Griffiths' conjecture for algebraic surfaces of general type having positive Segre class (see also [1] for a simplified proof by Brunella). By employing Ahlfors currents, Duval [4] gave a quantitative version of the classical Brody's Lemma and obtained a characterization of complex hyperbolicity in terms of linear isoperimetric inequality for holomorphic discs. Using such currents, some geometric refinement of the classical Cartan's Second Main Theorem [5], as well as the high dimensional Weierstrass-Casorati Theorem [6] were obtained. The reader is also referred to [7] for recent key applications in complex dynamical systems.

Since Ahlfors currents and Nevanlinna currents encode geometric information of their original entire curves, several results in value distribution theory can be presented in terms of intersections of corresponding cohomology classes. For example, the First Main Theorem of Nevanlinna theory can be expressed as an inequality between the algebraic intersection and the geometric intersection (cf. [5]).

Note that in certain specific situations, Ahlfors currents (or Nevanlinna currents) from some holomorphic curve are unique [8,5,7,9], which subsequently leads to several interesting results. Therefore, it is natural and fundamental to ask generally

Question 1.1. *Are all Ahlfors currents associated to the same entire curve cohomologically equivalent?*

The study of such currents is itself of independent interest. By Siu's decomposition Theorem [10], an Ahlfors current T can be written as the sum $T = T_{\mathrm{Sing}} + T_{\mathrm{Diff}}$, where the singular part $T_{\mathrm{Sing}} = \sum_{\ell \in I} c_\ell \cdot [C_\ell]$ is some positive linear combination ($c_\ell > 0$; $I \subset \mathbb{Z}_+$, could be $\varnothing$) of currents of integration on irreducible algebraic curves C_ℓ, and where the diffuse part T_{Diff} is a positive closed $(1, 1)$–current having zero Lelong number along any algebraic curve. If the singular part T_{Sing} is nontrivial, Duval [11] showed that any irreducible curve C_ℓ above must be rational or elliptic (see also [12] for a local version). In [13], da Costa

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gave an example of entire curve in the projective plane whose associated Ahlfors current is supported in some line. This construction can be modified to produce Ahlfors currents supported on a rational or an elliptic curve [14, Theorem 2.6.1]. On the other hand, we would like to mention the following unsolved question, which had been considered by Brunella [1, page 200].

Question 1.2. *Is there any Ahlfors current from an entire curve such that both of its singular part and diffuse part are nontrivial?*

In this paper, we answer the above two questions by constructing explicit examples.

Theorem 1.3. *There exists an entire curve producing cohomologically different Ahlfors currents.*

By Siu's decomposition, Ahlfors currents with nontrivial singular parts can be distinguished as different types by the data $(|I| \in \mathbb{Z}_+ \cup \{\infty\}, T_{\mathrm{Diff}}$ is trivial / nontrivial).

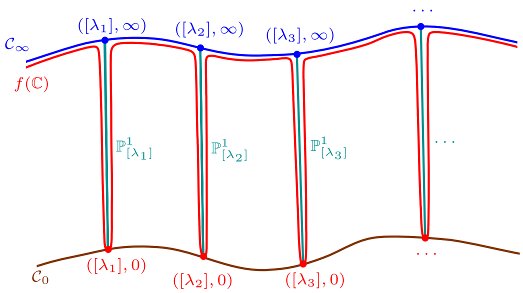
Theorem 1.4. *There exists an entire curve producing all types of Ahlfors currents with nontrivial singular parts.*

Remark 1.5. The above two results also hold true for Nevanlinna currents, see Subsection 7.2.

Lastly, it is natural to seek Ahlfors (Nevanlinna) currents with trivial singular part. Examples of such currents are known to exist on $\mathbb{P}^2(\mathbb{C})$, by looking at the Levi-flat real hypersurface in $(\mathbb{C}^*)^2$ defined by the equation $|x| = |y|^\alpha$, where α is an irrational real number (cf. [15, page 262]). Indeed, this real hypersurface is foliated by entire curves, while its closure in $\mathbb{P}^2(\mathbb{C})$ contains no algebraic curve. In [8] there are more examples of holomorphic curves whose associated Ahlfors (Nevanlinna) currents are diffuse and unique. In Section 7, we show new examples of diffuse Ahlfors currents on the product of two elliptic curves and on $\mathbb{P}^2(\mathbb{C})$, see Propositions 7.1, 7.2.

We now outline the ideas and the structure of this paper. As a matter of fact, our source of inspiration is an example due to da Costa [13] about a nondegenerate entire curve clustering to a line in $\mathbb{P}^2(\mathbb{C})$ (see also [5] for more discussions). In Section 2, we start with an elliptic curve $\mathcal{C} = \mathbb{C}/\Gamma$ equipped with a negative line bundle $\mathcal{L}$. For some large integer $m \gg 1$, we construct a section s_m of $\pi_0^*\mathcal{L}^m$ having large exponential growth of order 2, where $\pi_0 : \mathbb{C} \to \mathcal{C}$ is the canonical projection. The surface X is obtained by taking the geometric projectivization $\mathbb{P}(\mathcal{L}^m \oplus \mathbb{C}) =: X$ of the vector bundle $\mathcal{L}^m \oplus \mathbb{C}$ on $\mathcal{C}$. Thus the section s_m induces a holomorphic map $f_0 : \mathbb{C} \to X$ clustering to the curve $\mathcal{C}_\infty = \mathcal{C} \times [1 \oplus 0]$. To generate Ahlfors currents with larger singular supports, we hence modify the original section s_m by multiplying it with a Weierstrass canonical product $\psi(z) = \prod_{\lambda\in\Lambda}\left(1 - \frac{z}{\lambda}\right)e^{\frac{z}{\lambda}+\frac{z^2}{2\lambda^2}}$, whose zero locus Λ is distributed in a delicate pattern, to make sure that the new section $\psi \cdot s_m$ induces an entire curve $f : \mathbb{C} \longrightarrow X$ producing Ahlfors currents with more singularities. Indeed, for every $\lambda \in \Lambda$, since $\psi \cdot s_m(\lambda) = 0$, $f(\lambda)$ touches the curve $\mathcal{C}_0 := \mathcal{C} \times [0 \oplus 1]$, defined by the zero section of $\pi_0^*\mathcal{L}^m$, at $([\lambda], [0 \oplus 1])$.

The idea is that, the image of a small neighborhood of $\lambda \in \Lambda \subset \mathbb{C}$ by f shall contribute moderate area $O(1)$ near the fiber $\mathbb{P}^1_{[\lambda]} \subset X$ over $[\lambda] \in \mathcal{C}$, and once there are sufficiently many $\lambda' \in \Lambda$ mapping to the same class $[\lambda]$ by π_0, the area of the image of f should spend a positive portion about $\mathbb{P}^1_{[\lambda]}$, hence the Ahlfors currents should charge positive mass there. See the picture below for illustration.



Nevertheless, to make sense of this idea, we need to show, first of all, that the growth of ψ is neither too rapid nor too slow, which will be accomplished in Section 3, by means of the Stirling formula as well as the symmetry of the lattice Γ. Consequently, in Section 4, we can manipulate Jensen's formula to evaluate various areas, which distinguish the singularities of the Ahlfors currents. In Section 5, we present an algorithm for constructing the zero locus Λ, which is designed for the proofs of the main theorems in Section 6. In Section 7, we provide new examples of diffuse Ahlfors currents. Moreover, we show cohomologically elaborate Ahlfors currents on surfaces obtained by blowing-up X.

Convention: Throughout this paper, K denotes positive numbers which are uniformly bounded from both sides $0 < K_1 < \mathsf{K} < K_2 < \infty$. Further, notation $\mathsf{K}_{\star_1,\star_2,\star_3}$ indicates dependence on parameters $\star_1, \star_2, \star_3$. The notation $\mathbb{D}(a, r) := \{z \in \mathbb{C} : |z - a| < r\}$ means the disc centered at $a \in \mathbb{C}$ with the radius $r > 0$. When $a = 0 \in \mathbb{C}$, we write $\mathbb{D}_r$ instead of $\mathbb{D}(0, r)$. For a point $[a]$ in a torus $\mathbb{C}/\Gamma$, we denote by $\mathbb{D}([a], r) \subset \mathbb{C}/\Gamma$ the image of $\mathbb{D}(a, r)$ under the projection $\mathbb{C} \to \mathbb{C}/\Gamma$. The differential operator d^c stands for $\frac{\sqrt{-1}}{4\pi}(\overline{\partial} - \partial)$.

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2. Construction

Fix a smooth elliptic curve $\mathcal{C} = \mathbb{C}/\Gamma$, where the lattice $\Gamma := \mathbb{Z} \oplus \mathbb{Z}\sqrt{-1}$ is chosen to simplify the arguments later. We can find a negative line bundle $\mathcal{L}$ on $\mathcal{C}$ equipped with some hermitian metric h' having strictly negative curvature. Now comparing with the Kähler form $\mathrm{d}\,\mathrm{d}^c\left(|z|^2\right)$ on $\mathcal{C}$ descending from the canonical projection $\pi_0 : \mathbb{C} \longrightarrow \mathbb{C}/\Gamma$, the curvature of h' is cohomologous to $-2\alpha\,\mathrm{d}\,\mathrm{d}^c\left(|z|^2\right)$ for some positive constant α, namely their difference is of the form $\mathrm{d}\,\mathrm{d}^c\,\varphi$ for some smooth real function φ on $\mathbb{C}/\Gamma$. Therefore, replacing the initial metric h' by $h'e^{\varphi} =: h$, the curvature becomes $\Theta_h = -2\alpha\ \mathrm{d}\,\mathrm{d}^c\left(|z|^2\right)$. Noting that the line bundle $\pi_0^*\mathcal{L}$ on $\mathbb{C}$ is holomorphically trivial, it has a nowhere vanishing holomorphic section k, which by Lelong-Poincaré equation satisfies that $\mathrm{d}\,\mathrm{d}^c\left(\log\|k\|^2_{\pi_0^*h}\right) = 2\alpha\ \mathrm{d}\,\mathrm{d}^c\left(|z|^2\right)$. Hence $\log\|k\|^2_{\pi_0^*h} - 2\alpha\,|z|^2$ is a harmonic function on $\mathbb{C}$, hence can be written as the real part of some holomorphic function g. Therefore, the modified section $s := e^{-g/2}\,k$ of $\pi_0^*\mathcal{L}$ has exponential growth of order two $\|s\|_{\pi_0^*h} = \|e^{-g/2}\,k\|_{\pi_0^*h} = e^{\alpha|z|^2}$. The above construction is based on an idea of [13].

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We now amplify the negativity of $\mathcal{L}$, by introducing $\mathcal{L}_m := \mathcal{L}^{\otimes m}$ for some big multiplicity $m \geqslant 1$ to be determined. For the metric $h_m := h^{\otimes m}$ of $\mathcal{L}_m$, the section $s_m := s^{\otimes m}$ has large exponential growth

$$\|s_m\|_{h_m} = e^{m\alpha|z|^2}. \tag{2.1}$$

Now we introduce the complex surface $X := \mathbb{P}(\mathcal{L}_m \oplus \mathbb{C})$ obtained by the geometric projectivization of the rank 2 vector bundle $\mathcal{L}_m \oplus \mathbb{C}$ over $\mathcal{C}$. Denote by $\pi_1 : X \longrightarrow \mathcal{C}$ the canonical projection. By the fiberwised identification $\mathcal{L}_m \cong \mathcal{L}_m \oplus 1 \subset \mathbb{P}(\mathcal{L}_m \oplus \mathbb{C})$, the total space of $\mathcal{L}_m$ can be embedded into X as an open subset, whose complement is the elliptic curve $\mathcal{C}_\infty := \mathcal{C} \times [1 \oplus 0] \subset \mathbb{P}(\mathcal{L}_m \oplus \mathbb{C})$ at "infinity".

Next, we introduce an auxiliary holomorphic function

$$\psi(z) := \prod_{\lambda \in \Lambda} \Big(1 - \frac{z}{\lambda}\Big) e^{\frac{z}{\lambda} + \frac{z^2}{2\lambda^2}}$$

obtained by Weierstrass canonical product, where the zero locus Λ will be chosen carefully by the following sophisticated reasoning, to make sure that the global section $\psi \cdot s_m$ of $\pi_0^* \mathcal{L}_m$ together with the inclusion $\iota : \mathcal{L}_m \hookrightarrow X$ induce an entire curve $f : \mathbb{C} \longrightarrow X$ producing complicated Ahlfors currents.

First of all, we would like to have the estimate $\log|\psi(z)| \leqslant O(|z|^2)$, at least for $|z|$ around r_i for some specific radii $r_i \nearrow \infty$, in order to bound the area of $f(\mathbb{D}_{r_i})$ by $O(r_i^2)$.

Secondly, we require that the cardinality $|\Lambda \cap \mathbb{D}_{r_i}| = O(r_i^2)$, so that the image $f(\mathbb{D}_{r_i})$ intersects the curve $\mathcal{C}_0 := \mathcal{C} \times [0 \oplus 1] \subset X$ defined by the zero section of $\mathcal{L}_m$ frequently enough.

Lastly, we require that each time when the image of the entire curve f intersects $\mathcal{C}_0$ for $\lambda \in \Lambda$ with $|\lambda| \gg 1$, it contributes $O(1)$ area near the fiber $\mathbb{P}^1_{[\lambda]} := \pi_1^{-1}([\lambda])$.

Thus we declare that

(i) near each annulus $\mathbf{A}_{r_i} := \{z \in \mathbb{C} : \frac{r_i}{2} \leqslant |z| \leqslant r_i\}$, the zero locus Λ is a mild perturbation of $\mathbf{A}_{r_i} \cap c\,\Gamma$, where $c \geqslant 5$ is some positive integer to be determined. More precisely

$$\Lambda = \cup_{i \geqslant 1} B_{r_i} \qquad \text{where} \qquad B_{r_i} := \cup_{\mu \in \mathbf{A}_{r_i} \cap c\Gamma} \{\mu + x_\mu\}.$$

Here at the moment we only tell that all x_μ's take values in the fundamental domain

$$\mathcal{D} := \{x + y\sqrt{-1} \,:\, 0 \leqslant x, y < 1\},$$

and later in Section 5, we will elaborate on the choices of x_μ's for delicate reasons.

(ii) $\{r_i\}_{i \geqslant 1}$ grow very rapidly, say

$$r_1 \geqslant 2020 \cdot c, \qquad r_{i+1} \geqslant {r_i}^4 \quad (\forall\, i \geqslant 1). \tag{2.2}$$

3. Preparations

Lemma 3.1. *One has a uniform estimate* $|\sum_{\lambda \in B_{r_i}} \frac{1}{\lambda}| \leqslant \mathsf{K}/c^2$ *for all* $i = 1, 2, \ldots$.

Proof. In the special case that all $x_\mu = 0$, by the symmetry of Γ that $(-1) \cdot \Gamma = \Gamma$ and that of $\mathbf{A}_r$, the sum $\sum_{\mu \in \mathbf{A}_{r_i} \cap c\Gamma} \frac{1}{\mu + 0}$ is always 0.

In general, for every $\mu \in \mathbf{A}_{r_i} \cap c\Gamma$, one has the estimate $|\frac{1}{\mu + x_\mu} - \frac{1}{\mu + 0}| \leqslant \mathsf{K}/r_i^2$. Noting that the cardinality $|B_{r_i}| \leqslant \mathsf{K} \cdot (r_i/c)^2$, we conclude that $|\sum_{\lambda \in B_{r_i}} \frac{1}{\lambda}| \leqslant \mathsf{K} \cdot (r_i/c)^2 \cdot \mathsf{K}/r_i^2 = \mathsf{K}/c^2$. □

Lemma 3.2. *One has a uniform estimate* $|\sum_{\lambda \in B_{r_i}} \frac{1}{\lambda^2}| \leqslant \mathsf{K}/(c^2\, r_i)$ *for all* $i = 1, 2, \ldots$.

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Proof. The argument goes much the same way as the preceding one, by using the rotational symmetry of Γ that $\sqrt{-1}\cdot\Gamma=\Gamma$. Indeed, we have the identity $\sum_{\mu\in\mathbf{A}_{r_i}\cap c\Gamma}\frac{1}{(\mu+0)^2}=0$. Moreover, for every $\mu\in\mathbf{A}_{r_i}\cap c\Gamma$, we have $|\frac{1}{(\mu+x_\mu)^2}-\frac{1}{(\mu+0)^2}|\leqslant\mathsf{K}/r_i^3$. The remaining argument is clear. □

We make a convention that $\log 0=-\infty$.

Proposition 3.3. *For every $i\geqslant 2$ and for $r_i/3\leqslant|z|\leqslant 3r_i$, one has*

$$\log|\psi(z)|\leqslant\mathsf{K}\cdot r_i^2/c^2. \tag{3.1}$$

To bound the area of $f(\mathbb{D}_{r_i})$ by $\mathsf{K}\cdot r_i^2$, it is crucial to have the above estimate. In fact, by classical complex analysis (cf. [16, Chapter 4]), we can check that ψ is well-defined and that the infinite product is uniformly convergent in bounded domains, and that the exponential growth order of ψ is, by applying Borel's formula [16, page 30, Theorem 3], exactly 2, *i.e.* $\log|\psi(z)|\leqslant\mathsf{K}_\epsilon\cdot|z|^{2+\epsilon}$ for any $\epsilon>0$ and for large $|z|$. Nevertheless, for the critical case that $\epsilon=0$ we need more effort.

Proof. We first study $I:=\prod_{\ell=1}^{i-1}\prod_{\lambda\in B_{r_\ell}}(1-\frac{z}{\lambda})e^{\frac{z}{\lambda}+\frac{z^2}{2\lambda^2}}$ concerning smaller annuli compared with $\mathbf{A}_{r_i}$. Note that $\prod_{\ell=1}^{i-1}\prod_{\lambda\in B_{r_\ell}}|1-\frac{z}{\lambda}|\leqslant(1+3r_i)^{|B_{r_1}|+\cdots+|B_{r_{i-1}}|}\leqslant(1+3r_i)^{\mathsf{K}\cdot r_i/c^2}$. Hence by Lemmas 3.1, 3.2, we receive that $\log|I|\leqslant\log(1+3r_i)^{\mathsf{K}\cdot r_i/c^2}+\sum_{\ell=1}^{i-1}(|\sum_{\lambda\in B_{r_\ell}}\frac{1}{\lambda}||z|+|\sum_{\lambda\in B_{r_\ell}}\frac{1}{\lambda^2}||\frac{z^2}{2}|)\leqslant\mathsf{K}\cdot r_i^2/c^2$.

Secondly, we observe $II:=\prod_{\lambda\in B_{r_i}}(1-\frac{z}{\lambda})e^{\frac{z}{\lambda}+\frac{z^2}{3\lambda^2}}$ concerning the annulus $\mathbf{A}_{r_i}$. Note that each term $|1-\frac{z}{\lambda}|\leqslant\mathsf{K}$ by our construction, and that $|B_{r_i}|\leqslant\mathsf{K}\cdot(r_i/c)^2$. Now using Lemmas 3.1, 3.2, we receive that $\log|II|\leqslant\mathsf{K}\cdot r_i^2/c^2$.

Lastly, we analyze $III:=\prod_{\ell\geqslant i+1}\prod_{\lambda\in B_{r_\ell}}(1-\frac{z}{\lambda})e^{\frac{z}{\lambda}+\frac{z^2}{2\lambda^2}}$ concerning larger annuli compared with $\mathbf{A}_{r_i}$. Now the key point is that, for each $\lambda\in B_{r_\ell}$, one has $|\frac{z}{\lambda}|\leqslant\frac{3r_i}{r_\ell/3}\ll 1$. Therefore we can apply the Taylor expansion of $\log(1-\frac{z}{\lambda})$ to achieve desired estimates. Indeed, noting that $\log\big((1-\frac{z}{\lambda})e^{\frac{z}{\lambda}+\frac{z^2}{2\lambda^2}}\big)=-\sum_{n\geqslant 3}\frac{1}{n}(\frac{z}{\lambda})^n$, hence

$$\Big|\log\prod_{\lambda\in B_{r_\ell}}(1-\frac{z}{\lambda})e^{\frac{z}{\lambda}+\frac{z^2}{2\lambda^2}}\Big|\leqslant\sum_{\lambda\in B_{r_\ell}}\sum_{n\geqslant 3}\frac{1}{n}\Big|\frac{z}{\lambda}\Big|^n\leqslant\mathsf{K}\cdot r_\ell^2/c^2\cdot\sum_{n\geqslant 3}\frac{1}{n}\Big|\frac{3r_i}{r_\ell/3}\Big|^n\leqslant\mathsf{K}/c^2\cdot\frac{r_i^3}{r_\ell}\cdot\mathsf{K}.$$

Since the sequence $\{r_\ell\}_{\ell>i}$ grows very rapid by our construction (2.2), there holds $\sum_{\ell>i}\frac{r_i^3}{r_\ell}<\mathsf{K}$. Thus the above estimate yields $|\log III|\leqslant\mathsf{K}/c^2$.

Combining all the above estimates about I, II, III, we receive the desired inequality (3.1). □

By our construction of Λ, it intersects each disc $\mathbb{D}(z,1)$ at most once. Therefore we introduce

$$\psi_1(z):=\prod_{\lambda\in\Lambda\setminus\mathbb{D}(z,1)}\Big(1-\frac{z}{\lambda}\Big)e^{\frac{z}{\lambda}+\frac{z^2}{2\lambda^2}} \tag{3.2}$$

to capture the asymptotic behavior of ψ away from its zero locus Λ.

Proposition 3.4. *For every $z\in\mathbb{C}$ with large $|z|$, one has*

$$\log|\psi_1(z)|\geqslant-\mathsf{K}\cdot|z|^2/c^2. \tag{3.3}$$

Proof. Fix a positive small number $\eta=\frac{1}{100}$. For every $i\geqslant 1$, we introduce the slightly larger annulus $\widetilde{\mathbf{A}}_{r_i}:=\{x\in\mathbb{C}:(1-\eta)r_i/2\leqslant|x|\leqslant(1+\eta)r_i\}\supseteq\mathbf{A}_{r_i}$, to make sure that $\widetilde{\mathbf{A}}_{r_i}\supseteq B_i$.

Case (i): $|z|$ large with $z \notin \cup_{i\geqslant}\widetilde{\mathbf{A}}_{r_i}$. Then z lies between some two consequent annuli $\widetilde{\mathbf{A}}_{r_j}$ and $\widetilde{\mathbf{A}}_{r_{j+1}}$, i.e., $(1+\eta)r_j < |z| < (1-\eta)r_{j+1}/2$, and it is clear that $\psi(z) = \psi_1(z)$. Firstly, for each $\lambda \in \cup_{\ell=1}^{j} B_{r_\ell}$, we have $|1-\frac{z}{\lambda}| \geqslant |\frac{z}{\lambda}| - 1 \geqslant \eta' := \eta/2$. Thus $\prod_{\ell=1}^{j}\prod_{\lambda\in B_{r_\ell}} |1-\frac{z}{\lambda}| \geqslant \eta'^{\sum_{\ell=1}^{j}|B_{r_\ell}|} \geqslant \eta'^{\mathsf{K}\cdot r_j^2/c^2} \geqslant \eta'^{\mathsf{K}\cdot|z|^2/c^2}$. Next, thanks to Lemmas 3.1, 3.2, we have $\prod_{\ell=1}^{j}\prod_{\lambda\in B_{r_\ell}} |e^{\frac{z}{\lambda}+\frac{z^2}{2\lambda^2}}| \geqslant e^{-|z|\sum_{\ell=1}^{j}\mathsf{K}/c^2 - |z|^2\sum_{\ell=1}^{j}\mathsf{K}\cdot/(c^2 r_\ell)} \geqslant e^{-\mathsf{K}\cdot|z|^2/c^2}$. Lastly, by mimicking the estimate of III in the preceding proof, we receive that

$$\log\Big|\prod_{\ell\geqslant j+1}\prod_{\lambda\in B_{r_\ell}}(1-\frac{z}{\lambda})e^{\frac{z}{\lambda}+\frac{z^2}{2\lambda^2}}\Big| \geqslant -\sum_{\ell\geqslant j+1}\sum_{\lambda\in B_{r_\ell}}\sum_{n\geqslant 3}\frac{1}{n}\Big|\frac{z}{\lambda}\Big|^n \geqslant -\mathsf{K}\cdot|z|^2/c^2.$$

Summarizing the above estimates, we conclude that $\log|\psi_1(z)| = \log|\psi(z)| \geqslant -\mathsf{K}\cdot|z|^2/c^2$.

Case (ii): $|z|$ large with $z \in \widetilde{\mathbf{A}}_{r_j}$ for some j. By repeating the same arguments as above, we can show that, first of all, $\log|\prod_{\ell=1}^{j-1}\prod_{\lambda\in B_{r_\ell}}(1-\frac{z}{\lambda})e^{\frac{z}{\lambda}+\frac{z^2}{2\lambda^2}}| \geqslant -\mathsf{K}\cdot|z|^2/c^2$, and secondly, $\log|\prod_{\ell\geqslant j+1}\prod_{\lambda\in B_{r_\ell}}(1-\frac{z}{\lambda})e^{\frac{z}{\lambda}+\frac{z^2}{2\lambda^2}}| \geqslant -\mathsf{K}\cdot|z|^2/c^2$. By Lemmas 3.1, 3.2, we receive $\log|\prod_{\lambda\in B_{r_j}\setminus\mathbb{D}(z,1)} e^{\frac{z}{\lambda}+\frac{z^2}{2\lambda^2}}| \geqslant -\mathsf{K}\cdot|z|^2/c^2$. Thus the remaining problem is to show that $\log|\prod_{\lambda\in B_{r_j}\setminus\mathbb{D}(z,1)}(1-\frac{z}{\lambda})| \geqslant -\mathsf{K}\cdot|z|^2/c^2$.

To start with, we find a point μ_0 in $c\Gamma$ having the least Euclidean distance to z. Then for every $\lambda = \mu + x_\mu \in B_{r_j}$, we have $|\lambda - z| \geqslant |\mu - z| - |x_\mu| \geqslant \frac{1}{2}(|\mu - z| + |\mu_0 - z|) - \sqrt{2} \geqslant \frac{1}{2}|\mu-\mu_0| - \sqrt{2}$. If moreover assume that $\mu \neq \mu_0$, then we can continue to estimate $|\lambda - z| \geqslant \frac{1}{2}|\mu-\mu_0| - \sqrt{2} \geqslant \frac{1}{4}|\mu-\mu_0|$, whence $|1-\frac{z}{\lambda}| = \frac{|\lambda-z|}{|\lambda|} \geqslant \frac{1}{8}\frac{|\mu-\mu_0|}{|\mu|}$. Since $(\frac{1}{8})^{|B_{r_j}|} \geqslant \exp(-\mathsf{K}\cdot|r_j|^2/c^2)$, we only need to show that

$$\log\prod_{\mu_0\neq\mu\in\mathbf{A}_{r_j}\cap c\Gamma}\frac{|\mu-\mu_0|}{|\mu|} \geqslant -\mathsf{K}\cdot|r_j|^2/c^2 \geqslant -\mathsf{K}\cdot|z|^2/c^2. \tag{3.4}$$

For any positive number r', denote by $\Gamma_{\leqslant r'} \subset \Gamma$ the subset of points whose real and imaginary parts have absolute value $\leqslant r'$. Note that, for every $\mu \in \mathbf{A}_{r_j} \cap c\Gamma \setminus (\mu_0 + \Gamma_{\leqslant \eta r_j})$ far away from μ_0, we have

$$\frac{|\mu-\mu_0|}{|\mu|} > \frac{\eta r_j}{r_j} = \eta. \tag{3.5}$$

Thus these μ's, having cardinality $|\mathbf{A}_{r_j} \cap c\Gamma \setminus (\mu_0 + \Gamma_{\leqslant\eta r_j})| \leqslant \mathsf{K}\cdot|r_j|^2/c^2$, cause no trouble for (3.4).

Lastly, we handle $\mu \in \mathbf{A}_{r_j} \cap c\Gamma \cap (\mu_0 + \Gamma_{\leqslant\eta r_j})$ simultaneously. Note that $c\Gamma\cap\Gamma_{\leqslant\eta r_j}\setminus\{0\}$ can be decomposed into 2 horizontal parts consisting of $\pm\{\ell\cdot c + 0\cdot\sqrt{-1}\}_{\ell=1}^{[\frac{\eta}{c}r_j]}$, plus the remaining $4[\frac{\eta}{c}r_j]+2$ vertical parts consisting of $\pm\{i\cdot c + \ell\cdot c\sqrt{-1}\}_{\ell=1}^{[\frac{\eta}{c}r_j]}$ for $i = 0, \pm1, \pm2, \dots, \pm[\frac{\eta}{c}r_j]$. Each part contains consequential $[\frac{\eta}{c}r_j]$ points having absolute values $\geqslant 1\cdot c, 2\cdot c, \dots, [\frac{\eta}{c}r_j]\cdot c$ respectively. Hence

$$\prod_{0\neq\mu'\in c\Gamma\cap\Gamma_{\leqslant\eta r_j}}|\mu'| \geqslant \Big([\frac{\eta}{c}r_j]!\cdot c^{[\frac{\eta}{c}r_j]}\Big)^{4[\frac{\eta}{c}r_j]+2+2}. \tag{3.6}$$

Now it is time to apply the Stirling formula that for every positive integer n, one has

$$n! = n^n e^{-n}\sqrt{2\pi n}\, e^{\rho_n/12n}$$

for some $|\rho_n| \leqslant 1$. A straightforward computation yields

$$\begin{aligned}\log\prod_{\mu_0\neq\mu\in c\Gamma\cap(\mu_0+\Gamma_{\leqslant\eta r_j})}\frac{|\mu-\mu_0|}{|\mu|} &\geqslant \log\prod_{0\neq\mu'\in c\Gamma\cap\Gamma_{\leqslant\eta r_j}}\frac{|\mu'|}{2r_j}\\ [\text{by (3.6)}]\qquad &\geqslant \log\Big(\big([\frac{\eta}{c}r_j]!\cdot c^{[\frac{\eta}{c}r_j]}\big)^{4[\frac{\eta}{c}r_j]+4}\Big) - \log\Big((2r_j)^{4[\frac{\eta}{c}r_j]^2+4[\frac{\eta}{c}r_j]}\Big)\end{aligned}$$

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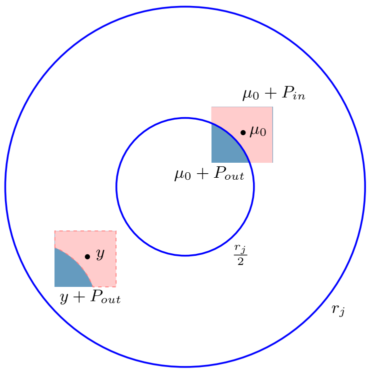
$$= \left[\log\left(([\frac{\eta}{c}r_j]!)^{4[\frac{\eta}{c}r_j]+4}\right) - \log\left((\frac{\eta}{c}r_j)^{4[\frac{\eta}{c}r_j]^2+4[\frac{\eta}{c}r_j]}\right)\right]$$

$$+\log\left((\frac{\eta}{2})^{4[\frac{\eta}{c}r_j]^2+4[\frac{\eta}{c}r_j]}\right)$$

$$\text{[by the Stirling formula]} \qquad \geqslant -\mathsf{K}\cdot|r_j|^2/c^2. \tag{3.7}$$

Now the remaining problem is that $c\Gamma\cap(\mu_0+\Gamma_{\leqslant\eta r_j})$ might exceed $\mathbf{A}_{r_j}$. Let us decompose $c\Gamma\cap(\mu_0+\Gamma_{\leqslant\eta r_j})$ with respect to $\mathbf{A}_{r_j}$ into two parts $(\mu_0+P_{in})\cup(\mu_0+P_{out})$, where the first (resp. second) part lies entirely in (resp. outside) $\mathbf{A}_{r_j}$. Since η is small, we can find some point $y\in\mathbf{A}_{r_j}\cap c\Gamma$ such that $y+P_{out}\subset\mathbf{A}_{r_j}$ stays away from $\mu_0+\Gamma_{\leqslant 8\eta r_j}$. See the picture below for illustration.



Lastly, we decompose $\mathbf{A}_{r_j}\cap c\Gamma$ into 3 disjoint parts, μ_0+P_{in}, $y+P_{out}$ and R the remaining. Note that $\prod_{\mu\in y+P_{out}}\frac{|\mu-\mu_0|}{|\mu|}\geqslant\prod_{\mu\in\mu_0+P_{out}}\frac{|\mu-\mu_0|}{|\mu|}$, because each factor on the left-hand-side $\geqslant\frac{8\eta r_j}{r_j}=8\eta$, while each factor on the right-hand-side $\leqslant\frac{\sqrt{2}\eta r_j}{r_i/4}=4\sqrt{2}\eta$. Thus $\prod_{\mu_0\neq\mu\in\mathbf{A}_{r_j}\cap c\Gamma}\frac{|\mu-\mu_0|}{|\mu|}\geqslant\prod_{\mu_0\neq\mu\in c\Gamma\cap(\mu_0+\Gamma_{\leqslant\eta r_j})}\frac{|\mu-\mu_0|}{|\mu|}\cdot\prod_{\mu\in R}\frac{|\mu-\mu_0|}{|\mu|}$. By the estimates (3.5), (3.7) and that the cardinality $|R|\leqslant\mathsf{K}\cdot r_j^2/c^2$, we conclude the proof. □

4. Estimates

4.1. $f(z)$ is close to $\mathcal{C}_\infty$ unless z is near Λ

Recalling (3.2), we first rewrite

$$||\psi\cdot s_m(z)||_{h_m}=||\psi_1\cdot s_m(z)||_{h_m}\cdot|\diamondsuit(z)|$$

to concentrate positivity to the first factor, where if $z\in\mathbb{D}(\lambda,1)$ for some $\lambda\in\Lambda$ then $\diamondsuit(z)=(1-\frac{z}{\lambda})e^{\frac{z}{\lambda}+\frac{z^2}{2\lambda^2}}$, otherwise $\diamondsuit(z)=1$. Now thanks to (2.1), (3.3), the left part satisfies that

$$||\psi_1\cdot s_m(z)||_{h_m}\geqslant\exp\left((m\cdot\alpha-\mathsf{K}/c^2)\cdot|z|^2\right).$$

A key trick in this paper is that we choose sufficiently large m and c such that

$$m\cdot\alpha-\mathsf{K}/c^2>0. \tag{4.1}$$

Thus for any $\epsilon>0$, for all large $|z|\gg 1$ with $\mathrm{dist}(z,\Lambda)\geqslant\epsilon$, there holds

$$||\psi\cdot s_m(z)||_{h_m}\gg 1, \tag{4.2}$$

i.e., $f(z)$ is very close to $\mathcal{C}_\infty$. Indeed, if $\mathrm{dist}(z,\Lambda) \geqslant 1$, then $\diamondsuit(z) = 1$ and there is nothing to prove; otherwise $\epsilon \leqslant |z-\lambda| < 1$ for some $\lambda \in \Lambda$, hence $|\diamondsuit(z)| \geqslant \frac{|z-\lambda|}{|\lambda|} \cdot \exp(-\frac{|z|}{|\lambda|} - \frac{|z|^2}{2|\lambda|^2}) \geqslant \frac{\epsilon}{|z|+1} \cdot \exp(-2)$ for $|z| \gg 1$, therefore $||\psi \cdot s(z)||_{h_m} = ||\psi_1 \cdot s(z)||_{h_m} \cdot |\diamondsuit(z)|$ is very large.

4.2. Bound the area $\int_{\mathbb{D}_{2r_i}} f^*\omega_X$ from above

From now on, we fix a Kähler form $\omega_{\mathcal{C}} = \frac{\sqrt{-1}}{2} \mathrm{d}\, z \wedge \mathrm{d}\, \overline{z}$ on $\mathcal{C}$ descending from the standard Euclidean area form on $\mathbb{C}$. The metric $h_m = h^{\otimes m}$ of $\mathcal{L}_m = \mathcal{L}^{\otimes m}$ together with the Euclidean metric $|\mathrm{d}\, z|$ on $\mathbb{C}$ provide a metric for the vector bundle $\mathcal{E} := \mathcal{L}_m \oplus \mathbb{C}$, and therefore it induces a metric on the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)$ on $\mathbb{P}(\mathcal{E}) = X$. Restricting to any fiber of $\pi_1 : X \longrightarrow \mathcal{C}$, the curvature form $\Theta_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)}$ is strictly negative due to the property of the Fubini-Study metric of the tautological line bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$. Therefore, by standard compactness argument, for sufficiently small $\epsilon_1 > 0$, we receive a Kähler form on X of the shape

$$\omega_X := \pi_1^*\omega_{\mathcal{C}} - \epsilon_1\, \Theta_{O_{\mathbb{P}(\mathcal{E})}(-1)}. \tag{4.3}$$

We can identify the total space $\mathcal{L}_m = \{(z,\xi) : \xi \in \mathcal{L}_m|_z\}$ with a Zariski open set of $\mathbb{P}(\mathcal{L}_m \oplus \mathbb{C})$, by mapping $(z,\xi) \mapsto (z, [\xi \oplus 1])$. Thus in the local coordinates (z,ξ), the curvature

$$\Theta_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)} = -\,\mathrm{d}\,\mathrm{d}^c \left(\log\left(\|\xi\|^2_{h_m} + 1\right)\right) \tag{4.4}$$

is of Fubini-Study shape. For r lies in $[\frac{1}{3}r_i, 3r_i]$ for some $i \geqslant 1$, by Jensen's formula, we receive

$$\begin{aligned} \int_1^r \frac{\mathrm{d}\, t}{t} \int_{\mathbb{D}_t} f^*\Theta_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)} &= -\frac{1}{4\pi}\int_0^{2\pi} \log\left(\|\psi \cdot s_m\|^2_{h_m} + 1\right)(re^{i\theta})\,\mathrm{d}\,\theta + \frac{1}{4\pi}\int_0^{2\pi} \log\left(\|\psi \cdot s_m\|^2_{h_m} + 1\right)(e^{i\theta})\,\mathrm{d}\,\theta \\ [\text{use (2.1), (3.1)}] \qquad &\geqslant -\mathsf{K} \cdot r_i^2. \end{aligned}$$

Noting that $\pi_1 \circ f = \pi_0$, where $\pi_0 : \mathbb{C} \to \mathbb{C}/\Gamma$ is the canonical projection, we have $\int_{\mathbb{D}_t} f^*\pi_1^*\omega_{\mathcal{C}} = \pi \cdot t^2$. Hence the Nevanlinna order function satisfies the estimate

$$T_{f,r}(\omega_X) = \int_1^r \frac{\mathrm{d}\, t}{t} \int_{\mathbb{D}_t} f^*\omega_X = \int_1^r \frac{\mathrm{d}\, t}{t} \int_{\mathbb{D}_t} f^*\pi_1^*\,\omega_{\mathcal{C}} - \epsilon_1 \int_1^r \frac{\mathrm{d}\, t}{t} \int_{\mathbb{D}_t} f^*\Theta_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)} \leqslant \mathsf{K} \cdot r_i^2. \tag{4.5}$$

Here is a useful observation

$$T_{f,3r_i}(\omega_X) = \int_1^{3r_i} \frac{\mathrm{d}\, t}{t} \int_{\mathbb{D}_t} f^*\omega_X \geqslant \int_{2r_i}^{3r_i} \frac{\mathrm{d}\, t}{t} \int_{\mathbb{D}_{2r_i}} f^*\omega_X = \log(3/2) \cdot \int_{\mathbb{D}_{2r_i}} f^*\omega_X. \tag{4.6}$$

Combining the two estimates above, we conclude that

$$\int_{\mathbb{D}_{2r_i}} f^*\omega_X \leqslant \mathsf{K} \cdot r_i^2. \tag{4.7}$$

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4.3. Bound the area $\int_{\mathbb{D}_{r_i/3}} f^*\omega_X$ from below

By Jensen's formula and (4.4), we have

$$\int_{r_i/4}^{r_i/3} \frac{\mathrm{d}\,t}{t} \int_{\mathbb{D}_t} f^*\Theta_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)} = -\frac{1}{4\pi}\int_0^{2\pi} \log\left(\|\psi\cdot s_m\|^2_{h_m}+1\right)\left(\frac{r_i}{3}e^{i\theta}\right)\mathrm{d}\,\theta + \frac{1}{4\pi}\int_0^{2\pi} \log\left(\|\psi\cdot s_m\|^2_{h_m}+1\right)\left(\frac{r_i}{4}e^{i\theta}\right)\mathrm{d}\,\theta$$

$$[\text{for } r_i \gg 1 \text{ and (4.2)}] \quad = -\frac{1}{4\pi}\int_0^{2\pi} \log\left(\|\psi\cdot s_m\|^2_{h_m}\right)\left(\frac{r_i}{3}e^{i\theta}\right)\mathrm{d}\,\theta + \frac{1}{4\pi}\int_0^{2\pi} \log\left(\|\psi\cdot s_m\|^2_{h_m}\right)\left(\frac{r_i}{4}e^{i\theta}\right)\mathrm{d}\,\theta + o(1). \tag{4.8}$$

By our construction, the holomorphic function ψ is nowhere vanishing on $\overline{\mathbb{D}}_{r_i/3}\setminus\mathbb{D}_{r_i/4}$, hence $\log|\psi|^2$ is harmonic on $\overline{\mathbb{D}}_{r_i/3}\setminus\mathbb{D}_{r_i/4}$. Therefore

$$-\frac{1}{4\pi}\int_0^{2\pi} \log|\psi|^2\left(\frac{r_i}{3}e^{i\theta}\right)\mathrm{d}\,\theta + \frac{1}{4\pi}\int_0^{2\pi} \log|\psi|^2\left(\frac{r_i}{4}e^{i\theta}\right)\mathrm{d}\,\theta = 0.$$

Hence we can continue to compute (4.8) as

$$\begin{aligned}\int_{r_i/4}^{r_i/3} \frac{\mathrm{d}\,t}{t} \int_{\mathbb{D}_t} f^*\Theta_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)} &= -\frac{1}{4\pi}\int_0^{2\pi} \log\left(\|s_m\|^2_{h_m}\right)\left(\frac{r_i}{3}e^{i\theta}\right)\mathrm{d}\,\theta + \frac{1}{4\pi}\int_0^{2\pi} \log\left(\|s_m\|^2_{h_m}\right)\left(\frac{r_i}{4}e^{i\theta}\right)\mathrm{d}\,\theta + o(1)\\ [\text{by (2.1)}] \qquad &= -\frac{1}{4\pi}\int_0^{2\pi} 2m\alpha\left|\frac{r_i}{3}\right|^2\mathrm{d}\,\theta + \frac{1}{4\pi}\int_0^{2\pi} 2m\alpha\left|\frac{r_i}{4}\right|^2\mathrm{d}\,\theta + o(1)\\ &= -\frac{7m\alpha}{144}r_i^2 + o(1).\end{aligned}$$

Thus

$$\int_{r_i/4}^{r_i/3} \frac{\mathrm{d}\,t}{t} \int_{\mathbb{D}_t} f^*\omega_X = \int_{r_i/4}^{r_i/3} \frac{\mathrm{d}\,t}{t} \int_{\mathbb{D}_t} f^*\pi_1^*\omega_{\mathcal{C}} - \epsilon_1\int_{r_i/4}^{r_i/3} \frac{\mathrm{d}\,t}{t} \int_{\mathbb{D}_t} f^*\Theta_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)} = \frac{7}{144}\left(\frac{\pi}{2}+\epsilon_1 m\alpha\right)r_i^2 + o(1). \tag{4.9}$$

Noting that

$$\int_{r_i/4}^{r_i/3} \frac{\mathrm{d}\,t}{t} \int_{\mathbb{D}_t} f^*\omega_X \leqslant \int_{r_i/4}^{r_i/3} \frac{\mathrm{d}\,t}{t} \int_{\mathbb{D}_{r_i/3}} f^*\omega_X = \log(4/3)\cdot\int_{\mathbb{D}_{r_i/3}} f^*\omega_X,$$

we conclude that for $r_i \gg 1$ there holds

$$\int_{\mathbb{D}_{r_i/3}} f^*\omega_X \geqslant \mathsf{K}\cdot r_i^2. \tag{4.10}$$

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4.4. Estimates of $\int_{\mathbb{D}(\lambda,\epsilon)} f^*\omega_X$

Mark the curve $\mathcal{C}_0 = \mathcal{C} \times [0 \oplus 1] \subset X$ induced by the zero section of $\mathcal{L}_m$. Contrasting to the phenomenon in Subsection 4.1, for every $\lambda \in \Lambda$, since $\psi(\lambda) = 0$, $f(\lambda)$ must lie in $\mathcal{C}_0$, which keeps certain positive distance to $\mathcal{C}_\infty = \mathcal{C} \times [1 \oplus 0]$. Let $\epsilon > 0$ be a small positive radius. Recall our convention that $\mathbb{D}([\lambda], \epsilon) \subset \mathcal{C}$ is the disc centered at $[\lambda]$ with the radius ϵ. Note that the image $f\big(\mathbb{D}(\lambda, \epsilon)\big)$ is contained in the small neighborhood $\pi_1^{-1}(\mathbb{D}([\lambda], \epsilon))$ of $\mathbb{P}^1_{[\lambda]} := \pi_1^{-1}([\lambda])$.

By Jensen's formula, for $\lambda \in \Lambda$ with $|\lambda| \gg 1$, we have

$$
\begin{aligned}
\int_\epsilon^{2\epsilon} \frac{\mathrm{d}\,t}{t} \int_{\mathbb{D}(\lambda,t)} f^*\Theta_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)} &= -\frac{1}{4\pi}\int_0^{2\pi} \log\big(\|\psi\cdot s_m\|^2_{h_m}+1\big)(\lambda+2\epsilon e^{i\theta})\,\mathrm{d}\,\theta \\
&\quad + \frac{1}{4\pi}\int_0^{2\pi} \log\big(\|\psi^2\cdot s_m\|^2_{h_m}+1\big)(\lambda+\epsilon e^{i\theta})\,\mathrm{d}\,\theta \\
\text{[by (4.2) and } |\lambda|\gg 1\text{]} \qquad &= -\frac{1}{4\pi}\int_0^{2\pi} \log\big(\|\psi\cdot s_m\|^2_{h_m}\big)(\lambda+2\epsilon e^{i\theta})\,\mathrm{d}\,\theta \\
&\quad + \frac{1}{4\pi}\int_0^{2\pi} \log\big(\|\psi^2\cdot s_m\|^2_{h_m}\big)(\lambda+\epsilon e^{i\theta})\,\mathrm{d}\,\theta + o(1) \\
&= \Big[-\frac{1}{4\pi}\int_0^{2\pi} \log\big(\|s_m\|^2_{h_m}\big)(\lambda+2\epsilon e^{i\theta})\,\mathrm{d}\,\theta + \frac{1}{4\pi}\int_0^{2\pi} \log\big(\|s_m\|^2_{h_m}\big)(\lambda+\epsilon e^{i\theta})\,\mathrm{d}\,\theta\Big] \\
&\quad + \Big[-\frac{1}{4\pi}\int_0^{2\pi} \log|\psi|^2(\lambda+2\epsilon e^{i\theta})\,\mathrm{d}\,\theta + \frac{1}{4\pi}\int_0^{2\pi} \log|\psi|^2(\lambda+\epsilon e^{i\theta})\,\mathrm{d}\,\theta\Big] + o(1).
\end{aligned}
\tag{4.11}
$$

By (2.1), the first bracket $[\cdots]$ in (4.11) can be computed as

$$
-\frac{1}{4\pi}\int_0^{2\pi} 2m\alpha|\lambda+2\epsilon e^{i\theta}|^2\,\mathrm{d}\,\theta + \frac{1}{4\pi}\int_0^{2\pi} 2m\alpha|\lambda+\epsilon e^{i\theta}|^2\,\mathrm{d}\,\theta = -3m\alpha\epsilon^2. \tag{4.12}
$$

Recalling (3.2), we can rewrite $\psi(z) = (\psi_1 \cdot e^{\frac{z}{\lambda}+\frac{z^2}{2\lambda}}) \cdot (1-\frac{z}{\lambda})$, where the factor $\psi_2 := \psi_1 \cdot e^{\frac{z}{\lambda}+\frac{z^2}{2\lambda}}$ is nowhere vanishing for $z \in \mathbb{D}(\lambda, 2\epsilon)$. Hence $\log|\psi_2|^2$ is a harmonic function on $\mathbb{D}(\lambda, 2\epsilon)$, and we receive

$$
-\frac{1}{4\pi}\int_0^{2\pi} \log|\psi_2|^2(\lambda+2\epsilon e^{i\theta})\,\mathrm{d}\,\theta + \frac{1}{4\pi}\int_0^{2\pi} \log|\psi_2|^2(\lambda+\epsilon e^{i\theta})\,\mathrm{d}\,\theta = 0.
$$

Thus we can calculate the second bracket $[\cdots]$ of (4.11) as

$$
-\frac{1}{4\pi}\int_0^{2\pi} \log\Big|1-\frac{\lambda+2\epsilon e^{i\theta}}{\lambda}\Big|^2\,\mathrm{d}\,\theta + \frac{1}{4\pi}\int_0^{2\pi} \log\Big|1-\frac{\lambda+\epsilon e^{i\theta}}{\lambda}\Big|^2\,\mathrm{d}\,\theta = -\log 2. \tag{4.13}
$$

Hence it follows from (4.11), (4.12), (4.13) that

$$\int_{\epsilon}^{2\epsilon} \frac{\mathrm{d}\,t}{t} \int_{\mathbb{D}(\lambda,t)} f^{*}\Theta_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)} = -3m\alpha\epsilon^{2} - \log 2 + o(1) \qquad (\text{for } |\lambda| \gg 1).$$

Note that $\int_{\mathbb{D}(\lambda,t)} f^{*}\pi_1^{*}\omega_{\mathcal{C}} = \pi \cdot t^{2}$. Therefore

$$\int_{\epsilon}^{2\epsilon} \frac{\mathrm{d}\,t}{t} \int_{\mathbb{D}(\lambda,t)} f^{*}\omega_{X} = \frac{\pi}{2} \cdot 3\epsilon^{2} + \epsilon_1\Big(3m\alpha\epsilon^{2} + \log 2 + o(1)\Big) \qquad (\text{for } |\lambda| \gg 1). \tag{4.14}$$

By the same trick as (4.6), we have

$$\int_{\epsilon}^{2\epsilon} \frac{\mathrm{d}\,t}{t} \int_{\mathbb{D}(\lambda,t)} f^{*}\omega_{X} \geqslant \int_{\epsilon}^{2\epsilon} \frac{\mathrm{d}\,t}{t} \int_{\mathbb{D}(\lambda,\epsilon)} f^{*}\omega_{X} \geqslant \log 2 \cdot \int_{\mathbb{D}(\lambda,\epsilon)} f^{*}\omega_{X}.$$

Combining the above two estimates, we conclude

$$\int_{\mathbb{D}(\lambda,\epsilon)} f^{*}\omega_{X} \leqslant \frac{1}{\log 2}\bigg(\frac{\pi}{2} \cdot 3\epsilon^{2} + \epsilon_1\Big(3m\alpha\epsilon^{2} + \log 2 + o(1)\Big)\bigg) \qquad (\text{for } |\lambda| \gg 1). \tag{4.15}$$

Next, we provide an lower bound for $\int_{\mathbb{D}(\lambda,\epsilon)} f^{*}\omega_{X}$. Substituting ϵ by $\frac{\epsilon}{2}$ in (4.14), we receive that

$$\int_{\frac{\epsilon}{2}}^{\epsilon} \frac{\mathrm{d}\,t}{t} \int_{\mathbb{D}(\lambda,t)} f^{*}\omega_{X} = \frac{\pi}{2} \cdot \frac{3\epsilon^{2}}{4} + \epsilon_1\Big(\frac{3m\alpha\epsilon^{2}}{4} + \log 2 + o(1)\Big) \qquad (\text{for } |\lambda| \gg 1). \tag{4.16}$$

Note that

$$\int_{\frac{\epsilon}{2}}^{\epsilon} \frac{\mathrm{d}\,t}{t} \int_{\mathbb{D}(\lambda,t)} f^{*}\omega_{X} \leqslant \int_{\frac{\epsilon}{2}}^{\epsilon} \frac{\mathrm{d}\,t}{t} \int_{\mathbb{D}(\lambda,\epsilon)} f^{*}\omega_{X} = \log 2 \cdot \int_{\mathbb{D}(\lambda,\epsilon)} f^{*}\omega_{X}.$$

Hence it follows from the above two estimates that

$$\int_{\mathbb{D}(\lambda,\epsilon)} f^{*}\omega_{X} \geqslant \frac{1}{\log 2}\bigg(\frac{\pi}{2} \cdot \frac{3\epsilon^{2}}{4} + \epsilon_1\Big(\frac{3m\alpha\epsilon^{2}}{4} + \log 2 + o(1)\Big)\bigg) \qquad (\text{for } |\lambda| \gg 1). \tag{4.17}$$

Note that the right-hand-sides of (4.15) and (4.17) have the same limit $\epsilon_1 > 0$, as $\lambda \in \Lambda$ tends to infinity and ϵ tends to zero.

4.5. Area estimates of $f(\mathbb{C})$ near horizontal curves

An irreducible algebraic curve $D \subset X$ is said to be vertical if $\pi_1(D)$ is a point; otherwise it is called horizontal, in the sense that $\pi_1(D) = \mathcal{C}$.

Firstly, for a vertical curve $\mathbb{P}^1_{[y]} = \pi_1^{-1}([y])$, by the estimates (4.15) and (4.17), the area of $f(\mathbb{D}_r)$ near $\mathbb{P}^1_{[y]}$, as $r \to \infty$, is mostly determined by asymptotic growth of $|\mathbb{D}_r \cap \Lambda|$.

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Next, for the horizontal curve $D = \mathcal{C}_\infty$, by Subsection 4.1, $f(\mathbb{D}_r)$ shall concentrate a large portion of area near $\mathcal{C}_\infty$ as $r \to \infty$.

Lastly, for any other irreducible horizontal curve $D \neq \mathcal{C}_\infty$, we devote this subsection to prove that, roughly speaking, every time when $f(\mathbb{D}_r)$ intersects with D, it contributes negligible area about there.

To start with, we take a general point $d_0 \in D \setminus \mathcal{C}_\infty$ such that $\pi_1|_D$ is regular at d_0, i.e., some small open neighborhood $U \subset D$ of d_0 is a graph over $\pi_1(U)$ containing $\pi_1(d_0) =: c_0$. By shrinking U we may assume that U stays away from $\mathcal{C}_\infty$, and that $\pi_1(U)$ is a small disc $\mathbb{D}(c_0, 3\delta)$ for some $\delta > 0$, and that the line bundle $\mathcal{L}_m$ has a local trivialization $\mathcal{L}_m|_{\mathbb{D}(c_0,3\delta)} \cong \mathbb{D}(c_0, 3\delta) \times \mathbb{C}$, which extends to an identification

$$\vartheta : \pi_1^{-1}(\mathbb{D}(c_0, 3\delta)) \xrightarrow{\cong} \mathbb{D}(c_0, 3\delta) \times \mathbb{P}^1(\mathbb{C}) \tag{4.18}$$

by fiberwised compactification $\mathbb{C} \hookrightarrow \mathbb{P}^1(\mathbb{C})$ sending $z \mapsto [z : 1]$. Hence we can read the coordinates of U in the chart $\mathbb{D}(c_0, 3\delta) \times \mathbb{C}$ as the graph of some holomorphic map $u : \mathbb{D}(c_0, 3\delta) \to \mathbb{C}$.

Let p_1, p_2 be the projections of $\mathbb{D}(c_0, 3\delta) \times \mathbb{P}^1(\mathbb{C})$ to the two factors. Let ω_{FS} be the Fubini-Study form on $\mathbb{P}^1(\mathbb{C})$. By compactness argument, the metric $p_1^*\omega_{\mathcal{C}} + p_2^*\omega_{\mathrm{FS}}$ is comparable with $(\vartheta^{-1})^*\omega_X$ on $\mathbb{D}(c_0, \frac{5\delta}{2}) \times \mathbb{P}^1(\mathbb{C})$, namely

$$\mathsf{K}_{c_0,\delta,\vartheta}^{-1} \cdot (p_1^*\omega_{\mathcal{C}} + p_2^*\omega_{\mathrm{FS}}) \leqslant (\vartheta^{-1})^*\omega_X \leqslant \mathsf{K}_{c_0,\delta,\vartheta} \cdot (p_1^*\omega_{\mathcal{C}} + p_2^*\omega_{\mathrm{FS}}). \tag{4.19}$$

Fix a positive number $\epsilon \ll \delta$. Then the neighborhood $\pi_1^{-1}(\mathbb{D}(c_0, \delta)) \cap D$ of d_0 in the coordinates reads as

$$U_1 := \{(z, w) : z \in \mathbb{D}(c_0, \delta), w = u(z)\},$$

and it is contained in the small open neighborhood

$$U_1^\epsilon := \{(z, w) : z \in \mathbb{D}(c_0, \delta + \epsilon), |w - u(z)| < \epsilon\}.$$

Fix a positive small number $\delta' < \delta/2$. By Subsection 4.1, for $|z| \gg 1$ large with $\mathrm{dist}(z, \Lambda) > \delta'$, one sees that $f(z)$ is very close to $\mathcal{C}_\infty$, hence it is outside U_1^ϵ. Thus for bounding the area of $f(\mathbb{D}_r) \cap U_1^\epsilon$ from above by $o(1) \cdot r^2$, we only need to show that, for $\lambda \in \Lambda$ with $|\lambda| \gg 1$, the area $f(\mathbb{D}(\lambda, \delta')) \cap U_1^\epsilon$ is negligible $o(1)$.

Observation 4.1. *Set $f_2 := p_2 \circ \vartheta \circ f$. For $\lambda \in \Lambda$ with $|\lambda| \gg 1$ and with $[\lambda] \in \mathbb{D}(c_0, \delta + \delta')$, one has*

$$(\vartheta \circ f)^{-1}(U_1^\epsilon) \cap \mathbb{D}(\lambda, \delta') \subset f_2^{-1}\Big(\mathbb{D}\big(u([\lambda]), 2\epsilon\big)\Big) \cap \mathbb{D}(\lambda, \delta').$$

Proof. By continuity of u and by compactness of $\overline{\mathbb{D}}(c_0, \frac{5}{2}\delta)$, there exists some positive number $\delta_\epsilon < \delta'$ such that, for any two points $x_1, x_2 \in \overline{\mathbb{D}}(c_0, \frac{5}{2}\delta)$ with $|x_1 - x_2| < \delta_\epsilon$, there holds $|u(x_1) - u(x_2)| < \epsilon$. By Subsection 4.1, for all $\lambda \in \Lambda$ with $|\lambda| \gg 1$, the image of $\mathbb{D}(\lambda, \delta') \setminus \mathbb{D}(\lambda, \delta_\epsilon)$ under $\vartheta \circ f$ is outside U_1^ϵ, therefore

$$(\vartheta \circ f)^{-1}(U_1^\epsilon) \cap \mathbb{D}(\lambda, \delta') \subset (\vartheta \circ f)^{-1}(U_1^\epsilon) \cap \mathbb{D}(\lambda, \delta_\epsilon).$$

By definition, every element z in the right-hand-side satisfies that $|f_2(z) - u([z])| < \epsilon$ and $|z - \lambda| < \delta_\epsilon$. Thus $|f_2(z) - u([\lambda])| \leqslant |f_2(z) - u([z])| + |u([z]) - u([\lambda])| < \epsilon + \epsilon = 2\epsilon$, which concludes the proof. □

Now for every $v \in \mathbb{C}$ having absolute value $|v| \leqslant R := \max\{|u(z)| + \epsilon : z \in \overline{\mathbb{D}}(c_0, \frac{5}{2}\delta)\} < \infty$, for $\lambda \in \Lambda$ with $|\lambda| \gg 1$ and with $[\lambda] \in \mathbb{D}(c_0, \delta + \delta')$, consider the restricted holomorphic function

$$f_2 : \mathbb{D}(\lambda, \delta') \longrightarrow \mathbb{C}.$$

Since $|f_2| \gg 1$ (in particular $|f_2| > R$) on $\partial\mathbb{D}(\lambda, \delta')$ by Subsection 4.1, by the Argument Principle, the number of solutions of the equation $f_2(y) = v$ on the disc $\mathbb{D}(\lambda, \delta')$, counting multiplicities, equals to

$$\frac{1}{2\pi\sqrt{-1}} \int\limits_{z\in\partial\mathbb{D}(\lambda,\delta')} \frac{(f_2 - v)'}{f_2 - v}(z)\,\mathrm{d}\,z.$$

Noting that the above quantity takes integer value, and that it varies continuously with respect to $v \in \mathbb{D}_R$, it must be a constant for every $v \in \mathbb{D}_R$. Now checking the special value $v = 0$, we see that the number of solution is just 1. Thus Observation 4.1 implies that, for $\lambda \in \Lambda$ with $|\lambda|$ sufficiently large and with $[\lambda] \in \mathbb{D}(c_0, \delta + \delta')$, we have

$$\mathrm{Area}\,\Big((\vartheta \circ f)^{-1}(U_1^\epsilon) \cap \mathbb{D}(\lambda, \delta')\Big)_{f_2^*\omega_{\mathrm{FS}}} \leqslant \mathrm{Area}\,\big(u([\lambda]), 2\epsilon\big)_{\omega_{\mathrm{FS}}} \leqslant \mathsf{K}\cdot\epsilon^2. \tag{4.20}$$

Also, by Subsection 4.1, for every positive number $\epsilon' > 0$, for $\lambda \in \Lambda$ with sufficiently large $|\lambda|$, we have

$$(\vartheta \circ f)^{-1}(U_1^\epsilon) \cap \mathbb{D}(\lambda, \delta') \subset \mathbb{D}(\lambda, \epsilon').$$

Therefore

$$\mathrm{Area}\,\Big((\vartheta \circ f)^{-1}(U_1^\epsilon) \cap \mathbb{D}(\lambda, \delta')\Big)_{\pi_1^*\omega_C} \leqslant \mathrm{Area}\,\Big(\mathbb{D}(\lambda, \epsilon')\Big)_{\pi_1^*\omega_C} = \pi\cdot{\epsilon'}^2. \tag{4.21}$$

Summarizing (4.19), (4.20), (4.21), for $\lambda \in \Lambda$ with $|\lambda| \gg 1$ and $[\lambda] \in \mathbb{D}(c_0, \delta + \delta')$, we have

$$\mathrm{Area}\,\Big((\vartheta \circ f)^{-1}(U_1^\epsilon) \cap \mathbb{D}(\lambda, \delta')\Big)_{f^*\omega_X} \leqslant \mathsf{K}_{c_0,\delta,\vartheta}\cdot(\mathsf{K}\,\epsilon^2 + \pi\,{\epsilon'}^2). \tag{4.22}$$

4.6. Area of f near $\lambda \in \Lambda$ revisit

An alternative way to interpret (4.17) is the following

Observation 4.2. *Let $\delta' > 0$ be a small positive number. Let U be an open neighborhood of $\mathcal{C}_\infty$ such that its closure $\overline{U}$ stays away from $\mathcal{C}_0$. Then one has the estimate*

$$\mathrm{Area}\,\Big((X \setminus U) \cap f\big(\mathbb{D}(\lambda, \delta')\big)\Big)_{\omega_X} \geqslant \mathsf{K}_U \qquad (\forall\lambda \in \Lambda \text{ with } |\lambda| \gg 1). \tag{4.23}$$

This strengthens (4.17) in a qualitative sense, and will be helpful for discussing diffuse parts later. Before going to the proof of the above result, recall the following special case of Wirtinger's inequality.

Proposition 4.3 *(Cf. [17, page 7]). Let C be a proper holomorphic curve in the ball $B(0, \epsilon) \subset \mathbb{C}^n$ passing through 0. Then with the standard Euclidean metric, one has* $\mathrm{Area}(C) \geqslant \pi\epsilon^2$. □

Proof of Observation 4.2. By compactness, $\mathcal{C}_0$ can be covered by finitely many open neighborhoods U_i, being disjoint with U, with charts $\vartheta_i : U_i \to V_i \subset \mathbb{C}^2$. By shrinking U_i's if necessary, we can assume that every pull-back by ϑ_i of the standard Euclidean metric on $\mathbb{C}^2$ is comparable with ω_X. Again by the compactness of $\mathcal{C}_0$, for every point $c \in \mathcal{C}_0$, certain chart V_i of $U_i \ni c$ contains a sufficiently large ball centered at $\vartheta_i(c)$ with a uniform radius $r > 0$. Now, by Subsection 4.1, for $\lambda \in \Lambda$ with $|\lambda| \gg 1$, for $c = f(\lambda) \in \mathcal{C}_0$, in the chart V_i we see that $\vartheta_i\big(f(\mathbb{D}(\lambda, \delta')) \cap U_i\big)$ contains a proper holomorphic curve in the ball $B(\vartheta_i(c), r)$, having positive area $\geqslant \pi r^2$ by Proposition 4.3. The desired conclusion then follows from the comparability of metrics. □

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5. Algorithm

First of all, we require that m, c satisfy the condition (4.1).

Next, we choose distinct points in a strip of $\mathcal{D}$

$$\{y_i\}_{i\in\mathbb{Z}_+} \subset \{x + y\sqrt{-1} : 1/6 \leqslant x < 1/3, 0 \leqslant y < 1\}. \tag{5.1}$$

A collection of $N \geqslant 1$ points

$$b_1, \ldots, b_N \in \mathcal{D}_{\mathsf{R}} := \{x + y\sqrt{-1} : 1/2 \leqslant x < 1, 0 \leqslant y < 1\}$$

is said to be *distributed sparsely*, if for any disc $\mathbb{D}(a, r)$, the following cardinality estimate holds

$$\big|\mathbb{D}(a, r) \cap \{b_1, \ldots, b_N\}\big| \leqslant \max\{1, \mathsf{K} \cdot r^2 N\}. \tag{5.2}$$

For instance, this can be reached by choosing distinct points

$$b_1, \ldots, b_N \in \Big\{\frac{[\sqrt{N}] + 1 + \ell_1}{2[\sqrt{N}] + 2} + \frac{\ell_2}{[\sqrt{N}] + 1}\sqrt{-1}\Big\}_{0\leqslant \ell_1, \ell_2 \leqslant [\sqrt{N}]}.$$

A key observation is that, every point $b \in \mathcal{D}_{\mathsf{R}}$ keeps a uniform positive distance to $\{y_i\}_{i\in\mathbb{Z}_+}$

$$\mathrm{dist}(b, y_i) \geqslant 1/6 \qquad (\forall\, i \,\geqslant\, 1). \tag{5.3}$$

Let $\mathcal{S} = \{I \subset \mathbb{Z}_+ : I \text{ is a finite nonempty set, or } I = \varnothing, \text{ or } I = \mathbb{Z}_+\}$. Then $\mathcal{S}$ is countable, i.e., there exists some bijection $\sigma : \mathcal{S} \to \mathbb{Z}_+$. On the other hand, we can decompose $\mathbb{Z}_+$ into some infinite disjoint union $\cup_{i\in\mathbb{Z}_+}\mathcal{Z}_i$, where each component $\mathcal{Z}_i$ contains infinitely many integers. For every $I \in \mathcal{S}$, write all the elements of $\mathcal{Z}_{\sigma(I)}$ in the increasing order as $Z_1^I < Z_2^I < Z_3^I < \cdots$. Thus we can rearrange

$$\mathbb{Z}_+ = \cup_{I\in\mathcal{S}}\mathcal{Z}_{\sigma(I)} = \cup_{I\in\mathcal{S}} \cup_{j\geqslant 1} \{Z_j^I\}.$$

For every positive integer $i = Z_j^I$, we now choose all the $x_\mu \in \mathcal{D}$ for $\mu \in \mathbf{A}_{r_i} \cap c\Gamma$ as follows.

- Case I: $I = \varnothing$.
  We require that all the x_μ's are distributed sparsely in $\mathcal{D}_{\mathsf{R}}$.
- Case II: $I = \{i_1, \ldots, i_k\}$ is some finite set of $k \geqslant 1$ elements, and $j \geqslant 1$ is an odd integer.
  We choose all x_μ from $\{y_{i_1}, \ldots, y_{i_k}\}$, such that, for every $\ell = 1, \ldots, k$, $x_\mu = y_{i_\ell}$ for at least $[\frac{|\mathbf{A}_{r_i}\cap c\Gamma|}{k}]$ times.
- Case II': $I = \{i_1, \ldots, i_k\}$ is some finite set of $k \geqslant 1$ elements, and $j \geqslant 1$ is an even integer.
  We choose some $x_\mu = y_{i_\ell}$ for $[\frac{|\mathbf{A}_{r_i}\cap c\Gamma|}{2k}]$ times, where $\ell = 1, \ldots, k$, and we require the remaining x_μ's to be distributed sparsely in $\mathcal{D}_{\mathsf{R}}$.
- Case III: $I = \mathbb{Z}_+$, and $j \geqslant 1$ is odd.
  Fix a sequence of positive numbers $\{\alpha_j\}_{j=1}^\infty$ with $\sum_{j=1}^\infty \alpha_j = 1$. We choose all x_μ from $\{y_\ell\}_{\ell=1}^\infty$, such that, for every $\ell \geqslant 1$, $x_\mu = y_\ell$ for at least $[\alpha_\ell \cdot |\mathbf{A}_{r_i} \cap c\Gamma|]$ times.
- Case III' $I = \mathbb{Z}_+$, and $j \geqslant 1$ is even.
  For every $\ell \geqslant 1$, we choose some $x_\mu = y_\ell$ for $[\frac{\alpha_\ell}{2} \cdot |\mathbf{A}_{r_i} \cap c\Gamma|]$ times; and we choose the remaining x_μ's to be distributed sparsely in $\mathcal{D}_{\mathsf{R}}$.

6. Proofs

We are now in position to prove the main results. Recall that from a given sequence of discs of increasing radii $r_i \nearrow \infty$, after a perturbation and passing to some subsequence, we can always receive an Ahlfors current for f.

Observation 6.1. *From the sequence of radii $\{\frac{r_i}{3}\}_{i \geqslant 1}$, one receives a singular Ahlfors current T of the shape*

$$T = c_\infty \cdot [\mathcal{C}_\infty].$$

Proof. Note that all points in $\mathbb{D}_{r_i/3} \setminus \mathbb{D}_{r_{i-1}+2}$ keep positive distance $\geqslant 2-\sqrt{2}$ to Λ. Thus for any small open neighborhood U of $\mathcal{C}_\infty$, by Subsection 4.1, for $i \gg 1$, for every $z \in \mathbb{D}_{r_i/3} \setminus \mathbb{D}_{r_{i-1}+2}$, we have $f(z) \in U$. Note that for $i \gg 1$, by (4.7) and (4.10), one sees that the area of $f(\mathbb{D}_{r_{i-1}+2})$ is negligible comparing with that of $f(\mathbb{D}_{r_i/3})$, namely

$$\int\limits_{\mathbb{D}_{r_{i-1}+2}} f^*\omega_X \leqslant \mathsf{K} \cdot r_{i-1}^2 = o(1) \cdot r_i^2 \leqslant o(1) \cdot \int\limits_{\mathbb{D}_{r_i/3}} f^*\omega_X.$$

Thus T charges zero mass outside U. Since this holds true for any open neighborhood $U \supset \mathcal{C}_\infty$, we conclude that T must be supported on $\mathcal{C}_\infty$. $\square$

Observation 6.2. *From the sequence of radii $\{r_{Z_j^\varnothing}\}_{j \geqslant 1}$, one gets an Ahlfors current T having the shape*

$$T = a_\infty \cdot [\mathcal{C}_\infty] + T_{\text{Diff}},$$

where a_∞ is some positive number and T_{Diff} is a nontrivial diffuse part.

Proof. *Step 1*: T charges positive mass along $\mathcal{C}_\infty$.

Indeed, for any open neighborhood U of $\mathcal{C}_\infty$, it follows from the preceding proof and the estimate (4.10) that, for $j \gg 1$ and for $i = Z_j^\varnothing$, we have $f(\mathbb{D}_{r_i/3} \setminus \mathbb{D}_{r_{i-1}+2}) \subset U$ and $\int_{\mathbb{D}_{r_i/3} \setminus \mathbb{D}_{r_{i-1}+2}} f^*\omega_X \geqslant \mathsf{K} \cdot r_i^2$. On the other hand, by (4.7) we know that $\int_{\mathbb{D}_{r_i}} f^*\omega_X \leqslant \mathsf{K} \cdot r_i^2$. Thus T charges U by positive mass $\geqslant \mathsf{K} > 0$. Since this holds true for any U, we conclude that T charges positive mass along $\mathcal{C}_\infty$.

Step 2: T does not charge any other algebraic curve.

If $D \neq \mathcal{C}_\infty$ is an irreducible horizontal curve, using the same notations as that of Subsection 4.5, by the estimate (4.22), and by choosing $\epsilon' \leqslant \epsilon$, we know that T charges the neighborhood $\vartheta^{-1}(U_1^\epsilon)$ by a small mass $\leqslant \mathsf{K}_{c_0,\delta,\vartheta} \cdot \epsilon^2$. Letting $\epsilon \to 0$, we receive that T charges no mass on $U_1 \subset D$. Thus T cannot charge positive mass on D.

If $D = \mathbb{P}_a^1$ is an irreducible vertical curve, for an open neighborhood U of $\mathcal{C}_\infty$, we claim that T charges no mass on $D \setminus U$. Indeed, for any small $\epsilon > 0$, by Subsection 4.1, for $|z| \gg 1$ with $f(z) \in \pi_1^{-1}(\mathbb{D}(a,\epsilon)) \setminus U$, there must be some $\lambda \in \Lambda$ such that $z \in \mathbb{D}(\lambda, \epsilon)$. Note also that $\pi_1(z) = [z]$, we get $[\lambda] \in \mathbb{D}(a, 2\epsilon)$. By (5.2), we have

$$|\pi_0^{-1}(\mathbb{D}(a,2\epsilon)) \cap \Lambda \cap \mathbb{D}_{r_{Z_j^\varnothing}}| \leqslant \mathsf{K}\epsilon^2 \cdot r_{Z_j^\varnothing}^2.$$

Hence by the estimates (4.15) and (4.10), such points $z \in \mathbb{D}_{r_{Z_j^\varnothing}}$ with $f(z) \in \pi_1^{-1}(\mathbb{D}(a,\epsilon)) \setminus U$ constitute only small portion of area measured by $f^*\omega_X$, thus T charges mass $\leqslant \mathsf{K} \cdot \epsilon^2$ over $\pi_1^{-1}(\mathbb{D}(a,\epsilon)) \setminus U$. Letting $\epsilon \to 0$, we conclude the claim. Since this holds for any open neighborhood U, we receive that T does not charge $\mathbb{P}_a^1$, which finishes this step.

Step 3: T has positive mass outside $\mathcal{C}_\infty$.

Take any small open neighborhood U of $\mathcal{C}_\infty$ such that $\overline{U} \cap \mathcal{C}_0$ is empty. Note that for every $\lambda \in \Lambda \cap \mathbf{A}_{r_{Z_j^\varnothing}}$ where $j \gg 1$, by (4.23), the image of f about λ contributes $\geqslant \mathsf{K}$ area outside U. Moreover, by our construction $|\Lambda \cap \mathbf{A}_{r_{Z_j^\varnothing}}| \geqslant \mathsf{K} \cdot r^2_{Z_j^\varnothing}$, thus the total area of $f(\mathbb{D}_{r_{Z_j^\varnothing}}) \setminus U$ is $\geqslant \mathsf{K} \cdot r^2_{Z_j^\varnothing}$. Lastly, by (4.7), we conclude that T charges positive mass outside U. □

Observation 6.3. *Fix any finite subset $I = \{i_1, \dots, i_k\} \subset \mathbb{Z}_+$ having cardinality $k \geqslant 1$. Then from the sequence of radii $\{r_{Z^I_{2j-1}}\}_{j \geqslant 1}$, one receives an Ahlfors current T having the shape*

$$T = a_\infty \cdot [\mathcal{C}_\infty] + \sum_{\ell=1}^{k} a_{i_\ell} \cdot [\mathbb{P}^1_{[y_{i_\ell}]}],$$

where $a_\infty, a_{i_1}, \dots, a_{i_k}$ are some positive numbers.

Proof. For any small open neighborhood U of $\mathcal{C}_\infty \cup \mathbb{P}^1_{[y_{i_1}]} \cup \cdots \cup \mathbb{P}^1_{[y_{i_k}]}$, by Subsection 4.1, for $j \gg 1$, for all points $z \in \mathbb{D}_{r_{Z^I_{2j-1}}} \setminus \mathbb{D}_{r_{Z^I_{2j-1}-1}+2}$ we have $f(z) \in U$. Indeed, choose a very small $\epsilon > 0$ such that U contains $\pi_1^{-1}(\mathbb{D}([y_{i_\ell}], \epsilon))$ for every $\ell = 1, \dots, k$. Then if $\mathrm{dist}(z, \Lambda) \geqslant \epsilon$, we know that $f(z)$ is very close to $\mathcal{C}_\infty$, whence $f(z) \in U$; otherwise $\mathrm{dist}(z, \Lambda) < \epsilon$, that is $z \in \mathbb{D}(\lambda, \epsilon)$ for some $\lambda \in \pi_0^{-1}([y_{i_\ell}])$ ($\ell = 1, \dots, k$) by our construction of Λ, hence $f(z) \in \pi_1^{-1}(\mathbb{D}([y_{i_\ell}], \epsilon)) \subset U$.

Therefore, by the same argument as that of Observation 6.1, one sees that T is supported in $\mathcal{C}_\infty \cup \mathbb{P}^1_{[y_{i_1}]} \cup \cdots \cup \mathbb{P}^1_{[y_{i_k}]}$. It remains to check that T charges positive mass in each of these components.

Indeed, first of all, our algorithm guarantees that

$$|\Lambda \cap \mathbf{A}_{r_{Z^I_{2j-1}}} \cap \pi_0^{-1}([y_{i_\ell}])| \geqslant \mathsf{K} \cdot r^2_{Z^I_{2j-1}} \qquad (\ell = 1, \dots, k).$$

By (4.17), for any fixed small $\epsilon > 0$, for large $j \gg 1$ and $i = r_{Z^I_{2j-1}}$, for any $\lambda \in \Lambda \cap \mathbf{A}_{r_i} \cap \pi_0^{-1}([y_{i_\ell}])$, the holomorphic disc $f(\mathbb{D}(\lambda, \epsilon))$ is contained in $\pi_1^{-1}(\mathbb{D}([y_{i_\ell}], \epsilon))$ with area $\int_{\mathbb{D}(\lambda,\epsilon)} f^*\omega_X \geqslant \mathsf{K}$ bounded from below by some uniformly positively constant independent of ϵ. Thus the total area of such discs is $\geqslant \mathsf{K} \cdot r_i^2$. Noting that (4.7) implies $\int_{\mathbb{D}(r_i)} \leqslant \mathsf{K} \cdot r_i^2$, thus T charges mass $\geqslant \mathsf{K}$ on $\pi_1^{-1}(\mathbb{D}([y_{i_\ell}], \epsilon))$. Letting $\epsilon \to 0$, we conclude that T charges positive mass on $\mathbb{P}^1_{[y_{i_\ell}]}$. Lastly, by the same argument as the Step 1 of Observation 6.2, we see that T charges positive mass on $\mathcal{C}_\infty$. Thus we conclude the proof. □

Observation 6.4. *Fix any finite subset $I = \{i_1, \dots, i_k\} \subset \mathbb{Z}_+$ having cardinality $k \geqslant 1$. Then from the sequence of radii $\{r_{Z^I_{2j}}\}_{j \geqslant 1}$, one receives an Ahlfors current T having the shape*

$$T = a_\infty \cdot [\mathcal{C}_\infty] + \sum_{\ell=1}^{k} a_{i_\ell} \cdot [\mathbb{P}^1_{[y_{i_\ell}]}] + T_{\mathrm{Diff}},$$

where a_∞, a_{i_ℓ} ($1 \leqslant \ell \leqslant k$) are some positive constants and where T_{Diff} is a nontrivial diffuse part.

Proof. *Step 1*: T charges positive mass along $\mathcal{C}_\infty, \mathbb{P}^1_{[y_{i_1}]}, \cdots, \mathbb{P}^1_{[y_{i_k}]}$.

This follows from the same arguments as in the preceding proof.

Step 2: T does not charge any other algebraic curve.

We can check it by using the same arguments as in the Step 2 of Observation 6.2.

Step 3: T has positive mass outside $\mathcal{C}_\infty \cup \mathbb{P}^1_{[y_{i_1}]} \cup \cdots \cup \mathbb{P}^1_{[y_{i_k}]}$.

The argument is similar to the Step 3 of Observation 6.2. The key point is that, by our algorithm

$$|(\mathcal{D}_{\mathsf{R}} + \Gamma) \cap \Lambda \cap \mathbb{D}_{r_{Z^I_{2j}}}| \geqslant \mathsf{K} \cdot r^2_{Z^I_{2j}},$$

and for every $\lambda \in (\mathcal{D}_{\mathsf{R}} + \Gamma) \cap \Lambda \cap \mathbb{D}_{r_{z^I_{2j}}}$, $[\lambda]$ keeps uniform distances $\geqslant \frac{1}{6} = \epsilon > 0$ to $[y_{i_1}], \dots, [y_{i_k}]$. Assuming moreover that $j \gg 1$, in the same notation as Observation 4.2 we receive

$$\operatorname{Area}\Big((X \setminus U) \cap f\big(\mathbb{D}(\lambda, \epsilon/2)\big)\Big)_{\omega_X} \geqslant \mathsf{K}_U.$$

Note that $f\big(\mathbb{D}(\lambda, \epsilon/2)\big)$ stays away from $\pi_1^{-1}(\mathbb{D}([y_{i_\ell}], \epsilon/2))$ for all $\ell = 1, \dots, k$. Thus by the same argument as the preceding proof, we see that T charges positive mass outside $U \cup \big(\cup_{\ell=1}^k \pi_1^{-1}(\mathbb{D}([y_{i_\ell}], \epsilon/2))\big)$. □

By much the same arguments, we have the following two results.

Observation 6.5. *From the sequence of radii $\{r_{z^{\mathbb{Z}_+}_{2j-1}}\}_{j \geqslant 1}$, one receives an Ahlfors current T having the shape*

$$T = a_\infty \cdot [\mathcal{C}_\infty] + \sum_{\ell=1}^{\infty} a_\ell \cdot [\mathbb{P}^1_{[y_\ell]}],$$

where a_∞, a_ℓ $(\ell \geqslant 1)$ are positive numbers. □

Observation 6.6. *From the sequence of radii $\{r_{z^{\mathbb{Z}_+}_{2j}}\}_{j \geqslant 1}$, one receives an Ahlfors current T having the shape*

$$T = a_\infty \cdot [\mathcal{C}_\infty] + \sum_{\ell=1}^{\infty} a_\ell \cdot [\mathbb{P}^1_{[y_\ell]}] + T_{\mathrm{Diff}},$$

where a_∞, a_ℓ $(\ell \geqslant 1)$ are positive numbers, and where T_{Diff} is a nontrivial diffuse part. □

Thus we prove Theorems 1.3, 1.4.

7. Examples

7.1. Diffuse Ahlfors currents

Let $\mathcal{A} = \mathbb{C}/\Lambda \times \mathbb{C}/\Lambda$ be the surface obtained as the product of two elliptic curves where Λ is a lattice. Fix a reference metric $\omega_{\mathcal{A}} := \mathrm{d}\,\mathrm{d}^c\,|z_1|^2 + \mathrm{d}\,\mathrm{d}^c\,|z_2|^2$ on $\mathcal{A}$. Choose an irrational number $\lambda \in \mathbb{R} \setminus \mathbb{Q}$. Consider the holomorphic curve $f : \mathbb{C} \longrightarrow \mathcal{A}$ given by $f(z) = ([z], [\lambda z])$.

Proposition 7.1. *Any Ahlfors current T of f is diffuse.*

Proof. Since there is no nonconstant holomorphic map from $\mathbb{P}^1(\mathbb{C})$ to an elliptic curve, $\mathcal{A}$ contains no rational curve. Hence by a theorem of Duval [11], it suffices to check that T charges zero mass along any elliptic curve in $\mathcal{A}$.

Fact *(Cf. [18, Prop. 1.3.2]). Let $\Phi : \mathbb{C}/\Gamma_1 \longrightarrow \mathbb{C}/\Gamma_2$ be a holomorphic map between complex tori. Then there exist complex numbers m, b with $m\Gamma_1 \subset \Gamma_2$, such that $\Phi([z]) = [mz + b]$.*

Therefore, any nonconstant holomorphic map $\iota : \mathbb{C}/\Gamma_3 \to \mathbb{C}/\Lambda \times \mathbb{C}/\Lambda$ from an elliptic curve to $\mathcal{A}$ can be written explicitly as $\iota([z]) = ([m_1 z + b_1], [m_2 z + b_2])$ for some complex numbers m_1, m_2, b_1, b_2, such that $(m_1, m_2) \neq (0, 0)$ and $m_1\Gamma_3, m_2\Gamma_3 \subset \Gamma$. Hence $m_2 - \lambda m_1 \neq 0$. We claim that the intersection numbers

$$|\iota(\mathbb{C}/\Gamma_3) \cap f(\mathbb{D}_r)| \leqslant \mathsf{K} \cdot r^2 \tag{7.1}$$

for $r \gg 1$. Indeed, we can find a large disc $\mathbb{D}_R$ containing a fundamental domain of Γ_3. Then for $z \in \mathbb{D}_r, y \in \mathbb{D}_R$ with $([z], [\lambda z]) = ([m_1 y + b_1], [m_2 y + b_2])$, we receive that

$$z - (m_1 y + b_1) = \lambda_1, \qquad \lambda z - (m_2 y + b_2) = \lambda_2 \tag{7.2}$$

for some $\lambda_1, \lambda_2 \in \Lambda$ having absolute values less than $r + \mathsf{K}, |\lambda| \cdot r + \mathsf{K}$ respectively.

Since $m_2 - \lambda m_1 \neq 0$, we can solve the linear equation (7.2) as

$$z = \frac{m_2(\lambda_1 + b_1) - m_1(\lambda_2 + b_2)}{m_2 - \lambda m_1}, \qquad y = \frac{\lambda(\lambda_1 + b_1) - (\lambda_2 + b_2)}{m_2 - \lambda m_1}.$$

Noting that $y \in \mathbb{D}_R$, for any fixed λ_1, the cardinality of possible choices of

$$\lambda_2 \in \big((-m_2 + \lambda m_1) \cdot \mathbb{D}_R + \lambda(\lambda_1 + b_1) - b_2\big) \cap \Lambda$$

is $\leqslant \mathsf{K}$. Thus the cardinality of possible choices of such $(\lambda_1, \lambda_2) \in \Lambda \times \Lambda$ is $\leqslant \mathsf{K} \cdot (r + \mathsf{K})^2 \cdot \mathsf{K} \leqslant \mathsf{K} \cdot r^2$. Hence the estimate (7.1) is proved.

By the compactness of $\iota(\mathbb{C}/\Gamma_3)$, and by shrinking neighborhood U of $\iota(\mathbb{C}/\Gamma_3)$ if necessary, each intersection point corresponds to a small area $o(1)$ component of $f(\mathbb{D}_r) \cap U$, thus the total area of $f(\mathbb{D}_r) \cap U$ is $\leqslant o(1)\mathsf{K} \cdot r^2$. However, the area growth of $f(\mathbb{D}_r)$ is $\mathsf{K}_\lambda \cdot r^2$. Thus any obtained Ahlfors current of f charges mass $\leqslant o(1)\mathsf{K}$ on U. By shrinking U, we know that T charges zero mass along $\iota(\mathbb{C}/\Gamma_3)$. Hence we conclude the proof. $\square$

Take a holomorphic surjective map $\pi_2 : \mathcal{A} \longrightarrow \mathbb{P}^2(\mathbb{C})$, which induces an entire curve

$$f_2 := \pi_2 \circ f : \ \mathbb{C} \ \longrightarrow \ \mathbb{P}^2(\mathbb{C}).$$

Since $\pi_2^* \omega_{\mathrm{FS}} \geqslant 0$ is closed, by the geometry of $\mathcal{A}$, in the cohomology class $[\pi_2^* \omega_{\mathrm{FS}}]$ we can find a harmonic representative

$$\omega = a_1 \sqrt{-1}\, \mathrm{d}\, z_1 \wedge \mathrm{d}\, \overline{z_1} + a_2 \sqrt{-1}\, \mathrm{d}\, z_2 \wedge \mathrm{d}\, \overline{z_2} + a_3 \sqrt{-1}\, \mathrm{d}\, z_1 \wedge \mathrm{d}\, \overline{z_2} + a_4 \sqrt{-1}\, \mathrm{d}\, z_2 \wedge \mathrm{d}\, \overline{z_1} \geqslant 0$$

for some constants a_1, a_2, a_3, a_4. Thus $f^* \omega = K \sqrt{-1}\, \mathrm{d}\, z \wedge \mathrm{d}\, \overline{z} \geqslant 0$ where

$$K = a_1 + a_2 \lambda^2 + \lambda(a_3 + a_4) \geqslant 0.$$

Since $a_1, a_2, a_3 + a_4$ cannot vanish simultaneously, at most one λ in $\mathbb{R} \setminus \mathbb{Q}$ can make $K = 0$. Now we only choose $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ such that $K > 0$.

Proposition 7.2. *Any Ahlfors current T_2 of f_2 is diffuse.*

It is interesting to see that f_2 is tangent to a multi-valued vector field induced by the push-forward of the constant vector field $(1, \lambda)$ on $\mathcal{A}$.

Proof. Assume that T_2 is obtained by an increasing radii $\{r_i\}_{i \geqslant 1} \nearrow \infty$. By our chosen metric $\omega_{\mathcal{A}}$, the "Length-Area" condition of Ahlfors' lemma is automatically satisfied, thus by passing to some subsequence $\{r_{i_k}\}_{k \geqslant 1}$ we can receive an Ahlfors current T of f.

Noting that $f^* \omega = K \sqrt{-1}\, \mathrm{d}\, z \wedge \mathrm{d}\, \overline{z} > 0$, by closedness of T and by $\mathrm{Area}(f(\mathbb{D}_r))_{\omega_{\mathcal{A}}} = \mathsf{K} \cdot r^2$, we receive

$$T(\pi_2^* \omega_{\mathrm{FS}}) = T(\omega) \geqslant \mathsf{K} > 0.$$

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By the construction of T, we receive that

$$\operatorname{Area}\big(f_2(\mathbb{D}_{r_{i_k}})\big)_{\omega_{\mathrm{FS}}} \geqslant \mathsf{K} \cdot r_{i_k}^2 \qquad (k \gg 1).$$

Fix some K such that $\pi_2^* \omega_{\mathrm{FS}} \leqslant \mathsf{K} \cdot \omega_{\mathcal{A}}$. For any irreducible curve $C \subset \mathbb{P}^2(\mathbb{C})$, for any open neighborhood U of C, we have

$$\operatorname{Area}\big(f_2(\mathbb{D}_{r_{i_k}}) \cap U\big)_{\omega_{\mathrm{FS}}} = \operatorname{Area}\big(f(\mathbb{D}_{r_{i_k}}) \cap \pi_2^{-1}(U)\big)_{\pi_2^*\omega_{\mathrm{FS}}} \leqslant \mathsf{K} \cdot \operatorname{Area}\big(f(\mathbb{D}_{r_{r_{i_k}}}) \cap \pi_2^{-1}(U)\big)_{\omega_{\mathcal{A}}}.$$

Since T charges no mass along $\pi_2^{-1}(C)$ by Proposition 7.1, by shrinking U, the right-hand-side above is $\leqslant o(1) \cdot r_{i_k}^2$. Thus T_2 charges no mass along C. Since C is arbitrary, we conclude the proof. □

7.2. Singular Nevanlinna currents on X

Replacing "Ahlfors currents" by "Nevanlinna currents" in Observations 6.1–6.6, the same statements still hold true by much the same arguments. Indeed, every upper or lower bound about $\int_{\mathbb{D}(\lambda,\epsilon)} f^*\omega_X$ or $\int_{\mathbb{D}_{r_i}} f^*\omega_X$ has a corresponding one about order function. A remaining technical detail we would like to mention is the following

Observation 7.3. *For every $\ell \geqslant 1$, there exists some positive $\beta_\ell < 1$ such that, for $j \gg 1$ and $i = Z_j^{\mathbb{Z}_+}$,*

$$|\Lambda \cap \mathbf{A}_{r_i} \cap \mathbb{D}_{\beta_\ell r_i} \cap \pi_0^{-1}([y_\ell])| \geqslant \mathsf{K}_\ell \cdot r_i^2.$$

It will be helpful to show that certain Nevanlinna currents of f charge positive mass along $\mathbb{P}^1_{[y_\ell]}$.

Proof. Note that for $j \gg 1$ we have $|\Lambda \cap \mathbf{A}_{r_i} \cap \pi_0^{-1}([y_\ell])| > \frac{\alpha_\ell}{3} \cdot \mathsf{K} \cdot r_i^2$. Moreover, for any fixed $\beta < 1$, for $j \gg 1$, we have $|\Lambda \cap \mathbf{A}_{r_i} \setminus \mathbb{D}_{\beta r_i})| \leqslant \mathsf{K} \cdot (1-\beta) r_i^2$. By these two estimates, we can conclude the proof. □

Therefore we can replace "Ahlfors currents" by "Nevanlinna currents" in the statements of Theorems 1.3, 1.4. Also, by much the same proofs, Propositions 7.1, 7.2 also hold true for Nevanlinna currents.

7.3. Singular Ahlfors currents on blow-ups of X

We sketch a construction of elaborate (in the sense of cohomology classes) singular Ahlfors currents, on the blow-ups of X having Picard numbers $\geqslant 3$.

For any given positive integer $n \geqslant 1$, recall the collection of points $y_1, \dots, y_n$ given in (5.1), let $\mathcal{X}$ be the blow-up of X at these points with the corresponding exceptional divisors $E_1, \dots, E_n$. Let $\mathsf{p} : \mathcal{X} \to X$ be the projection. We now use the section $\psi^2 \cdot s_m$ instead of $\psi \cdot s_m$ to induce an entire curve $\mathsf{f} : \mathbb{C} \longrightarrow X$. By lifting we thus receive an entire curve $\zeta : \mathbb{C} \longrightarrow \mathcal{X}$. We strengthen our choices of m, c in (4.1) by the condition $m \cdot \alpha - 2\mathsf{K}/c^2 > 0$, to make sure that the same clustering phenomenon as in Subsection 4.1 holds true for f and $\mathcal{C}_\infty$. Let e_i be the intersection point of the strict transformation $\widetilde{\mathcal{C}_0}$ of $\mathcal{C}_0$ with E_i $(i = 1, \dots, n)$. The purpose of using ψ^2 instead of ψ is to make sure that, for $\lambda \in \Lambda$ with $[\lambda] = [y_i]$, we have the certain value $\zeta(\lambda) = e_i$.

It is well-known that, there exist some hermitian metrics h_i of the line bundles $\mathcal{O}(-E_i)$ and some small positive constant $\epsilon_2 \ll 1$ such that $\omega_{\mathcal{X}} := \mathsf{p}^*\omega_X + \epsilon_2 \sum_{\ell=1}^n \Theta_{h_\ell}$ is a Kähler form on $\mathcal{X}$ (cf. [19, Proposition 3.24]). Moreover, comparing the lifting $\zeta : \mathbb{C} \longrightarrow \mathcal{X}$ with $\mathsf{f} : \mathbb{C} \longrightarrow X$, we have

$$T_{\zeta,r}(\omega_{\mathcal{X}}) := \int_1^r \frac{\mathrm{d}\,t}{t} \int_{\mathbb{D}_t} \zeta^*\omega_{\mathcal{X}} \leqslant T_{\mathsf{f},r}(\omega_X) + O(1),$$

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(cf. [14, page 64, Observation 2.5.1]). Thus we can use the same arguments for (4.7) to conclude that

$$\int\limits_{\mathbb{D}_{2r_i}} \zeta^* \omega_{\mathcal{X}} \leqslant \mathsf{K} \cdot r_i^2.$$

For $\lambda \in \Lambda$ with $[\lambda] = [y_i]$, computing in local coordinates around e_i, for any small $\epsilon > 0$, for any open neighborhood U of E_i, assuming further that $|\lambda| \gg 1$, then the area of $\zeta(\mathbb{D}(\lambda, \epsilon)) \cap U$ is uniformly positively bounded (independent of U and ϵ) from below by using Propositions 3.4 and 4.3.

Therefore, as an analogue of Observation 6.3, for the finite subset $I = \{1, \ldots, n\}$, from the sequence of radii $\{r_{Z^I_{2j-1}}\}_{j \geqslant 1}$, after a perturbation and passing to a subsequence, we can receive an Ahlfors current

$$T = a_\infty \cdot [\widetilde{\mathcal{C}_\infty}] + \sum_{\ell=1}^{n} a_\ell \cdot [\widetilde{\mathbb{P}^1_{[y_\ell]}}] + \sum_{\ell=1}^{n} b_\ell \cdot [E_i],$$

where $\widetilde{\mathcal{C}_\infty}$, $\widetilde{\mathbb{P}^1_{[y_\ell]}}$ stand for the strict transformations of $\mathcal{C}_\infty$, $\mathbb{P}^1_{[y_\ell]}$, and where $a_\infty, a_\ell, b_\ell > 0$ ($\ell = 1, \ldots, n$) are some positive numbers.

Similarly, we have the counterparts of other Observations 6.1–6.6.

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