JOURNAL OF
COMMUTATIVE ALGEBRA
Volume 14 (2022), No. 1, 61–76

ALGORITHMS FOR CHECKING ZERO-DIMENSIONAL COMPLETE INTERSECTIONS

MARTIN KREUZER, LE NGOC LONG AND LORENZO ROBBIANO

Given a 0-dimensional affine K-algebra $R = K[x_1, \dots, x_n]/I$, where I is an ideal in a polynomial ring $K[x_1, \dots, x_n]$ over a field K, or, equivalently, given a 0-dimensional affine scheme, we construct effective algorithms for checking whether R is a complete intersection at a maximal ideal, whether R is locally a complete intersection, and whether R is a strict complete intersection. These algorithms are based on Wiebe's characterization of 0-dimensional local complete intersections via the 0-th Fitting ideal of the maximal ideal. They allow us to detect which generators of I form a regular sequence resp. a strict regular sequence, and they work over an arbitrary base field K. Using degree filtered border bases, we can detect strict complete intersections in certain families of 0-dimensional ideals.

1. Introduction

Regular sequences and complete intersections play a fundamental role in commutative algebra and algebraic geometry. Given an ideal I in a polynomial ring $P = K[x_1, \dots, x_n]$ over a field K, e.g., the vanishing ideal of an affine or projective scheme, it is therefore an important algorithmic task to check whether I is a complete intersection ideal, that is, whether I can be generated by $\operatorname{ht}(I)$ polynomials. Equivalently, we call $R = P/I$ a complete intersection ring in this case. Several approaches have been developed to tackle this problem effectively for special classes of ideals.

For instance, if the ring $R = P/I$ is local, based on a description of the structure of the R-algebra $\operatorname{Tor}^R(K, K)$ in [17], characterizations of the complete intersection property using the Hilbert series of $\operatorname{Tor}^R(K, K)$ were developed in [2] and [7]. However, this approach requires the calculation of the Hilbert series of a noncommutative algebra. Furthermore, in the local setting, complete intersections were characterized in [18] using the freeness of the conormal module I/I^2 and the finiteness of the projective dimension of I. Again, these conditions are not easy to check algorithmically. In a similar vein, if the base field K has characteristic zero, one can use techniques based on the Kähler differential module $\Omega^1_{R/K}$, as in [3], or on Kähler differents, as in [9], but they are neither general enough for our purposes nor do they lend themselves to a generalization for families of ideals. Finally, let us mention that effective algorithms have been developed in [4] for checking the complete intersection property of toric ideals defining affine semigroup rings.

In this paper we are interested in algorithms for checking the locally complete intersection property and the strict complete intersection property for a 0-dimensional ideal I in $P = K[x_1, \dots, x_n]$, where

2010 AMS *Mathematics subject classification:* primary 13C40; secondary 13H10, 13P99, 14M10, 14Q99.
Keywords and phrases: zero-dimensional affine algebra, zero-dimensional scheme, complete intersection, locally complete intersection, strict complete intersection, fitting ideal, border basis.
Received by the editors on March 22, 2019, and in revised form on August 5, 2019.

DOI: 10.1216/jca.2022.14.61

K is an arbitrary field. In other words, we want to check these properties for a 0-dimensional affine K-algebra of the form P/I. In the language of algebraic geometry, the scheme $\mathbb{X} = \operatorname{Spec}(P/I)$ is then a 0-dimensional affine scheme, and $R = P/I$ is its affine coordinate ring. Notice that every 0-dimensional projective scheme can be embedded into a basic affine open subset of $\mathbb{P}^n$ after possibly extending the field K slightly. Then we can readily move between the languages of affine and projective geometry by using homogenization and dehomogenization. For reasons which will become clear shortly, we prefer the affine setting in this paper. In any case, 0-dimensional affine K-algebras $R = P/I$ have been used in many areas besides algebraic geometry, for instance in algebraic statistics, algebraic biology, etc.

Our main algorithms check whether a 0-dimensional affine K-algebra $R = P/I$ is locally a complete intersection or a strict complete intersection. They are based on a characterization of local complete intersections by H. Wiebe in [19], where it is shown that a 0-dimensional local ring is a complete intersection if and only if the 0-th Fitting ideal of its maximal ideal is nonzero. Given a 0-dimensional affine K-algebra $R = P/I$ whose defining ideal I has a primary decomposition $I = \mathfrak{Q}_1 \cap \cdots \cap \mathfrak{Q}_s$, this characterization can be applied right off the bat, because the localizations are of the form $P/\mathfrak{Q}_i$, and hence again 0-dimensional local affine K-algebras. For checking the strict complete intersection property, we use the graded ring $\operatorname{gr}_{\mathcal{F}}(R)$ with respect to the degree filtration $\mathcal{F}$ instead. It has a presentation $\operatorname{gr}_{\mathcal{F}}(R) = P/\operatorname{DF}(I)$, where the degree form ideal $\operatorname{DF}(I)$ is homogeneous and 0-dimensional; see [11, Definition 4.2.13].

Now let us describe the contents of this paper in detail. In Section 2 we start by recalling some basic properties of a 0-dimensional scheme $\mathbb{X}$ embedded in an affine space $\mathbb{A}^n_K$, where K is an arbitrary field. In particular, besides recalling the degree filtration and the affine Hilbert function of the affine coordinate ring $R_{\mathbb{X}} = P/I_{\mathbb{X}}$ of $\mathbb{X}$, where $I_{\mathbb{X}}$ is the vanishing ideal of $\mathbb{X}$ in $P = K[x_1, \ldots, x_n]$ and K is a field, we recall the degree form ideal $\operatorname{DF}(I_{\mathbb{X}})$, the associated graded ring $\operatorname{gr}_{\mathcal{F}}(R_{\mathbb{X}})$, and mention their connection to the Rees algebra $\mathcal{R}_{\mathcal{F}}(R_{\mathbb{X}})$ and to Macaulay bases of $I_{\mathbb{X}}$.

Then the main part of the paper starts in Section 3. After defining what we mean by a complete intersection at a maximal ideal and by the property of being locally a complete intersection, we recall Wiebe's characterization mentioned above; see [19, Satz 3]. The central tool for computing the 0-th Fitting ideal needed in this characterization is given in Proposition 3.3. For a maximal ideal $\mathfrak{M}$ of P and an $\mathfrak{M}$-primary ideal $\mathfrak{Q}$, it suffices to write the generators of $\mathfrak{Q}$ in terms of a regular sequence generating $\mathfrak{M}$ and to compute the maximal minors of the coefficient matrix. As an immediate consequence, we obtain Algorithm 3.4 for checking if a 0-dimensional affine K-algebra $R_{\mathbb{X}} = P/I_{\mathbb{X}}$ is a complete intersection at a maximal ideal, and by combining this with the computation of a primary decomposition of $I_{\mathbb{X}}$, we get an algorithm for checking whether $R_{\mathbb{X}}$ is locally a complete intersection. Moreover, the nonzero minors provide us with regular sequences generating $\mathfrak{Q}\, P_{\mathfrak{M}}$, as Proposition 3.7 shows. Suitable examples at the end of the section illustrate the merits and some hidden features of these algorithms.

In Section 4 further notions are recalled. More precisely, strict regular sequences, strict Gorenstein rings, and strict complete intersections enter the game. In particular, the ideal $I_{\mathbb{X}}$ is called a strict complete intersection if its degree form ideal $\operatorname{DF}(I_{\mathbb{X}})$ is generated by a homogeneous regular sequence. Since the graded ring $\operatorname{gr}_{\mathcal{F}}(R_{\mathbb{X}}) = P/\operatorname{DF}(I_{\mathbb{X}})$ is 0-dimensional and local, we can use Proposition 3.3 and Wiebe's result again to construct Algorithm 4.4 for checking the strict complete intersection property. Moreover, we also get a description of which generators of $I_{\mathbb{X}}$ form a strict regular sequence (see Corollary 4.5) and illustrate everything via some explicit examples.

In the last section of the paper, we present a second algorithm for checking the strict complete intersection property via border bases which generalizes to certain families of 0-dimensional ideals. Based on the notion of a degree filtered K-basis of $R_{\mathbb{X}}$, we define degree filtered $\mathcal{O}$-border bases. They have several nice characterizations, the most useful one here being the property that the degree forms of the border basis polynomials form an $\mathcal{O}$-border basis of the degree form ideal (see Proposition 5.3). Then we obtain Algorithm 5.4 which checks for the strict complete intersection property using a degree filtered border basis. This version allows us to detect all strict complete intersections within certain families of 0-dimensional ideals, as illustrated by Example 5.5 and applied further in [15]. Finally, in Remark 5.6, we compare the algorithms of this paper with the methods based on Jacobian matrices, Kähler differentials, and Kähler differents mentioned above.

All examples in this paper were computed using the computer algebra system CoCoA; see [6]. Unless explicitly stated otherwise, we adhere to the notation and definitions provided in [10; 11; 13].

2. Zero-dimensional affine schemes

In the following we always work over an arbitrary field K and let $\mathbb{A}^n_K$ be the affine n-space over K. We fix a coordinate system, so that the affine coordinate ring of $\mathbb{A}^n_K$ is given by the polynomial ring $P = K[x_1, \ldots, x_n]$. Thus a 0-dimensional subscheme $\mathbb{X}$ of $\mathbb{A}^n_K$ is defined by a 0-dimensional ideal $I_{\mathbb{X}}$ in P, and its affine coordinate ring is $R_{\mathbb{X}} = P/I_{\mathbb{X}}$. Consequently, the vector space dimension $\mu = \dim_K(R_{\mathbb{X}})$ is finite and equal to the length of the scheme $\mathbb{X}$.

Since we are keeping the coordinate system fixed at all times, we have further invariants of $\mathbb{X}$. Recall that the *degree filtration* $\widetilde{\mathcal{F}} = (F_i P)_{i \in \mathbb{Z}}$ on P is given by $F_i P = \{f \in P \setminus \{0\} \mid \deg(f) \le i\} \cup \{0\}$ for all $i \in \mathbb{Z}$. Then the induced filtration $\mathcal{F} = (F_i R_{\mathbb{X}})_{i \in \mathbb{Z}}$, where $F_i R_{\mathbb{X}} = F_i P/(F_i P \cap I_{\mathbb{X}})$, is called the *degree filtration* on $R_{\mathbb{X}}$. It is easy to see that the degree filtration on $R_{\mathbb{X}}$ is increasing, exhaustive and *orderly* in the sense that every element $f \in R_{\mathbb{X}} \setminus \{0\}$ has an *order* $\mathrm{ord}_{\mathcal{F}}(f) = \min\{i \in \mathbb{Z} \mid f \in F_i R_{\mathbb{X}} \setminus F_{i-1} R_{\mathbb{X}}\}$.

Definition 2.1. Let $\mathbb{X}$ be a 0-dimensional subscheme of $\mathbb{A}^n_K$ as above:

(a) The map $\mathrm{HF}^a_{\mathbb{X}} : \mathbb{Z} \to \mathbb{Z}$ given by $i \mapsto \dim_K(F_i R_{\mathbb{X}})$ is called the *affine Hilbert function* of $\mathbb{X}$.

(b) The number $\mathrm{ri}(R_{\mathbb{X}}) = \min\{i \in \mathbb{Z} \mid \mathrm{HF}^a_{\mathbb{X}}(j) = \mu \text{ for all } j \ge i\}$ is called the *regularity index* of $\mathbb{X}$.

(c) The first difference function $\Delta\,\mathrm{HF}^a_{\mathbb{X}}(i) = \mathrm{HF}^a_{\mathbb{X}}(i) - \mathrm{HF}^a_{\mathbb{X}}(i-1)$ of $\mathrm{HF}^a_{\mathbb{X}}$ is called the *Castelnuovo function* of $\mathbb{X}$, and the number $\Delta_{\mathbb{X}} = \Delta\,\mathrm{HF}^a_{\mathbb{X}}(\mathrm{ri}(R_{\mathbb{X}}))$ is the *last difference* of $\mathbb{X}$.

It is well-known that $\mathrm{HF}^a_{\mathbb{X}}$ satisfies $\mathrm{HF}^a_{\mathbb{X}}(i) = 0$ for $i < 0$ and

$$1 = \mathrm{HF}^a_{\mathbb{X}}(0) < \mathrm{HF}^a_{\mathbb{X}}(1) < \cdots < \mathrm{HF}^a_{\mathbb{X}}(\mathrm{ri}(R_{\mathbb{X}})) = \mu$$

as well as $\mathrm{HF}^a_{\mathbb{X}}(i) = \mu$ for $i \ge \mathrm{ri}(R_{\mathbb{X}})$. The affine Hilbert function of $\mathbb{X}$ is related to the following objects.

Definition 2.2. Let $\mathbb{X}$ be a 0-dimensional subscheme of $\mathbb{A}^n_K$ as above:

(a) For every polynomial $f \in P \setminus \{0\}$, its homogeneous component of highest degree is called the *degree form* of f and is denoted by $\mathrm{DF}(f)$.

(b) The ideal $\mathrm{DF}(I_{\mathbb{X}}) = \langle \mathrm{DF}(f) \mid f \in I_{\mathbb{X}} \setminus \{0\} \rangle$ is called the *degree form ideal* of $I_{\mathbb{X}}$.

(c) The ring $\mathrm{gr}_{\widetilde{\mathcal{F}}}(P) = \bigoplus_{i \in \mathbb{Z}} F_i P / F_{i-1} P$ is called the *associated graded ring* of P with respect to $\widetilde{\mathcal{F}}$.

(d) The ring $\mathrm{gr}_{\mathcal{F}}(R_{\mathbb{X}}) = \bigoplus_{i\in\mathbb{Z}} F_i R_{\mathbb{X}}/F_{i-1}R_{\mathbb{X}}$ is called the *associated graded ring* of $R_{\mathbb{X}}$ with respect to $\mathcal{F}$.

(e) For an element $f \in R_{\mathbb{X}} \setminus \{0\}$ of order $d = \mathrm{ord}_{\mathcal{F}}(f)$, the residue class $\mathrm{LF}(f) = f + F_{d-1}R_{\mathbb{X}}$ in $\mathrm{gr}_{\mathcal{F}}(R_{\mathbb{X}})$ is called the *leading form* of f with respect to $\mathcal{F}$.

We observe that in our setting the associated graded ring $\mathrm{gr}_{\mathcal{F}}(R_{\mathbb{X}})$ is a 0-dimensional local ring whose maximal ideal is generated by the residue classes of the indeterminates. Its K-vector space dimension is given by

$$\dim_K(\mathrm{gr}_{\mathcal{F}}(R_{\mathbb{X}})) = \sum_{i=0}^{\infty} \dim_K(F_i R_{\mathbb{X}}/F_{i-1}R_{\mathbb{X}}) = \dim_K(R_{\mathbb{X}}).$$

For actual computations involving the associated graded ring and leading forms, we can use the following observations.

Remark 2.3. Notice that there is a canonical isomorphism of graded K-algebras $\mathrm{gr}_{\widetilde{\mathcal{F}}}(P) \cong P$ which allows us to identify $\mathrm{gr}_{\mathcal{F}}(R_{\mathbb{X}}) \cong P/\mathrm{DF}(I_{\mathbb{X}})$; see [11, Propositions 6.5.8 and 6.5.9].

In order to represent the leading form of a nonzero element $f \in R_{\mathbb{X}}$ as a residue class in $P/\mathrm{DF}(I_{\mathbb{X}})$, we first have to represent f by a polynomial $F \in P$ with $\deg(F) = \mathrm{ord}_{\mathcal{F}}(f)$. This can, for instance, be achieved by taking any representative $\widetilde{F} \in P$ of f and computing the normal form $F = \mathrm{NF}_{\sigma, I_{\mathbb{X}}}(\widetilde{F})$ with respect to a degree compatible term ordering σ. Then the degree form $\mathrm{DF}(F)$ represents the leading form $\mathrm{LF}_{\mathcal{F}}(f)$ with respect to the isomorphism $\mathrm{gr}_{\mathcal{F}}(R_{\mathbb{X}}) \cong P/\mathrm{DF}(I_{\mathbb{X}})$.

The affine Hilbert function of $\mathbb{X}$ and the (usual) Hilbert function of $\mathrm{gr}_{\mathcal{F}}(R_{\mathbb{X}})$ are connected by

$$\mathrm{HF}^a_{\mathbb{X}}(i) = \sum_{j=0}^{i} \mathrm{HF}_{\mathrm{gr}_{\mathcal{F}}(R_{\mathbb{X}})}(j) \quad \text{and} \quad \Delta\,\mathrm{HF}^a_{\mathbb{X}}(i) = \mathrm{HF}_{\mathrm{gr}_{\mathcal{F}}(R_{\mathbb{X}})}(i)$$

for all $i \geq 0$. Since the Hilbert function of $\mathrm{gr}_{\mathcal{F}}(R_{\mathbb{X}})$ can be calculated using a suitable Gröbner basis of $I_{\mathbb{X}}$ (see [11], Section 4.3), this formula allows us to compute the affine Hilbert function of $\mathbb{X}$.

The ring $R_{\mathbb{X}}$ and its graded ring $\mathrm{gr}_{\mathcal{F}}(R_{\mathbb{X}})$ are connected by the following flat family. (For further details and proofs; see [11, Section 4.3.B].)

Remark 2.4. Let x_0 be a further indeterminate:

(a) The ring $\mathcal{R}_{\widetilde{\mathcal{F}}}(P) = \bigoplus_{i\in\mathbb{Z}} F_i P \cdot x_0^i$ is called the *Rees algebra* of P with respect to $\widetilde{\mathcal{F}}$. Every nonzero homogeneous element of the Rees algebra is of the form $f \cdot x_0^{d+j}$ with $d = \deg(f)$ and $j \geq 0$. By identifying it with the polynomial $f^{\mathrm{hom}} \cdot x_0^j$, we obtain an isomorphism of graded $K[x_0]$-algebras $\mathcal{R}_{\widetilde{\mathcal{F}}}(P) \cong K[x_0, x_1, \dots, x_n] =: \overline{P}$. Here f^{hom} is the usual homogenization $f^{\mathrm{hom}} = x_0^{\deg(f)} \cdot f\bigl(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\bigr)$.

(b) Similarly, using the degree filtration $\mathcal{F}$ on $R_{\mathbb{X}}$, we have the *Rees algebra* $\mathcal{R}_{\mathcal{F}}(R_{\mathbb{X}}) = \bigoplus_{i\in\mathbb{Z}} F_i R_{\mathbb{X}} \cdot x_0^i$. Now we let $I^{\mathrm{hom}}_{\mathbb{X}} = \langle f^{\mathrm{hom}} \mid f \in I_{\mathbb{X}} \setminus \{0\}\rangle$ be the *homogenization* of $I_{\mathbb{X}}$. Then the above isomorphism induces an isomorphism of graded $K[x_0]$-algebras $\mathcal{R}_{\mathcal{F}}(R_{\mathbb{X}}) \cong \overline{P}/I^{\mathrm{hom}}_{\mathbb{X}} =: R^{\mathrm{hom}}_{\mathbb{X}}$.

Geometrically speaking, the ring $R^{\mathrm{hom}}_{\mathbb{X}}$ is the homogeneous coordinate ring of the 0-dimensional scheme obtained by embedding $\mathbb{X} = \mathrm{Spec}(P/I_{\mathbb{X}}) \subset \mathbb{A}^n_K$ into the projective n-space via $\mathbb{A}^n_K \cong D_+(x_0) \subset \mathbb{P}^n$.

is the defining ideal of the universal family of all subschemes of length four of the affine plane which have the property that their coordinate ring admits $\overline{\mathcal{O}}$ as a vector space basis. The parameters c_{21}, c_{23}, c_{32}, c_{34}, c_{41}, c_{42}, c_{43}, and c_{44} are free. In other words, the family is parametrized by an 8-dimensional affine space. Since the degree form ideal $\mathrm{DF}(I)$ is generated by the degree forms of f_1, f_2, f_3, f_4, we have $\mathrm{DF}(I) = \langle y^2 - c_{41}xy,\, x^2 - c_{42}xy,\, xy^2,\, x^2y\rangle$. To compute the locus of strict complete intersections in $\mathbb{A}^8_K$, we write the generators of $\mathrm{DF}(I)$ in the form

$$(-c_{41}y)x + (y)y, \quad (x)x + (-c_{42}x)y, \quad (y^2)x + (0)y, \quad (xy)x + (0)y$$

We get the matrix

$$W = \begin{pmatrix} -c_{41}y & x & y^2 & xy \\ y & c_{42}x & 0 & 0 \end{pmatrix}$$

Then the only nonzero maximal minor of W modulo $\mathrm{DF}(I)$ is given by $(1 - c_{41}c_{42})xy$. In conclusion, outside the hypersurface in $\mathbb{A}^8_K$ defined by $1 - c_{41}c_{42} = 0$, all ideals define a strict complete intersection scheme.

Finally, we note that one can also use the Kähler different of $R_{\mathbb{X}}$ to check whether this ring is a strict complete intersection or locally a complete intersection. However, this approach introduces constraints on the characteristic of the base field. Let us formulate the characterizations underlying the algorithms using Kähler differents and leave the details to the interested reader.

Remark 5.6. Let $\mathbb{X}$ be a 0-dimensional subscheme of $\mathbb{A}^n_K$, and let $R_{\mathbb{X}} = P/I_{\mathbb{X}}$ be the affine coordinate ring of $\mathbb{X}$:

(a) The module of Kähler differentials $\Omega^1_{R_{\mathbb{X}}/K}$ is given by the presentation

$$\Omega^1_{R_{\mathbb{X}}/K} \cong \bigoplus_{i=1}^{n} P\,dx_i \Big/ \Big(I_{\mathbb{X}} \cdot \bigoplus_{i=1}^{n} P\,dx_i + \Big\langle \sum_{i=1}^{n} \partial f/\partial x_i \, dx_i \mid f \in I_{\mathbb{X}} \Big\rangle \Big).$$

(b) The Kähler different $\vartheta_{R_{\mathbb{X}}}$ of the K-algebra $R_{\mathbb{X}}$ is the ideal in $R_{\mathbb{X}}$ generated by residue classes of the maximal minors of the *Jacobian matrix* $\mathrm{Jac}(f_1, \ldots, f_r) = (\partial f_i/\partial x_j)_{i,j}$, where $\{f_1, \ldots, f_r\}$ is a system of generators of $I_{\mathbb{X}}$. For further details about Kähler differents; see [16, Section 10].

(c) The Kähler different $\vartheta_{\mathrm{gr}_{\mathcal{F}}(R_{\mathbb{X}})}$ of the associated graded ring $\mathrm{gr}_{\mathcal{F}}(R_{\mathbb{X}}) \cong P/\mathrm{DF}(I_{\mathbb{X}})$ is defined similarly.

(d) Suppose that $\mathrm{char}(K)$ does not divide μ. Then $\mathbb{X}$ is a strict complete intersection if and only if $\vartheta_{\mathrm{gr}_{\mathcal{F}}(R_{\mathbb{X}})}$ is nonzero.

(e) Again, suppose that $\mathrm{char}(K)$ does not divide μ. Then $\mathbb{X}$ is locally a complete intersection if and only if the image of $\vartheta_{R_{\mathbb{X}}}$ in every local ring of $R_{\mathbb{X}}$ is nonzero. (For instance, if we know the principal idempotents $f_1, \ldots, f_s$ of $R_{\mathbb{X}}$, it suffices to check that $f_i \cdot \vartheta_{R_{\mathbb{X}}} \neq \langle 0\rangle$ for $i = 1, \ldots, s$.)

The following easy example shows that the assumption on the characteristic of K is necessary for the approach via Kähler differents to work, while the approach based on Wiebe's result (see Proposition 3.2) works in general.

Example 5.7. Let p be a prime number, let $K = \mathbb{F}_p$, let $P = K[x]$, and let $\mathbb{X}$ be the 0-dimensional subscheme of $\mathbb{A}^1_K$ defined by $I_{\mathbb{X}} = \langle x^p\rangle$.

When we use Algorithm 4.4 to check whether $\mathbb{X}$ is a strict complete intersection, we find the matrix $W = (x^{p-1})$ whose determinant yields the relation $\bar{x}^{p-1} \in \operatorname{gr}_{\mathcal{F}}(R_{\mathbb{X}}) \setminus \{0\}$. Hence we conclude that $\mathbb{X}$ is a strict complete intersection.

However, the Jacobian matrix is $\operatorname{Jac}(x^p) = (0)$ and therefore we have $\vartheta_{\operatorname{gr}_{\mathcal{F}}(R_{\mathbb{X}})} = \langle 0 \rangle$. Thus the Kähler different does not yield the correct answer.

Acknowledgements

Kreuzer and Long were partially supported by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) grant number 101.04-2019.07. Robbiano thanks the University of Passau for its hospitality and support during part of the preparation of this paper.

References

[1] J. Abbott, A. M. Bigatti, E. Palezzato, and L. Robbiano, “Computing and using minimal polynomials”, *J. Symbolic Comput.* **100** (2020), 137–163.

[2] E. F. Assmus, Jr., “On the homology of local rings”, *Illinois J. Math.* **3** (1959), 187–199.

[3] L. L. Avramov and J. Herzog, “Jacobian criteria for complete intersections: the graded case”, *Invent. Math.* **117**:1 (1994), 75–88.

[4] I. Bermejo, I. García-Marco, and J. J. Salazar-González, “An algorithm for checking whether the toric ideal of an affine monomial curve is a complete intersection”, *J. Symbolic Comput.* **42**:10 (2007), 971–991.

[5] W. Bruns and J. Herzog, *Cohen–Macaulay rings*, Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, 1993.

[6] J. Abbott, A. Bigatti, and L. Robbiano, “CoCoA: a system for doing computations in commutative algebra”, (2009).

[7] T. H. Gulliksen and G. Levin, *Homology of local rings*, Queen’s Papers in Pure and Applied Mathematics **20**, Queen’s University, Kingston, Ontario, 1969.

[8] M. Kreuzer and E. Kunz, “Traces in strict Frobenius algebras and strict complete intersections”, *J. Reine Angew. Math.* **381** (1987), 181–204.

[9] M. Kreuzer and L. N. Long, “Characterizations of zero-dimensional complete intersections”, *Beitr. Algebra Geom.* **58**:1 (2017), 93–129.

[10] M. Kreuzer and L. Robbiano, *Computational commutative algebra, 1*, Springer, 2000.

[11] M. Kreuzer and L. Robbiano, *Computational commutative algebra, 2*, Springer, 2005.

[12] M. Kreuzer and L. Robbiano, “Deformations of border bases”, *Collect. Math.* **59**:3 (2008), 275–297.

[13] M. Kreuzer and L. Robbiano, *Computational linear and commutative algebra*, Springer, 2016.

[14] M. Kreuzer, L. N. Long, and L. Robbiano, “On the Cayley–Bacharach property”, *Comm. Algebra* **47**:1 (2019), 328–354.

[15] M. Kreuzer, L. N. Long, and L. Robbiano, “Computing subschemes of the border basis scheme”, *International Journal of Algebra and Computation* **30**:08 (2020), 1671–1716.

[16] E. Kunz, *Kähler differentials*, Vieweg & Sohn, Braunschweig, 1986.

[17] J. Tate, “Homology of Noetherian rings and local rings”, *Illinois J. Math.* **1** (1957), 14–27.

[18] W. V. Vasconcelos, “Ideals generated by R-sequences”, *J. Algebra* **6** (1967), 309–316.

[19] H. Wiebe, “Über homologische Invarianten lokaler Ringe”, *Math. Ann.* **179** (1969), 257–274.

MARTIN KREUZER: martin.kreuzer@uni-passau.de

Fakultät für Informatik und Mathematik, Universität Passau, Passau, Germany

LE NGOC LONG: lelong@hueuni.edu.vn
Fakultät für Informatik und Mathematik, Universität Passau, Passau, Germany

and

Department of Mathematics, Hue University of Education, Hue, Vietnam

LORENZO ROBBIANO: lorobbiano@gmail.com
Dipartimento di Matematica, Università di Genova, Genova, Italy

JCA — prepared by msp for the
Rocky Mountain Mathematics Consortium