Vietnam Journal of Mathematics (2022) 50:229–248
https://doi.org/10.1007/s10013-021-00501-y

ORIGINAL ARTICLE

Generators for the Mod-p Cohomology of the Steinberg Summand of Thom Spectra Over $\mathrm{B}(\mathbb{Z}/p)^n$-Odd Primary Cases

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Received: 29 June 2020 / Accepted: 14 December 2020 / Published online: 5 June 2021

Abstract

For a given odd prime number p, in this paper we construct a minimal generating set for the mod-p cohomology of the Steinberg summand of a family of Thom spectra over the classifying space of an elementary p-abelian group. This resolves the remaining cases for odd prime numbers of a problem studied previously by M. Inoue (Contemporary Mathematics, vol. 293, pp. 125–139, 2002) and (J. Lond. Math. Soc. 75: 317–329, 2007) and by the author (J. Algebra 381: 164–175, 2013).

Keywords Steenrod algebra · Steinberg representation · Modular invariants

Mathematics Subject Classification (2010) 55S10 · 55P42

1 Introduction

Fix p an odd prime number. Given n and k two natural numbers, we study in this paper the following module over the mod-p Steenrod algebra $\mathscr{A}$:

$$L_{n,k} = \mathrm{st}_n(\mathfrak{e}_n^k H^* V_n).$$

In this formula,

- $H^*V_n \cong \mathrm{E}(x_1, \ldots, x_n) \otimes \mathbb{F}_p[y_1, \ldots, y_n]$ is the mod-p cohomology of a rank n elementary abelian p-group V_n,
- $\mathfrak{e}_n = \mathrm{L}_n^{\frac{p-1}{2}}$, where $\mathrm{L}_n = \det((y_j^{p^{i-1}})_{1\le i,j\le n})$, and
- st_n is the Steinberg idempotent of the group ring $\mathbb{F}_p[GL_n(\mathbb{F}_p)]$.

The module $L_{n,k}$ is thus the image of the action of st_n on the principal ideal of H^*V_n generated by $\mathfrak{e}_n^k$. The class $\mathfrak{e}_n$, up to sign, is the Euler class of the vector bundle over the classifying space $\mathrm{B}V_n$ associated to the reduced real regular representation, $\bar{\rho}_n$, of V_n. If

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we let $L(n,k)$ denote the stable summand associated to the Steinberg module of the Thom spectrum over the classifying space BV_n associated to k copies of $\bar{\rho}_n$, then by Thom's isomorphism the module $L_{n,k}$ is the mod-p cohomology of $L(n,k)$. We refer the reader to [1, 2, 13–15, 17, 22] for the important role of $L(n,k)$ in stable homotopy theory.

The purpose of this paper is to give an explicit description of a minimal generating set for $L_{n,k}$ as a module over the mod-p Steenrod algebra. The cases where $k = 0, 1$ were treated by M. Inoue for the prime 2 in [9] and for odd primes in [10]. The case where p is the prime 2 and k an arbitrary natural number was considered by the author in [6]. This paper thus completes the remaining cases for odd primes.

We now state the main results of the paper. Given an $\mathscr{A}$-module M, put $\mathrm{Q}(M) = M/\mathscr{A}^+M$, where $\mathscr{A}^+$ denotes the augmentation ideal of $\mathscr{A}$. By definition, $\mathrm{Q}(M)$ is the largest quotient of M on which $\mathscr{A}$ acts trivially. The following result relates the quotient spaces $\mathrm{Q}(L_{n,k})$ for different values of n and k.

Theorem 1 1. *If $m = m'p$ with $m' \geq 1$, there is an isomorphism of graded vector spaces:*

$$\mathrm{Q}(\Sigma^n L_{n,2m-1}) \cong \Phi \mathrm{Q}(\Sigma^n L_{n,2m'-1}).$$

2. *If $m = m'p + r$ with $m' \geq 0$ and $1 \leq r \leq p-1$, there is a short exact sequence of graded vector spaces:*

$$0 \to \Phi\mathrm{Q}\left(\Sigma^n L_{n,2m'+1}\right) \to \mathrm{Q}\left(\Sigma^n L_{n,2m-1}\right) \to \mathrm{Q}\left(\Sigma^{2m(p-1)+n-1} L_{n-1,2mp-1}\right) \to 0.$$

3. *For all $m \geq 0$, there is a short exact sequence of graded vector spaces:*

$$0 \to \mathrm{Q}(L_{n,2m+1}) \to \mathrm{Q}(L_{n,2m}) \to \mathrm{Q}\left(\Sigma^{2m(p-1)} L_{n-1,2mp+1}\right) \to 0.$$

Here Σ and Φ denote respectively the suspension functor and the Frobenius functor of the category of $\mathscr{A}$-modules.

By induction, Theorem 1 leads to a construction of minimal generating sets for all $L_{n,k}$'s. To state the result, we need the numerical function $\gamma : \mathbb{N} \to \mathbb{N}$ defined by

$$\gamma(a) = \begin{cases} p\gamma(a') & \text{if } a = pa', \\ p(a'+1) & \text{if } a = pa' + r, 1 \leq r \leq p-1. \end{cases}$$

It is checked easily that $\gamma(0) = 0$ and $\gamma(p^s q) = p^{s+1}(q+1)$ if $(p,q) = 1$. For each natural number i, we will write γ^i for the i-fold composition of the function γ, with the convention that γ^0 is the identity function.

Consider the following classes which were introduced by L. E. Dickson [4] and H. Mui [18] in modular invariant theory:

$$\mathrm{L}_s = \begin{vmatrix} y_1 & \cdots & y_s \\ y_1^p & \cdots & y_s^p \\ \vdots & \ddots & \vdots \\ y_1^{p^{s-1}} & \cdots & y_s^{p^{s-1}} \end{vmatrix}, \qquad \mathrm{M}_s = \begin{vmatrix} x_1 & \cdots & x_s \\ y_1 & \cdots & y_s \\ \vdots & \ddots & \vdots \\ y_1^{p^{s-2}} & \cdots & y_s^{p^{s-2}} \end{vmatrix}.$$

Put $\mu_s = \frac{\mathrm{M}_s}{\mathrm{L}_s}$ and $\omega_s = \mathrm{L}_s^{p-1}$. So μ_s (resp. ω_s) is a class of degree $1-2p^{s-1}$ (resp. $2p^s-2$) in the localized ring $H^*V_s[\mathrm{L}_s^{\pm 1}]$. For $E = (e_1, \ldots, e_n) \in \{0,1\}^n$ and $I = (i_1, \ldots, i_n) \in \mathbb{N}^n$, put

$$\omega_{E;I} = \mathrm{st}_n\left(\mu_1^{e_1}\omega_1^{i_1-pi_2+e_2} \cdots \mu_{n-1}^{e_{n-1}}\omega_{n-1}^{i_{n-1}-pi_n+e_n}\mu_n^{e_n}\omega_n^{i_n}\right).$$

Theorem 2 *Let m be a positive integer.*

1. *The classes $\omega_{1^n; p^{n-1}\gamma^{i_1}(m),\dots,p^0\gamma^{i_n}(m)}$ with $i_1 \geq i_2 \geq \cdots \geq i_n \geq 0$ form a minimal generating set for $L_{n,2m-1}$.*
2. *The classes*

$$\begin{aligned} \omega_{1^n;\ p^{n-1}\gamma^{i_1}(m),\dots,p^0\gamma^{i_n}(m)}, &\qquad i_1 \geq i_2 \geq \cdots \geq i_n \geq 0, \\ \omega_{1^{n-1},0;\ p^{n-2}\gamma^{i_1}(mp-p+1),\dots,p^0\gamma^{i_{n-1}}(mp-p+1),m-1}, &\qquad i_1 \geq i_2 \geq \cdots \geq i_{n-1} \geq 0, \end{aligned}$$

   form a minimal generating set for $L_{n,2m-2}$.

Here and below, 1^n denotes the sequence $(1, \dots, 1)$ of length n.

As in the case $p = 2$ [6], two main tools employed in this paper are odd primary versions of Kameko's homomorphism [11] and Takayasu's short exact sequence [22]. As we may expect when working with odd primes, there are some technical points which need to have careful consideration. For example, the construction of Kameko's homomorphism makes use of the Frobenius functor Φ of $\mathscr{A}$-modules and, as opposed to the case $p = 2$, we have to take into account the fact that Φ does not commute with either the indecomposable functor Q or the tensor product when p is odd. Furthermore, when $p = 2$, the module $L_{n,k}$'s are related by a short exact sequence constructed by Takayasu, while in odd primary cases, these modules are related by two short exact sequences. This is similar to the fact that when localized at 2 there is a fibration of James which relates the spheres while at odd primes, the spheres are related by two fibrations discovered by Toda. Though we are not going to pursuit it here, it should be noted that this similarity could be explained by using work of Arone–Mahowald on the Goodwillie calculus of the identity functor [1].

The paper is organized as follows. In Section 2, we focus on the linear structure of $L_{n,k}$ and construct odd primary versions of Takayasu's short exact sequences. In Section 3, we consider an odd primary version of Kameko's homomorphism and prove Theorem 1. The proof of Theorem 2 will be given in Section 4.

2 The Modules $L_{n,k}$

We work in the category $\mathscr{M}$ whose objects are $\mathbb{Z}$-graded $\mathscr{A}$-modules and whose morphisms are $\mathscr{A}$-linear maps of degree zero. We need to recall some functors used in this paper.

2.1 The Suspension Functor Σ

The functor $\Sigma : \mathscr{M} \to \mathscr{M}$ is given on an $\mathscr{A}$-module M by

$$(\Sigma M)^d = M^{d-1}, \quad \theta(\Sigma x) = (-1)^{|\theta|}\Sigma\theta(x), \quad \theta \in \mathscr{A},$$

$\Sigma x \in \Sigma M$ denoting the element corresponding to $x \in M$. It is clear that the functor Σ is exact and commutes with the indecomposable functor Q:

$$\mathrm{Q}(\Sigma M) \cong \Sigma \mathrm{Q}(M).$$

2.2 The Frobenius Functor Φ

The functor $\Phi : \mathscr{M} \to \mathscr{M}$ is defined on an $\mathscr{A}$-module M by

$$(\Phi M)^d = \begin{cases} M^{2i} & \text{if } d = 2ip, \\ M^{2i+1} & \text{if } d = 2ip + 2, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, ΦM is concentrated in even degrees, and an element $x \in M$ of degree $2i$ (resp. $2i+1$) gives rise to $\Phi x \in \Phi M$ of degree $2ip$ (resp. $2ip+2$). The action of the Steenrod algebra is given by $\beta(\Phi x) = 0$ and

$$\mathscr{P}^i(\Phi x) = \begin{cases} \Phi(\mathscr{P}^{i/p} x) & \text{if } p \mid i, \\ \Phi(\beta P^{(i-1)/p} x) & \text{if } p \mid (i-1) \text{ and } |x| \text{ odd}, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 1 The functor Φ is usually defined in the category of unstable modules. In this case, the natural $\mathbb{F}_p$-linear map $\lambda_M : \Phi M \to M$ defined by $\lambda_M(\Phi x) = \beta^e \mathscr{P}^i(x)$, where $|x| = 2i + e$, is $\mathscr{A}$-linear and so λ_M is a homomorphism of unstable modules ([19, p. 27]).

Remark 2 Given an $\mathscr{A}$-module M, applying the exact functor Φ on the short exact sequence $0 \to \mathscr{A}^+ M \to M \to \mathrm{Q}(M) \to 0$ gives rise to an exact sequence $0 \to \Phi\mathscr{A}^+ M \to \Phi M \to \Phi\mathrm{Q}(M) \to 0$. Let $\mathscr{P}$ denote the subalgebra of $\mathscr{A}$ generated by the Steenrod powers $\mathscr{P}^i$, $i \geq 0$, and let $\mathscr{P}^+$ denote the augmentation ideal of $\mathscr{P}$. It is clear that $\mathscr{A}^+\Phi M = \mathscr{P}^+\Phi M \subset \Phi\mathscr{A}^+ M$ and so there is a natural surjection $\mathrm{Q}(\Phi M) \twoheadrightarrow \Phi\mathrm{Q}(M)$. If M is concentrated in even degrees, it is clear that this is an isomorphism. It is not the case in general if p is odd. For example, if $M = \Sigma F(1)$, where $F(1) = \mathbb{F}_p\langle x, y, y^p, \cdots \rangle$ is the free unstable module generated by an element of degree 1 (see [19, p. 23]), then $\mathrm{Q}(\Phi\Sigma F(1))$ is generated by $\Phi\Sigma x$ (of degree $2p$) and $\Phi\Sigma y$ (of degree $2p+2$) while $\Phi\Sigma\mathrm{Q}(F(1))$ is generated by $\Phi\Sigma x$.

Remark 3 Given two $\mathscr{A}$-modules M and N, there is also a natural $\mathscr{A}$-linear map

$$\varphi_{M,N} : \Phi M \otimes \Phi N \to \Phi(M \otimes N),$$

given by

$$\varphi_{M,N}(\Phi x \otimes \Phi y) = \begin{cases} \Phi(x \otimes y) & \text{if } |x| \text{and } |y| \text{ are even}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that this is not an isomorphism in general (this again can be seen by taking $M = N = F(1)$).

2.3 The Modules $L_{n,k}$

Recall that the mod-p cohomology of the group $V_n := (\mathbb{Z}/p)^n$ is given by

$$H^* V_n \cong \mathrm{E}(x_1, \ldots, x_n) \otimes \mathbb{F}_p[y_1, \ldots, y_n],$$

where $|x_i| = 1$, $|y_i| = 1$, and $\{x_1, \dots, x_n\}$ is an $\mathbb{F}_p$-basis of the $\mathbb{F}_p$-linear dual of V_n. The action of the Steenrod algebra $\mathscr{A}$ on H^*V_n is determined by the Cartan formula and the following:

$$\beta x_j = y_j, \quad \mathscr{P}^0 x_j = x_j, \quad \mathscr{P}^i x_j = 0, \quad i > 0, j \geq 1,$$
$$\beta y_j = 0, \quad \mathscr{P}^i y_j^d = \binom{d}{i} y_j^{d+(p-1)i}, \quad i, d \geq 0, j \geq 1.$$

The natural action of the general linear group $GL_n := GL_n(\mathbb{F}_p)$ on the $\mathbb{F}_p$-vector space $V_n \cong (\mathbb{F}_p)^n$ induces an action of GL_n on H^*V_n. In this paper we consider the *left* action of GL_n on H^*V_n specified as follows. For each matrix $g = (g_{i,j})_{1 \leq i,j \leq n} \in GL_n$ and each $F \in H^*V_n$, gF is given by

$$gF(x_1, \dots, x_n; y_1, \dots, y_n) = F(gx_1, \dots, gx_n; gy_1, \dots, gy_n),$$

where $gx_j = \sum_{i=1}^n g_{i,j} x_i$ and $gy_j = \sum_{i=1}^n g_{i,j} y_i$ for $1 \leq j \leq n$. It is well-known that the action of GL_n commutes with that of the Steenrod algebra on H^*V_n.

The Steinberg idempotent st_n of $\mathbb{F}_p[GL_n]$ is defined as follows. Let B_n be the subgroup of upper triangular matrices in GL_n and let Σ_n be the subgroup of permutation matrices. Then st_n is defined by [20]

$$\mathrm{st}_n = (-1)^n \bar{B}_n \widetilde{\Sigma}_n,$$

where $\bar{B}_n = \sum_{b \in B_n} b$ and $\widetilde{\Sigma}_n = \sum_{\sigma \in \Sigma_n} \mathrm{sgn}(\sigma)\sigma$.

Recall from the introduction that, for $1 \leq s \leq n$, we put

$$\mathrm{L}_s = \begin{vmatrix} y_1 & \cdots & y_s \\ y_1^p & \cdots & y_s^p \\ \vdots & \ddots & \vdots \\ y_1^{p^{s-1}} & \cdots & y_k^{p^{s-1}} \end{vmatrix}, \qquad \mathrm{M}_s = \begin{vmatrix} x_1 & \cdots & x_s \\ y_1 & \cdots & y_s \\ \vdots & \ddots & \vdots \\ y_1^{p^{s-2}} & \cdots & y_s^{p^{s-2}} \end{vmatrix},$$

and

$$\mu_s = \frac{\mathrm{M}_s}{\mathrm{L}_s}, \quad \omega_s = \mathrm{L}_s^{p-1}.$$

For $E = (e_1, \dots, e_n) \in \{0, 1\}^n$ and $I = (i_1, \dots, i_n) \in \mathbb{N}^n$, put

$$\sigma_{E;I} = \mu_1^{e_1} \omega_1^{i_1 - pi_2 + e_2} \cdots \mu_{n-1}^{e_{n-1}} \omega_{n-1}^{i_{n-1} - pi_n + e_n} \mu_n^{e_n} \omega_n^{i_n}$$

and

$$\omega_{E;I} = \mathrm{st}_n(\sigma_{E;I}).$$

Note that the classes μ_s and ω_s are GL_s-invariant where we consider GL_s as a subgroup of GL_n by sending a matrix g in GL_s to $\begin{pmatrix} g & 0 \\ 0 & \mathrm{Id}_{n-s} \end{pmatrix}$ in GL_n, Id_{n-s} denoting the identity matrix.

Definition 1 For $E = (e_1, \dots, e_n) \in \{0, 1\}^n$ and $I = (i_1, \dots, i_n) \in \mathbb{N}^n$, the couple $(E; I)$ is (n, k)*-admissible* if the following condition holds:

$$\begin{cases} e_j \in \{0, 1\}, & \text{if } 1 \leq j \leq n, \\ i_j - pi_{j+1} + e_{j+1} > 0, & \text{if } 1 \leq j \leq n-1, \\ 2i_n - e_n \geq k, & \text{otherwise.} \end{cases}$$

We also say that the class $\omega_{E;I}$ (or the class $\sigma_{\mathrm{E};I}$) is (n,k)-admissible if $(E;I)$ is (n,k)-admissible. We note that if $(E;I)$ is (n,k)-admissible, then $\omega_{E;I}$ is a B_n-invariant element of H^*V_n and

$$\deg(\omega_{E;I}) = 2(p-1)(i_1+\cdots+i_n) - (e_1+\cdots+e_n).$$

Remark 4 The couple $(E;I)$ is (n,k)-admissible if and only if the operation

$$\beta^{1^n-E}\mathscr{P}^I := \beta^{1-e_1}\mathscr{P}^{i_1}\cdots\beta^{1-e_n}\mathscr{P}^{i_n}$$

is *admissible* in the Steenrod algebra $\mathscr{A}$ and the *excess* of $\beta^{1-e_n}\mathscr{P}^{i_n}$, i.e. $2i_n - e_n$, is bigger than k. Recall also that an $\mathscr{A}$-module M is said to be *unstable* if $\beta^e\mathscr{P}^i(x)$ vanishes whenever the excess of $\beta^e\mathscr{P}^i$ is bigger than the degree $|x|$ of a homogeneous element x of M. We refer the reader to [19, Part I] for more information on unstable modules.

For $1 \le j \le n$, put $X_j = x_j/y_j$ and $Y_j = y_j^{p-1}$. We associate to a monomial $X_1^{e_1}Y_1^{i_1}\cdots X_n^{e_n}Y_n^{i_n}$ the sequence $(d_1,\ldots,d_n)$ where $d_j = 2(p-1)i_j - e_j$ is the degree of $X_j^{e_j}Y_j^{i_j}$. It is clear that the sequence $(d_1,\ldots,d_n)$ determines uniquely the sequence $(e_1,i_1,\ldots,e_n,i_n)$. The set of monomials $X_1^{e_1}Y_1^{i_1}\cdots X_n^{e_n}Y_n^{i_n}$ is ordered by using the right lexicographical order on the set of associated sequences $(d_1,\ldots,d_n)$.

The following result gives a link between the basis elements considered by Mitchell–Priddy [17] and the classes $\omega_{E;I}$'s considered here.

Theorem 3 *If $(E;I)$ is $(n,0)$-admissible then*

$$\beta^{1^n-E}\mathscr{P}^I(X_1\cdots X_n) = (-1)^{|I|}\widetilde{\Sigma}_n(\sigma_{E;I}).$$

Proof We prove the formula by induction on $n \ge 1$, following the approach of [7, Prop. A.2]. If $n=1$ then $\beta^{1-e}\mathscr{P}^i(X_1) = (-1)^i X_1^e Y_1^i$. Suppose the formula is true for $1,\ldots,n-1$. Put $E' = (e_2,\ldots,e_n)$ and $I' = (i_2,\ldots,i_n)$. We have

$$\begin{aligned}\beta^{1^{n-1}-E'}\mathscr{P}^{I'}(X_1\cdots X_n) &= P_{E',I'} + \sum_{1\le j\le n}(-1)^{j-1}X_j\beta^{1^{n-1}-E'}\mathscr{P}^{I'}\left(\frac{X_1\cdots X_n}{X_j}\right)\\ &\quad + \sum_{1\le j<k\le n}(-1)^{j+k-2}X_jX_k\beta^{1^{n-1}-E'}\mathscr{P}^{I'}\left(\frac{X_1\cdots X_n}{X_jX_k}\right)+\cdots,\end{aligned}$$

where $P_{E',I'}$ is an element of H^*V_n of degree $2(p-1)(i_2+\cdots+i_n)-(e_2+\cdots+e_n)$. By $(n,0)$-admissibility of $(E;I)$, we check that this degree is less than the excess of $\beta^{1-e_1}\mathscr{P}^{i_1}$, which is $2i_1+1-e_n$, and so $P_{E',I'}$ is killed by $\beta^{1-e_1}\mathscr{P}^{i_1}$ by instability. Writing

$$\beta^{1^{n-1}-E'}\mathscr{P}^{I'} = \beta^{1-e_2}\mathscr{P}^{i_2}\beta^{1^{n-2}-E''}\mathscr{P}^{I''},$$

where $E'' = (e_3,\ldots,e_n)$ and $I'' = (i_3,\ldots,i_n)$, we have by inductive hypothesis that $\beta^{1^{n-2}-E''}\mathscr{P}^{I''}(\frac{X_1\cdots X_n}{X_jX_k})$ is an element of H^*V_n of degree less than the excess of $\beta^{1-e_2}\mathscr{P}^{i_2}$. It follows that the terms in the second line of the above identity vanish, and so we obtain

$$\begin{aligned}\beta^{1^n-E}\mathscr{P}^I(X_1\cdots X_n) &= \sum_{1\le j\le n}(-1)^{j-1}\beta^{1-e_1}\mathscr{P}^{i_1}\left(X_j\beta^{1^{n-1}-E'}\mathscr{P}^{I'}\left(\frac{X_1\cdots X_n}{X_j}\right)\right)\\ &= \widetilde{C}_n\left(\beta^{1-e_1}\mathscr{P}^{i_1}\left(X_1\beta^{1^{n-1}-E'}\mathscr{P}^{I'}(X_2\cdots X_n)\right)\right)\\ &= (-1)^{|I'|}\widetilde{C}_n\widetilde{\Sigma}_{n-1}\left(\beta^{1-e_1}\mathscr{P}^{i_1}\left(X_1\sigma_{E';I'}(x_2,\ldots,x_n;y_2,\ldots,y_n)\right)\right).\end{aligned}$$

Here $\widetilde{C}_n = \sum_{j=1}^{n} \operatorname{sgn}(\pi_j)\pi_j$, with π_j being the permutation sending $(1, 2, \ldots, n)$ to $(j, 1, \ldots, j-1, j+1, \ldots, n)$ and Σ_{n-1} the subgroup of Σ_n of permutations which fix 1. It suffices now to prove that

$$\beta^{1-e_1}\mathscr{P}^{i_1}\left(X_1\sigma_{E';I'}(x_2,\ldots,x_n;y_2,\ldots,y_n)\right) = (-1)^{i_1}\sigma_{E;I}(x_1,\ldots,x_n;y_1,\ldots,y_n). \quad (1)$$

To this end, we use the stable version of the total Steenrod power as defined in the work of Hung and Sum [8]. For M an $\mathscr{A}$-module, let $S : M \to (\mathrm{E}(x) \otimes \mathbb{F}_p[y^{\pm 1}])\widehat{\otimes} M$ be the linear morphism

$$S(z) = \sum_{i\geq 0,\ e\in\{0,1\}} (-1)^{i+e} X^e Y^{-i} \otimes \beta^e \mathscr{P}^i(z),$$

where $X = x/y$ and $Y = y^{p-1}$. For our purpose, we consider the case where M is $\mathrm{E}(x_2,\ldots,x_n)\otimes\mathbb{F}_p[y_2,\ldots,y_n]$. In this case, as M is an $\mathscr{A}$-algebra, if follows from the Cartan formula that S is a homomorphism of algebras (in fact, it is a monomorphism of algebras [8, Prop. 2.6]). Following the computation in [8], we verify that

$$\begin{aligned} S\big(\mu_s(x_2,\ldots,x_{s+1};y_2,\ldots,y_{s+1})\big) &= Y^{p^{s-1}}\mu_{s+1}(x,x_2,\ldots,x_{s+1};y,y_2,\ldots,y_{s+1})\\ S\big(\omega_s(y_2,\ldots,y_{s+1})\big) &= Y^{-p^s}\omega_{s+1}(y,y_2,\ldots,y_{s+1}). \end{aligned}$$

We get then the following equalities

$$\begin{aligned} X^{e_1}Y^{i_1} &= Y^{-e_2+pi_2}\mu_1^{e_1}\omega_1^{i_1-pi_2+e_2},\\ S(\mu_1^{e_2}) &= Y^{e_2}\mu_2^{e_2},\\ S(\omega_1^{i_2-pi_3+e_3}) &= Y^{-p(i_2-pi_3+e_3)}\omega_2^{i_2-pi_3+e_3},\\ S(\mu_2^{e_3}) &= Y^{pe_3}\mu_3^{e_3},\\ S(\omega_2^{i_3-pi_4+e_4}) &= Y^{-p^2(i_3-pi_4+e_4)}\omega_3^{i_3-pi_4+e_4},\\ \ldots &\ldots \ldots\\ S(\mu_{n-2}^{e_{n-1}}) &= Y^{p^{n-3}e_{n-1}}\mu_{n-1}^{e_{n-1}},\\ S(\omega_{n-2}^{i_{n-1}-pi_n+e_n}) &= Y^{-p^{n-2}(i_{n-1}-pi_n+e_n)}\omega_{n-1}^{i_{n-1}-pi_n+e_n},\\ S(\mu_{n-1}^{e_n}) &= Y^{p^{n-2}e_n}\mu_n^{e_n},\\ S(\omega_{n-1}^{i_n}) &= Y^{-p^{n-1}i_n}\omega_n^{i_n}, \end{aligned}$$

where μ_s and ω_s on the left-hand sides (resp. right-hand sides) are in terms of the variables $x_2,\ldots,x_{s+1}$ and $y_2,\ldots,y_{s+1}$ (resp. $x, x_2,\ldots,x_s$ and $y, y_2,\ldots,y_s$). Taking into account the multiplicativity of S, we get

$$X^{e_1}Y^{i_1}S\left(\sigma_{E';I'}(x_2,\ldots,x_n;y_2,\ldots,y_n)\right) = \sigma_{E;I}(x,x_2,\ldots,x_n;y,y_2,\ldots,y_n),$$

and so

$$X^{e_1}Y^{i_1}S\left(X_1\sigma_{E';I'}(x_2,\ldots,x_n;y_2,\ldots,y_n)\right) = (-1)^{e_1}S(X_1)\sigma_{E;I}(x,x_2,\ldots,x_n;y,y_2,\ldots,y_n). \quad (2)$$

The coefficient of X on the left-hand side of this equality is equal to

$$c := (-1)^{i_1+1-e_1}\beta^{1-e_1}\mathscr{P}^{i_1}(X_1\sigma_{E';I'}(x_2,\ldots,x_n;y_2,\ldots,y_n)).$$

For the right-hand side of (2), we know that the action of the total Steenrod power on $X_1 = x_1/y_1$ is given by

$$S(X_1) = \sum_{e,i}(-1)^e X^e Y^{-i} X_1^{1-e} Y_1^i.$$

Furthermore, the $(n, 0)$-admissibility of $(E; I)$ permits us to write $\sigma_{E;I}$ as a finite sum

$$\sigma_{E;I}(x, x_2, \dots, x_n; y, y_2, \dots, y_n) = \sum_{e',i'} X^{e'} Y^{i'} f_{e',i'}$$

with $f_{e',i'} \in \mathrm{E}(x_2, \dots, x_n) \otimes \mathbb{F}[y_2, \dots, y_n]$. The right-hand side of (2) is thus equal to

$$\sum_{e,i} \sum_{e',i'} (-1)^{e_1+e+e'(1-e)} X^{e+e'} Y^{-i+i'} X_1^{1-e} Y_1^i f_{e',i'}.$$

It is easily to check that the coefficient of X in this sum is

$$c' := (-1)^{1+e_1} \sum_{e',i'} X_1^{e'} Y_1^{i'} f_{e',i'} = (-1)^{1+e_1} \sigma_{E;I}(x_1, x_2, \dots, x_n; y_1, y_2, \dots, y_n).$$

Comparing the coefficients c and c' yields the identity (1). The theorem is proved. □

The following result was proved in [5] using properties of the Steinberg idempotents. Using Lemma 3, this is similar to [17, Lemma 3.6].

Lemma 1 ([5, Lemme 2.11]) *For each $(n, 0)$-admissible couple (E, I), we have*

$$\omega_{e_1,\dots,e_n;i_1,\dots,i_n} = \omega_{e_1,\dots,e_{n-1};i_1,\dots,i_{n-1}} \cdot X_n^{e_n} Y_n^{i_n} + \sum_{\deg(X_n^e Y_n^i) > \deg(X_n^{e_n} Y_n^{i_n})} f_{e,i} \cdot X_n^e Y_n^i$$

for some $f_{e,i} \in \mathrm{E}(x_1, \dots, x_{n-1}) \otimes \mathbb{F}_p[y_1 \dots, y_{n-1}]$. As a consequence,

$$\omega_{e_1,\dots,e_n;i_1,\dots,i_n} = X_1^{e_1} Y_1^{i_1} \cdots X_n^{e_n} Y_n^{i_n} + \textit{monomials of higher order}.$$

The linear structure of $L_{n,k}$ is given by the following result. The case where k is odd was proved in [5] based on the work of Mitchell and Priddy [17]. The case where k is an arbitrary natural number was proved in [3] using the modular Hecke algebra [12] generated by Steinberg idempotents.

Theorem 4 *$L_{n,k}$ has a basis consisting of all (n, k)-admissible classes.*

Proof We sketch a proof which follows that of [5, Proposition 2.8]. By Lemma 1, we have

$$\omega_{e_1,\dots,e_n;i_1,\dots,i_n} = X_1^{e_1} Y_1^{i_1} \cdots X_n^{e_n} Y_n^{i_n} + \text{monomials of higher order}.$$

This implies the linear independence of the set of all (n, k)-admissible classes. To prove that this set is a basis, we make use of the formulas of the Poincaré series of $L_{n,0}$ (resp. $L_{n,1}$) computed by Mitchell-Priddy [17] in order to determine the Poincaré series of $L_{n,even}$ (resp. $L_{n,odd}$). □

The following theorem provides an odd primary versions of Takayasu's short exact sequence [22].

Theorem 5 *There are short exact sequences of $\mathscr{A}$-modules:*

1. $0 \to L_{n,2m+1} \xrightarrow{\iota_{n,2m+1}} L_{n,2m} \xrightarrow{\pi_{n,2m}} \Sigma^{2m(p-1)} L_{n-1,2mp+1} \to 0, \quad m \geq 0;$
2. $0 \to L_{n,2m} \xrightarrow{\iota_{n,2m}} L_{n,2m-1} \xrightarrow{\pi_{n,2m-1}} \Sigma^{2m(p-1)-1} L_{n-1,2mp-1} \to 0, \quad m \geq 1.$

Proof Given a $(n, 2m)$-admissible class $\omega_{E;I}$, the condition $2i_n - e_n \geq 2m$ implies that $\deg(X_n^{e_n} Y_n^{i_n}) \geq 2(p-1)m$, where the equality holds if and only if $(e_n, i_n) = (0, m)$. It follows from Lemma 1 that $L_{n,2m}$ is a submodule of $H^*V_{n-1} \otimes (H^*V_1)^{\geq 2m(p-1)}$, where for each $\mathscr{A}$-module M, $M^{\geq d}$ denotes the submodule of M consisting of elements of degree at least d. The projection on the bottom class $(H^*V_1)^{\geq 2m(p-1)} \to \Sigma^{2m(p-1)}\mathbb{Z}/p$ then gives rise to an $\mathscr{A}$-linear map

$$\pi_{n,2m} : L_{n,2m} \to \Sigma^{2m(p-1)} H^* V_{n-1}.$$

The image of $\omega_{E;I}$ under this map is given by

$$\pi_{n,2m}(\omega_{E;I}) = \begin{cases} \Sigma^{2m(p-1)} \omega_{e_1,\ldots,e_{n-1};i_1,\ldots,i_{n-1}} & \text{if } e_n = 0 \text{ and } i_n = m, \\ 0 & \text{otherwise.} \end{cases}$$

If $(e_n, i_n) = (0, m)$, then the $(n, 2m)$-admissibility of (E, I) implies that $2i_{n-1} - e_{n-1} \geq 2(mp+1) - 1 = 2mp + 1$, and so $\omega_{e_1,\ldots,e_{n-1};i_1,\ldots,i_{n-1}}$ is $(n-1, 2mp+1)$-admissible. The map $\pi_{n,2m}$ thus induces a surjection $L_{n,2m} \twoheadrightarrow \Sigma^{2m(p-1)} L_{n-1,2mp+1}$. The kernel of $\pi_{n,2m}$ has a basis consisting of all classes $\omega_{E;I}$ for which

$$(E; I) \text{ is } (n, 2m)\text{-admissible and } (e_n, i_n) \neq (0, m).$$

We check that

$$\begin{cases} 2i_n - e_n \geq 2m, \\ (e_n, i_n) \neq (0, m) \end{cases} \iff 2i_n - e_n \geq 2m + 1.$$

It follows that the kernel of $\pi_{n,2m}$ is exactly $L_{n,2m+1}$. The short exact sequence 1) is proved. The sequence 2) is proved similarly. □

3 Odd Primary Version of Kameko's Homomorphism and its Restriction to $L_{n,m}$

3.1 Odd Primary Version of Kameko's Homomorphism

We first recall the definition of Kameko's homomorphism when $p = 2$. The mod-2 cohomology ring of $(\mathbb{Z}/2)^n$ is given by the polynomial ring $P_n := \mathbb{F}_2[x_1, \ldots, x_n]$ with each x_i of degree 1. In order to study the space $\mathrm{Q}(P_n)$, Kameko introduced in his thesis [11] the homomorphism $\psi : P_n \to P_n$ by the formula

$$\psi(x_1^{i_1} \cdots x_n^{i_n}) = x_1^{(i_1-1)/2} \cdots x_n^{(i_n-1)/2},$$

where $x_j^{(i_j-1)/2}$ is zero if i_j is even. The homomorphism ψ is GL_n-linear. Though it is not $\mathscr{A}_2$-linear, it can be checked that $\psi \mathrm{Sq}^i = \mathrm{Sq}^{i/2}\psi$, where $\mathrm{Sq}^{i/2}$ is zero if i is odd. The most fundamental property of ψ discovered by Kameko is that ψ induces an isomorphism between $\mathrm{Q}^d(P_n)$ and $\mathrm{Q}^{\frac{d-n}{2}}(P_n)$ whenever $\mu(d) = n$ where $\mu(d)$ is the smallest number r for which it is possible to write $d = \sum_{1 \leq i \leq r}(2^{d_i} - 1)$ with $d_i > 0$.

In [6], we considered the homomorphism $\psi_n : \Sigma^n P_n \to \Phi\Sigma^n P_n$ defined by

$$\psi_n(\Sigma^n X) = \Phi\Sigma^n \psi(X),$$

where ψ is as above. The identity $\psi \mathrm{Sq}^i = \mathrm{Sq}^{i/2}\psi$ mentioned above can be interpreted as saying that ψ_n is $\mathscr{A}_2$-linear. One of the main results of [6] is that ψ_n induces an isomorphism of graded vector spaces

$$\mathrm{Q}(\Sigma^n L_{n,2k+1}) \cong \mathrm{Q}(\Phi\Sigma^n L_{n,k}).$$

It was also observed in [6] that if we let $\hat{P}_1$ denote the subspace of $\mathbb{F}_2[x_1^{\pm 1}]$ spanned by the classes x_1^i with $i \geq -1$, then $\psi_1 : \Sigma P_1 \to \Phi\Sigma P_1$ is the restriction of the $\mathscr{A}_2$-linear map $\hat{\psi}_1 : \Sigma\hat{P}_1 \to \Phi\Sigma\hat{P}_1$ which sends Σx_1^i to $\Phi\Sigma x_1^{(i-1)/2}$. The formula $\mathrm{Sq}^{i+1}(x_1^{-1}) = x_1^i$ implies that $\hat{\psi}_1$ is the unique $\mathscr{A}_2$-linear map which extends the isomorphism $\Sigma x_1^{-1} \mapsto \Phi\Sigma x_1^{-1}$ in degree zero. The map ψ_n is thus the restriction of $(\hat{\psi}_1)^{\otimes n}$ to $\Sigma^n P_n$ in an evident way.

We now present an odd primary version of the Kameko homomorphism. Let $H^*\mathbb{Z}/p$ be identified with $\mathrm{E}(x) \otimes \mathbb{F}_p[y]$ where $|x| = 1$ and $|y| = 2$. It is well-known that the localized ring $(H^*\mathbb{Z}/p)[y^{-1}]$ is equipped with an $\mathscr{A}$-module structure which is compatible with that of $H^*\mathbb{Z}/p$. In particular,

$$\mathscr{P}^i y^d = \binom{d}{i} y^{d+(p-1)i}, \quad i \geq 0,$$

for any $d \in \mathbb{Z}$ where the binomial coefficient $\binom{d}{i}$ is interpreted as the coefficient of t^i in the formal power series $(1+t)^d \in \mathbb{F}_p[[t]]$. For example, $\binom{-1}{i} = (-1)^i$ for any $i \in \mathbb{N}$.

Put $X = x/y$ and let $\hat{P}_1$ denote the $\mathscr{A}$-submodule of $(H^*\mathbb{Z}/p)[y^{-1}]$ generated by the classes $X^e y^i$ for which $e \in \{0, 1\}$ and $2i - e \geq -1$. We define the linear map $\hat{\psi}_1 : \Sigma\hat{P}_1 \to \Phi\Sigma\hat{P}_1$ by the formula

$$\hat{\psi}_1(\Sigma X^e y^i) = \begin{cases} \Phi\Sigma X y^j & \text{if } e = 1 \text{ and } i = pj, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Proposition 1 *$\hat{\psi}_1$ is $\mathscr{A}$-linear.*

Proof We need to verify

$$\theta\hat{\psi}_1(\Sigma X^e y^i) = \hat{\psi}_1\theta(\Sigma X^e y^i), \tag{4}$$

where $2i - e \geq -1$ and $\theta = \beta$ or $\theta = \mathscr{P}^\ell$. We consider the following cases:

Case $e = 0$. Both sides of (4) are zero.

Case $e = 1$ and $p \nmid i$. The left-hand side of (4) is zero. For the right-hand side, if θ is the Bockstein operation β, then

$$\hat{\psi}_1\left(\beta(\Sigma X y^i)\right) = -\hat{\psi}_1(\Sigma y^i) = 0.$$

If $\theta = \mathscr{P}^\ell$, then

$$\mathscr{P}^\ell(\Sigma X y^i) = \binom{i-1}{\ell}\Sigma X y^{i+\ell(p-1)}.$$

If $i - \ell$ is not divisible by p, then $\hat{\psi}_1$ sends this class to zero. If $i - \ell$ is divisible by p then, writing $i = pi' + r$ and $\ell = p\ell' + r$ with $1 \leq r \leq p-1$, we have

$$\binom{i-1}{\ell} = \binom{pi'+r-1}{p\ell'+r} = \binom{pi'}{p\ell'}\binom{r-1}{r} = 0 \tag{5}$$

by Lucas's theorem.

Case $e = 1$ and $i = pj \geq 0$. We have

$$\beta\hat{\psi}_1(\Sigma X y^{pj}) = \beta\Phi\Sigma X Y^j = 0$$

and

$$\hat{\psi}_1\beta(\Sigma X y^{pj}) = -\hat{\psi}_1(\Sigma y^{pj}) = 0,$$

so (4) holds if $\theta = \beta$. If $\theta = \mathscr{P}^\ell$, we have

$$\mathscr{P}^\ell\big(\hat{\psi}_1(\Sigma X y^{pj})\big) = \mathscr{P}^\ell(\Phi \Sigma X y^j) = \begin{cases} \binom{j-1}{k}\Phi \Sigma X y^{j+k(p-1)} & \text{if } \ell = pk, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\hat{\psi}_1\big(\mathscr{P}^\ell(\Sigma X y^{pj})\big) = \binom{pj-1}{\ell}\hat{\psi}_1(\Sigma X y^{pj+\ell(p-1)}) = \begin{cases} \binom{pj-1}{pk}\Phi \Sigma X y^{j+k(p-1)} & \text{if } \ell = pk, \\ 0 & \text{otherwise.} \end{cases}$$

The validity of (4) now follows from the identity

$$\binom{j-1}{k} = \binom{pj-1}{pk}. \tag{6}$$

This is true if $j = 0$ since $\binom{-1}{k} = (-1)^k$ and $\binom{-1}{pk} = (-1)^{pk} = (-1)^k$. If $j > 0$, the identity is obtained by an easy application of Lucas's theorem:

$$\binom{pj-1}{pk} = \binom{p(j-1)+p-1}{pk} = \binom{p(j-1)}{pk}\binom{p-1}{0} = \binom{j-1}{k}.$$

The proposition is proved. $\square$

The map $\hat{\psi}_1$ now gives rise to an $\mathscr{A}$-linear map:

$$\hat{\psi}_n : \Sigma^n \hat{P}_1^{\otimes n} \overset{\cong}{\to} (\Sigma \hat{P}_1)^{\otimes n} \xrightarrow{\hat{\psi}_1^{\otimes n}} (\Phi \Sigma \hat{P}_1)^{\otimes n} \overset{\varphi}{\to} \Phi(\Sigma \hat{P}_1)^{\otimes n} \cong \Phi \Sigma^n \hat{P}_1^{\otimes n},$$

where φ is induced by the natural $\mathscr{A}$-linear map $\varphi_{M,N} : \Phi M \otimes \Phi N \to \Phi(M \otimes N)$. By (3), the only non-trivial action of $\hat{\psi}_n$ on monomials is given by

$$\hat{\psi}_n\left(\Sigma^n X_1 y_1^{pj_1} \cdots X_n y_n^{pj_n}\right) = \Phi \Sigma^n \left(X_1 y_1^{j_1} \cdots X_n y_n^{j_n}\right), \qquad j_1, \ldots, j_n \geq 0.$$

Restricting $\hat{\psi}_n$ to $\Sigma^n H^* V_n$ provides a homomorphism

$$\psi_n : \Sigma^n H^* V_n \to \Phi \Sigma^n H^* V_n$$

which may be seen as an odd primary version of the Kameko homomorphism.

Proposition 2 *ψ_n is GL_n-linear.*

Proof It suffices to prove the proposition for $n = 2$. To this end, we need to verify that ψ_2 commutes with $\tau := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\sigma := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $d_{a,b} := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $a, b \in \mathbb{F}_p$. The verification is easy for the matrices σ and $d_{a,b}$. We prove now that ψ_2 commutes with τ, that is,

$$\psi_2\left(\tau(X_1^{e_1} y_1^{i_1} X_2^{e_2} y_2^{i_2})\right) = \tau\left(\psi_2(X_1^{e_1} y_1^{i_1} X_2^{e_2} y_2^{i_2})\right), \qquad e_1, e_2 \in \{0, 1\}, \quad i_1, i_2 \geq 1.$$

If $e_1 e_2 = 0$, then $X_1 X_2$ does not appear in $\tau(X_1^{e_1} y_1^{i_1} X_2^{e_2} y_2^{i_2})$, so both $\psi_2 \tau$ and $\tau \psi_2$ vanish on $X_1^{e_1} y_1^{i_1} X_2^{e_2} y_2^{i_2}$. If $e_1 = e_2 = 1$, then

$$\tau\left(X_1 y_1^{i_1} X_2 y_2^{i_2}\right) = x_1 y_1^{i_1 - 1}(x_1 + x_2)(y_1 + y_2)^{i_2 - 1} = \sum_{\ell=0}^{i_2-1} \binom{i_2 - 1}{\ell} X_1 y_1^{i_1+\ell} X_2 y_2^{i_2-\ell}.$$

We consider the following cases:

Case $p \nmid i_1$ and $p \mid i_2$. The monomial $X_1 y_1^{i_1+\ell} X_2 y_2^{i_2-\ell}$ is sent by ψ_2 to zero since one of the powers $i_1 + \ell$, $i_2 - \ell$ is not divisible by p. Hence both $\psi_2 \tau$ and $\tau \psi_2$ vanish on $X_1^{e_1} y_1^{i_1} X_2^{e_2} y_2^{i_2}$.

Case $p \mid i_1$ *and* $p \nmid i_2$. This is similar to the previous case.

Case $p \nmid i_1$ *and* $p \nmid i_2$. The map ψ_2 is non-trivial on $X_1 y_1^{i_1+\ell} X_2 y_2^{i_2-\ell}$ only if $i_1+\ell$ and $i_2-\ell$ are both divisible by p. But if $i_2-\ell$ is divisible by p and i_2 is not divisible by p, then the coefficient $\binom{i_2-1}{\ell}$ is zero by (5). So again both $\psi_2\tau$ and $\tau\psi_2$ vanish on $X_1 y_1^{i_1} X_2 y_2^{i_2}$.

Case $p \mid i_1$ *and* $p \mid i_2$. Put $i_1 = pj_1$ and $i_2 = pj_2$. We have (omitting the functors Φ and Σ^2 in the writing of ψ_2 for simplicity)

$$\begin{aligned}\psi_2\left(\tau(X_1 y_1^{i_1} X_2 y_2^{i_2})\right) &= \sum_{\ell=0}^{pj_2-1} \binom{pj_2-1}{\ell} \psi_2\left(X_1 y_1^{pj_1+\ell} X_2 y_2^{pj_2-\ell}\right)\\ &= \sum_{k=0}^{j_2-1} \binom{pj_2-1}{pk} X_1 y_1^{j_1+k} X_2 y_2^{j_2-k}.\end{aligned}$$

On the other hand,

$$\tau\left(\psi_2(X_1 y_1^{i_1} X_2 y_2^{i_2})\right) = \tau\left(X_1 y_1^{j_1} X_2 y_2^{j_2}\right) = \sum_{k=0}^{j_2-1} \binom{j_2-1}{k} X_1 y_1^{j_1+k} X_2 y_2^{j_2-k}.$$

By (6), we see that the coefficients $\binom{pj_2-1}{pk}$ and $\binom{j_2-1}{k}$ are equal, and so $\psi_2\tau$ and $\tau\psi_2$ take the same value on the class $X_1 y_1^{i_1} X_2 y_2^{i_2}$. The proposition is proved. □

The following is a direct consequence of Theorem 3 and Proposition 2.

Proposition 3 *For each* $(n, 0)$*-admissible couple* $(E; I)$*,*

$$\psi_n(\Sigma^n \omega_{E;I}) = \begin{cases} \Phi\Sigma^n \omega_{E;J} & \text{if } E = 1^n \text{ and } I = pJ,\\ 0 & \text{otherwise.}\end{cases}$$

Proof Put $\alpha_{E;I} = \beta^{1^n-E}\mathscr{P}^I(X_1\cdots X_n)$. The $\mathscr{A}$-linearity of $\hat{\psi}_n$ yields

$$\begin{aligned}\psi_n(\alpha_{E;I}) = \beta^{1^n-E}\mathscr{P}^I\left(\hat{\psi}_n(X_1\cdots X_n)\right) &= \beta^{1^n-E}\mathscr{P}^I\left(\Phi\Sigma^n(X_1\cdots X_n)\right)\\ &= \begin{cases}\mathscr{P}^I\left(\Phi\Sigma^n(X_1\cdots X_n)\right) & \text{if } E = 1^n,\\ 0 & \text{if } E \neq 1^n\end{cases}\\ &= \begin{cases}\Phi\Sigma^n\alpha_{E,J} & \text{if } E = 1^n \text{ and } I = pJ,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

By Theorem 3, we have $\omega_{E;I} = (-1)^{|I|}\bar{B}_n(\alpha_{E;I})$ and so the result follows from the GL_n-linearity of ψ_n. □

3.2 Restriction of ψ_n to $L_{n,k}$

Denote by $\psi_{n,k}$ the restriction of ψ_n to $L_{n,k}$. The following result will play an essential role in constructing by double induction a minimal generating set for $L_{n,k}$.

Theorem 6 1. *If* $m = m'p$ *with* $m' \geq 1$*, then the following hold:*

(a) *ψ_n induces a commutative diagram*

$$\begin{tikzcd} \Sigma^n L_{n,2m-1} \arrow[r, hook, "\iota_{n,2m-1}"] \arrow[d, "\psi_{n,2m-1}"'] & \Sigma^n L_{n,2m-2} \arrow[dl, "\psi_{n,2m-2}"] \\ \Phi\Sigma^n L_{n,2m'-1}. & \end{tikzcd}$$

(b) *$\psi_{n,2m-1}$ induces an isomorphism*

$$\mathrm{Q}\left(\Sigma^n L_{n,2m-1}\right) \cong \Phi\mathrm{Q}\left(\Sigma^n L_{n,2m'-1}\right).$$

(c) *The induced map $\mathrm{Q}(\iota_{n,2m-1})$ is injective.*

2. *If $m = m'p + r$ with $m' \geq 0$ and $1 \leq r \leq p-1$, then the following hold:*

(a) *ψ_n induces a commutative diagram*

$$\begin{tikzcd} \Sigma^n L_{n,2m} \arrow[r, hook, "\iota_{n,2m}"] \arrow[d, "\psi_{n,2m}"'] & \Sigma^n L_{n,2m-1} \arrow[r, hook, "\iota_{n,2m-1}"] \arrow[dl, "\psi_{n,2m-1}"'] & \Sigma^n L_{n,2m-2} \arrow[dll, "\psi_{n,2m-2}"] \\ \Phi\Sigma^n L_{n,2m'+1}. & & \end{tikzcd}$$

(b) *$\psi_{n,2m-1}$ induces an isomorphism*

$$\mathrm{Im}(\mathrm{Q}(\iota_{n,2m})) \cong \Phi\mathrm{Q}(\Sigma^n L_{n,2m'+1}).$$

(c) *The induced map $\mathrm{Q}(\iota_{n,2m-1})$ is injective on $\mathrm{Im}(\mathrm{Q}(\iota_{n,2m}))$.*

The proof of this theorem will be given in Section 3.3 below. Note that the first part of Theorem 1 is included in the first part of this theorem. The remaining parts of Theorem 1 are given in the following corollaries.

Corollary 1 (Theorem 1(2)) *If $m = m'p + r$ with $m' \geq 0$ and $1 \leq r \leq p-1$, then there is a short exact sequence of graded vector spaces*

$$0 \to \Phi\mathrm{Q}(\Sigma^n L_{n,2m'+1}) \to \mathrm{Q}(\Sigma^n L_{n,2m-1}) \to \mathrm{Q}\left(\Sigma^{2m(p-1)+n-1} L_{n-1,2mp-1}\right) \to 0.$$

Proof The short exact sequence

$$0 \to L_{n,2m} \xrightarrow{\iota_{n,2m}} L_{n,2m-1} \to \Sigma^{2m(p-1)-1} L_{n-1,2mp-1} \to 0$$

induces an exact sequence

$$\mathrm{Q}\left(\Sigma^n L_{n,2m}\right) \xrightarrow{\mathrm{Q}(\iota_{n,2m})} \mathrm{Q}\left(\Sigma^n L_{n,2m-1}\right) \to \mathrm{Q}\left(\Sigma^{2m(p-1)+n-1} L_{n-1,2mp-1}\right) \to 0.$$

By Theorem 6(2.b), we have $\mathrm{Im}(\mathrm{Q}(\iota_{n,2m})) \cong \Phi\mathrm{Q}(\Sigma^n L_{n,2m'+1})$. The corollary follows. □

Corollary 2 (Theorem 1(3)) *For all $m \geq 1$, the indecomposable functor Q preserves the exactness of the short exact sequence*

$$0 \to L_{n,2m-1} \xrightarrow{\iota_{n,2m-1}} L_{n,2m-2} \xrightarrow{\pi_{n,2m-2}} \Sigma^{2(m-1)(p-1)} L_{n-1,2(mp-p+1)p-1} \to 0.$$

Proof It suffices to prove that the induced map $\mathrm{Q}(\iota_{n,2m-1})$ is injective. If $m \equiv 0 \pmod p$, this is Theorem 6(1.a). Suppose now that $m \not\equiv 0 \pmod p$. As XY^m is $\mathscr{A}$-indecomposable in $L_{1,2m-2}$, there is a non trivial $\mathscr{A}$-linear map

$$\pi' : L_{1,2m-2} \to \Sigma^{2m(p-1)-1}\mathbb{Z}/p,$$

which extends the map $\pi_{1,2m-1} : L_{1,2m-1} \to \Sigma^{2m(p-1)-1}\mathbb{Z}/p$. It follows that for $n > 1$, the map

$$\pi_{n,2m-1} : L_{n,2m-1} \to \Sigma^{2m(p-1)-1} L_{n-1,2mp-1}$$

can also be extended to $L_{n,2m-2}$ as indicated in the following commutative diagram:

$$\begin{tikzcd}
L_{n,2m-1} \arrow[r, hook] \arrow[d, "\pi_{n,2m-1}"] & L_{n,2m-2} \arrow[r, hook] \arrow[dl, dashed, "\pi''"] & L_{n-1,1} \otimes L_{1,2m-2} \arrow[d, "\mathrm{Id}\otimes\pi'"] \\
\Sigma^{2m(p-1)-1} L_{n-1,2mp-1} \arrow[rr, hook] & & L_{n-1,1} \otimes \Sigma^{2m(p-1)-1}\mathbb{Z}/p.
\end{tikzcd}$$

We get then a commutative diagram where the row is exact:

$$\begin{tikzcd}
\mathrm{Q}(\Sigma^n L_{n,2m}) \arrow[r, "\mathrm{Q}(\iota_{n,2m})"] & \mathrm{Q}(\Sigma^n L_{n,2m-1}) \arrow[r, "\mathrm{Q}(\pi_{n,2m-1})"] \arrow[d, "\mathrm{Q}(\iota_{n,2m-1})"'] & \mathrm{Q}(\Sigma^{2m(p-1)+n-1} L_{n-1,2mp-1}) \arrow[r] & 0. \\
& \mathrm{Q}(\Sigma^n L_{n,2m-2}) \arrow[ur, "\mathrm{Q}(\pi'')"'] & &
\end{tikzcd}$$

It follows that $\mathrm{Ker}\,\mathrm{Q}(\iota_{n,2m-1}) \subseteq \mathrm{Ker}\,\mathrm{Q}(\pi_{n,2m-1}) = \mathrm{Im}\,\mathrm{Q}(\iota_{n,2m})$. But $\mathrm{Q}(\iota_{n,2m-1})$ is injective on $\mathrm{Im}\,\mathrm{Q}(\iota_{n,2m})$ by Theorem 6(2.c), so we must have $\mathrm{Ker}\,\mathrm{Q}(\iota_{n,2m-1}) = 0$. The injectivity of $\mathrm{Q}(\iota_{n,2m-1})$ is proved. □

3.3 Proof of Theorem 6

We need some preparatory results. We recall that the dual of the Steenrod algebra is computed by J. Milnor in the fundamental work [16]:

$$\mathscr{A}^* \cong \mathrm{E}(\tau_0, \tau_1, \tau_2, \dots) \otimes \mathbb{F}_p[\xi_1, \xi_2, \xi_3, \dots], \qquad |\tau_i| = 2p^i - 1,\ |\xi_i| = 2p^i - 2.$$

Let $\mathscr{Q}_i$, $i \geq 0$, (resp. $\mathscr{P}_i$, $i \geq 1$) denote the dual of τ_i (resp. ξ_i) with respect to the basis of $\mathscr{A}^*$ consisting of all monomials $\tau_{s_1} \cdots \tau_{s_k} \xi_1^{r_1} \cdots \xi_m^{r_m}$ with $0 \leq s_1 < s_2 < \cdots < s_k$ and $r_1, \dots, r_m \geq 0$. The operations $\mathscr{Q}_i$ and $\mathscr{P}_i$ are primitive in $\mathscr{A}$ (which is a Hopf algebra) and can also be defined inductively as follows (see Corollaries 2 and 5 in [16]):

$$\mathscr{Q}_0 = \beta, \quad \mathscr{Q}_{i+1} = [\mathscr{P}^{p^i}, \mathscr{Q}_i],$$
$$\mathscr{P}_1 = \mathscr{P}^1, \quad \mathscr{P}_{i+1} = [\mathscr{P}^{p^i}, \mathscr{P}_i],$$

$[a, b]$ denoting the graded commutator $ab - (-1)^{|a||b|}ba$. The actions of $\mathscr{Q}_i$ and $\mathscr{P}_i$ on $H^*V_n \cong \mathrm{E}(x_1, \dots, x_n) \otimes \mathbb{F}_p[y_1, \dots, y_n]$ are given by the rules

$$\mathscr{Q}_i(ab) = \mathscr{Q}_i(a)b + (-1)^{|a|}a\mathscr{Q}_i(b), \quad \mathscr{Q}_i(x_j) = y_j^{p^i}, \quad \mathscr{Q}_i(y_j) = 0, \quad i \geq 0,$$

$$\mathscr{P}_i(ab) = \mathscr{P}_i(a)b + a\mathscr{P}_i(b), \quad \mathscr{P}_i(y_j) = y_j^{p^i}, \quad \mathscr{P}_i(x_j) = 0, \quad i \geq 1,$$

where a, b are homogeneous elements of H^*V_n.

The following can be proved easily by using the above rules.

Lemma 2 *For* $1 \leq i \leq s$, *we have*

1. $\mathscr{Q}_{i-1}(\mathrm{M}_s) = \begin{cases} (-1)^{s-1}\mathrm{L}_s & \textit{if } i = s, \\ 0 & \textit{if } i \neq s. \end{cases}$
2. $\mathscr{P}_i(\mathrm{L}_s) = \begin{cases} (-1)^{s-1}\mathrm{L}_s^p & \textit{if } i = s, \\ 0 & \textit{if } i \neq s. \end{cases}$

For a proof the reader may refer to the proofs of Lemmas 2.2 and 2.3 in [21] (where the action of primitive Milnor operations on modular invariants are computed explicitly).

We also need the following property of Steinberg idempotents. Recall that for $s \leq n$, we consider GL_s as a subgroup of GL_n by sending $g \in GL_s$ to $\begin{pmatrix} g & 0 \\ 0 & \mathrm{Id}_{n-s} \end{pmatrix}$.

Lemma 3 ([12]) $\mathrm{st}_n \mathrm{st}_s = \mathrm{st}_n$.

The following propositions, which will play key roles in the proof of Theorem 6, provide some criteria for a class $\omega_{E;I}$ to be $\mathscr{A}$-decomposable (a.k.a. *hit*) in $L_{n,k}$. Recall that an element of an $\mathscr{A}$-module M is $\mathscr{A}$*-decomposable* in M if it belongs to $\mathscr{A}^+M$, $\mathscr{A}^+$ denoting the augmentation ideal of $\mathscr{A}$.

Proposition 4 *Suppose* $(E; I)$ *is* (n, k)*-admissible.*

1. *If* $e_s = 0$ *for some* $1 \leq s \leq n-1$*, then* $\omega_{E;I}$ *is hit in* $L_{n,k}$.
2. *If* $e_n = 0$ *and* $2i_n - 1 \geq k$*, then* $\omega_{E;I}$ *is hit in* $L_{n,k}$.

Proof Let s be the first index for which $e_s = 0$. Put

$$u = \mu_1\omega_1^{j_1} \cdots \mu_{s-1}\omega_{s-1}^{j_{s-1}} \quad \text{and} \quad v = \mu_s\omega_s^{j_s}\mu_{s+1}^{e_{s+1}}\omega_{s+1}^{j_{s+1}} \cdots \mu_n^{e_n}\omega_n^{j_n},$$

where $j_r = i_r - pi_{r+1} + e_{r+1}$ for $1 \leq r \leq n-1$ and $j_n = i_n$. The conditions given in the proposition assure that $\mathrm{st}_n(uv)$ is (n, k)-admissible and so is a basis element of $L_{n,k}$. [This is clear if $s < n$. If $s = n$, then $v = \mu_n\omega_n^{i_n}$, so the power of μ_n in uv is 1 and that of ω_n is i_n, and so $\mathrm{st}_n(uv)$ is (n, k)-admissible by the condition $2i_n - 1 \geq k$.]

By Lemma 2(1), we have

$$\mathscr{Q}_{s-1}(v) = (-1)^{s-1}\omega_s^{j_s}\mu_{s+1}^{e_{s+1}}\omega_{s+1}^{j_{s+1}} \cdots \mu_n^{e_n}\omega_n^{j_n}.$$

This shows in particular that $\omega_{E;I}$ is hit in $L_{n,k}$ if $s = 1$. Suppose $s > 1$ and $\omega_{E;I}$ is hit if $e_r = 0$ for some $1 \leq r \leq s-1$. We have

$$\begin{aligned} \omega_{E;I} &= (-1)^{s-1}\mathrm{st}_n\big(u\mathscr{Q}_{s-1}(v)\big) \\ &= \mathrm{st}_n\big(\mathscr{Q}_{s-1}(uv) - \mathscr{Q}_{s-1}(u)v\big) \\ &\equiv \mathrm{st}_n\big(\mathscr{Q}_{s-1}(u)v\big) \pmod{\mathscr{A}^+L_{n,m}}. \end{aligned}$$

Since v is GL_{s-1}-invariant and $\mathrm{st}_n = \mathrm{st}_n\mathrm{st}_{s-1}$ by Lemma 3, we get

$$\mathrm{st}_n\big(\mathscr{Q}_{s-1}(u)v\big) = \mathrm{st}_n\big((\mathrm{st}_{s-1}\mathscr{Q}_{s-1}(u))v\big).$$

As $\mathrm{st}_{s-1}(\mathscr{Q}_{s-1}(u))$ is an element of $L_{s-1,1}$, it is a linear combination of $\mathrm{st}_{s-1}(\mu_1^{\varepsilon_1}\omega_1^{t_1} \cdots \mu_{s-1}^{\varepsilon_{s-1}}\omega_{s-1}^{t_{s-1}})$ with $t_1, \ldots, t_{s-1} > 0$ and $\varepsilon_1, \ldots, \varepsilon_{s-1} \in \{0, 1\}$ by Theorem 4. By comparing degrees, we see that

$$\varepsilon_1 + \cdots + \varepsilon_{s-1} \equiv s \pmod 2$$

and so $\varepsilon_r = 0$ for some $1 \leq r \leq s-1$. By inductive hypothesis, the class

$$\mathrm{st}_n\left(\mu_1^{\varepsilon_1}\omega_1^{t_1} \cdots \mu_{s-1}^{\varepsilon_{s-1}}\omega_{s-1}^{t_{s-1}} \cdot \mu_s\omega_s^{j_s}\mu_{s+1}^{e_{s+1}}\omega_{s+1}^{j_{s+1}} \cdots \mu_n^{e_n}\omega_n^{j_n}\right)$$

is hit in $L_{n,k}$, and so $\omega_{E;I}$ is also hit in $L_{n,k}$. The proposition is proved. □

Proposition 5 *Suppose* $(1^n; I)$ *is* (n,k)*-admissible.*

1. *If* $i_s \not\equiv 0 \pmod p$ *for some* $1 \le s \le n-1$, *then* $\omega_{1^n;I}$ *is hit in* $L_{n,k}$.
2. *If* $i_n \not\equiv 0 \pmod p$ *and* $2(i_n-1)-1 \ge k$ *then* $\omega_{1^n;I}$ *is hit in* $L_{n,k}$.

Proof Using the Milnor operation $\mathscr{P}_s$, the proof is similar to that of the previous proposition. We first observe that

$$\mu_1 \cdots \mu_n = (-1)^{\frac{n(n-1)}{2}} \mathrm{L}_n^{-1} x_1 \cdots x_n.$$

The class $\omega_{1^n;I}$ is then rewritten as

$$\omega_{1^n;I} = (-1)^{\frac{n(n-1)}{2}} \mathrm{st}_n \left(\omega_1^{i_1 - p i_2 + 1} \cdots \omega_{n-1}^{i_{n-1} - p i_n + 1} \omega_n^{i_n} \mathrm{L}_n^{-1} x_1 \cdots x_n \right).$$

Let s be the first index for which $i_s \not\equiv 0 \pmod p$. Put

$$u = (-1)^{\frac{n(n-1)}{2}} \omega_1^{j_1} \cdots \omega_{s-1}^{j_{s-1}} \quad \text{and} \quad v = \omega_s^{j_s - 1} \omega_{s+1}^{j_{s+1}} \cdots \omega_n^{j_n} \mathrm{L}_n^{-1} x_1 \cdots x_n,$$

where $j_r = i_r - p i_{r+1} + 1$ for $1 \le r \le n-1$ and $j_n = i_n$. The power of L_s in v is equal to $(p-1)(i_s - p i_{s+1})$ if $1 \le s \le n-1$ and to $(p-1)(j_n - 1) - 1$ if $s = n$, so this power is $\equiv -i_s \pmod p$. The conditions given in the proposition assure that $\mathrm{st}_n(uv)$ is (n,k)-admissible and so is a basis element of $L_{n,k}$. [Again this is clear if $s < n$. If $s = n$, then the power of μ_n in uv is 1 and that of ω_n is $i_n - 1$, and so $\mathrm{st}_n(uv)$ is (n,k)-admissible by the condition $2(i_n - 1) - 1 \ge k$.]

By Lemma 2(b), we have

$$\mathscr{P}_s(v) = (-1)^s i_s \omega_s^{j_s} \cdots \omega_n^{j_n} \mathrm{L}_n^{-1} x_1 \cdots x_n \neq 0.$$

This shows in particular that $\omega_{1^n;I}$ is hit in $L_{n,k}$ if $s = 1$. Suppose $s > 1$ and $\omega_{1^n;I}$ is hit if $i_r \not\equiv 0 \pmod p$ for some $1 \le r \le r-1$. We have

$$(-1)^s i_s \omega_{1^n;I} = \mathrm{st}_n(u \mathscr{P}_s v) = \mathrm{st}_n \left(\mathscr{P}_s(uv) - \mathscr{P}_s(u) v \right) \equiv -\mathrm{st}_n(\mathscr{P}_s(u) v) \pmod{\mathscr{A}^+ L_{n,k}}.$$

As $\mathrm{st}_{s-1}(\mathscr{P}_s(u))$ is an element of $L_{s-1,1}$ which is free of $x_1, \ldots, x_s$, it can be written as a linear combination of $\mathrm{st}_{s-1}(\omega_1^{t_1} \cdots \omega_{s-1}^{t_{s-1}})$ with $t_1, \ldots, t_{s-1} > 0$ by Theorem 4. So it suffices to prove that the class

$$\mathrm{st}_n \left(\omega_1^{t_1} \cdots \omega_{s-1}^{t_{s-1}} \omega_s^{j_s - 1} \omega_{s+1}^{j_{s+1}} \cdots \omega_n^{j_n} \mu_1 \cdots \mu_n \right)$$

is hit in $L_{n,k}$. Suppose this class corresponds to a (n,k)-admissible couple $(1^n; T_1, \ldots, T_n)$. We first note that $T_r \equiv t_r - 1 \pmod p$ for all $1 \le r \le s-1$, so by inductive hypothesis, it suffices to prove $t_r \not\equiv 1$ for some $1 \le r \le s-1$. By comparing degrees, we have

$$2(p^s - 1) + \deg(\omega_1^{j_1} \cdots \omega_{s-1}^{j_{s-1}}) = \deg \left(\omega_1^{t_1} \cdots \omega_{s-1}^{t_{s-1}} \right),$$

which implies that

$$t_1 + \cdots + t_{s-1} \equiv j_1 + \cdots + j_{s-1} + 1 \pmod p.$$

Since $j_r = i_r - p i_{r+1} + 1 \equiv 1 \pmod p$ for all $1 \le r \le s-1$, we get $t_r \not\equiv 1 \pmod p$ for some $1 \le r \le s-1$. The proposition is proved. □

We are now ready to prove Theorem 6.

Proof of Theorem 6

(1) Suppose $m \equiv 0 \pmod{p}$ and write $m = m'p$ with $m' \geq 1$. Let $\Sigma^n K$ denote the kernel of $\psi_{n,2m-1}$:

$$0 \to \Sigma^n K \to \Sigma^n L_{n,2m-1} \xrightarrow{\psi_{n,2m-1}} \Phi \Sigma^n L_{n,2m'-1}.$$

A basis class $\omega_{E;I}$ in K is of the form given either in Proposition 4 or in Proposition 5, and so is hit in $L_{n,2m-1}$. The induced map $\mathrm{Q}(\Sigma^n K) \to \mathrm{Q}(\Sigma^n L_{n,2m-1})$ is thus trivial.

By Proposition 3, we have $\mathrm{Im}(\psi_{n,2m-1}) = \Phi M$, where

$$M := \mathbb{F}_p\{\Sigma^n \omega_{1^n;I} : (1^n; I) \text{ is } (n, 2m'-1)\text{-admissible}\} \subset \Sigma^n L_{n,2m'-1}.$$

Since M is concentrated in even degrees, we have $\mathrm{Q}(\Phi M) = \Phi(M/\mathscr{P}^+ M)$. On the other hand, by Proposition 4, a class $\omega_{E;I}$ is hit in $L_{n,2m'-1}$ if $e_k = 0$ for some k. It follows that

$$M/\mathscr{P}^+ M \cong \Sigma^n L_{n,2m'-1}/\mathscr{A}^+ \Sigma^n L_{n,2m'-1} = \mathrm{Q}(\Sigma^n L_{n,2m'-1}).$$

We conclude $\mathrm{Q}(\Sigma^n L_{n,2m-1}) \cong \mathrm{Q}(\Phi M) \cong \Phi \mathrm{Q}(\Sigma^n L_{n,2m'-1})$.

(2) Suppose $m = m'p + r$ with $m' \geq 0$ and $1 \leq r \leq p-1$. By Proposition 3, we have

$$\mathrm{Im}(\psi_{n,2m}) = \mathrm{Im}(\psi_{n,2m-1}) = \mathrm{Im}(\psi_{n,2m-2}) = \Phi N,$$

where

$$N := \mathbb{F}_p\{\Sigma^n \omega_{1^n;I} : (1^n; I) \text{ is } (n, 2m'+1)\text{-admissible}\} \subset \Sigma^n L_{n,2m'+1}.$$

As above, we verify that $\mathrm{Q}(\Phi N) \cong \Phi(N/\mathscr{P}^+ N) \cong \Phi \mathrm{Q}(\Sigma^n L_{n,2m'+1})$.

Now let $\Sigma^n K$ denote the kernel of the map $\psi_{n,2m}$. We have the following commutative diagram in which, by abuse of notation, we write f in place of the induced map $\mathrm{Q}(f)$ for each $\mathscr{A}$-linear map f:

$$\begin{tikzcd}
\mathrm{Q}(\Sigma^n K) \arrow[d, "u"] & & \\
\mathrm{Q}(\Sigma^n L_{n,2m}) \arrow[r, "\iota_{n,2m}"] \arrow[d, "\psi_{n,2m}"] & \mathrm{Q}(\Sigma^n L_{n,2m-1}) \arrow[r, "\iota_{n,2m-1}"] \arrow[dl, "\psi_{n,2m-1}"'] & \mathrm{Q}(\Sigma^n L_{n,2m-2}). \arrow[dll, "\psi_{n,2m-2}"] \\
\mathrm{Q}(\Phi N) \arrow[d] & & \\
0 & &
\end{tikzcd}$$

The module K is generated by the classes $\omega_{E;I} \in L_{n,2m}$ for which either ($e_j = 0$ for some $1 \leq j \leq n$) or ($E = 1^n$ and $i_j \not\equiv 0 \pmod{p}$ for some $1 \leq j \leq n$). By Propositions 4 and 5, these elements are hit in $L_{n,2m-1}$ because ($2i_n \geq 2m$ implies $2i_n - 1 \geq 2m-1$) and ($2i_n - 1 \geq 2m$ implies $2(i_n - 1) - 1 \geq 2m - 1$). It follows that the composition $\iota_{n,2m} \circ u$ is trivial, that is $\mathrm{Im}(u) \subset \mathrm{Ker}(\iota_{n,2m})$.

We obtain then the following diagram

$$\begin{tikzcd}
\mathrm{Q}(\Sigma^n L_{n,2m})/\mathrm{Im}(u) \arrow[r, two heads] \arrow[d, "\cong"] & \mathrm{Im}(\iota_{n,2m}). \arrow[dl] \\
\mathrm{Q}(\Phi N) &
\end{tikzcd}$$

Hence $\mathrm{Im}(\iota_{n,2m}) \cong \mathrm{Q}(\Phi N) \cong \Phi\mathrm{Q}(\Sigma^n L_{n,2m'+1})$. This together with the commutative diagram above also gives the injectivity of $\iota_{n,2m-1}$ on $\mathrm{Im}(\iota_{n,2m})$. The theorem is proved. $\square$

4 Minimal Generating Set for $L_{n,k}$

4.1 Generators for $L_{n,1}$

We give first an inductive proof for the result of M. Inoue [10] on the linear structure of the space $\mathrm{Q}(L_{n,1})$.

Lemma 4 *Let $F_n(t)$ denote the Poincaré series of the graded vector space $\mathrm{Q}(\Sigma^n L_{n,1})$. Then $F_n(t)$ is given by*

$$F_{n,1}(t) = \sum_{j_1>\cdots>j_n\geq 0} t^{2p^{j_1}(p-1)+\cdots+2p^{j_n}(p-1)}.$$

Proof Note that if $P_M(t)$ is the Poincaré series of M then

$$P_{\Phi M}(t) = P_{M^+}(t^p) + t^{2-p}P_{M^-}(t^p),$$

where M^+, M^- denote respectively the subspace of even and odd degree elements of M. By the first two parts of Theorem 1, we have

$$\mathrm{Q}(\Sigma^n L_{n,1}) \cong \Phi\mathrm{Q}(\Sigma^n L_{n,1}) \oplus \Sigma^{2(p-1)}\Phi\mathrm{Q}(\Sigma^{n-1}L_{n-1,1}).$$

This shows in particular that the graded vector space $\mathrm{Q}(\Sigma^n L_{n,1})$ is concentrated in even degrees. It follows that the Poincaré series of $\Phi\mathrm{Q}(\Sigma^n L_{n,1})$ is $F_n(t^p)$ and so

$$F_n(t) = F_n(t^p) + t^{2(p-1)}F_{n-1}(t^p).$$

It is easy to verify that this equation determines uniquely $F_n(t)$ once we have obtained the formula for $F_{n-1}(t)$. The lemma now follows by induction on $n \geq 0$, starting from the case $n = 0$ where $L_{0,1} \cong \mathbb{F}_p$. $\square$

Theorem 7 (Cf. [10]) *The classes $\omega_{1^n;p^{j_1},p^{j_2},\ldots,p^{j_n}}$ with $j_1 > j_2 > \cdots > j_n \geq 0$ form a minimal generating set for $L_{n,1}$.*

Proof For $j_1 > \cdots > j_n \geq 0$, the class $\omega_{1^n;p^{j_1},\ldots,p^{j_n}}$ is $\mathscr{A}$-indecomposable because its expansion as a sum of monomials contains $x_1 y_1^{(p-1)p^{j_1}-1}\cdots x_n y_n^{(p-1)p^{j_n}-1}$. Moreover these classes occur in different degrees of $L_{n,1}$, so they are linearly independent in $\mathrm{Q}(L_{n,1})$. The theorem now follows from Lemma 4. $\square$

4.2 Generators for $L_{n,2m-1}$, $m \geq 1$

We now prove Theorem 2 by double induction on (n,m), $n \geq 0$, $m \geq 1$. The couples (n,m) are ordered by the left lexicographical order. The case $n = 0$ is clear and the case $m = 1$ is Theorem 7, so we can start the induction. We consider the following two cases:

Case $m = m'p$. By Theorem 6(1), the map $\psi_{n,2m-1}$ induces an isomorphism

$$\mathrm{Q}(\Sigma^n L_{n,2m-1}) \cong \Phi\mathrm{Q}(\Sigma^n L_{n,2m'-1}).$$

By inductive hypothesis for (n,m'), the classes

$$\omega_{1^n;p^{n-1}\gamma^{i_1}(m'),\ldots,\gamma^{i_n}(m')}$$

with $i_1 \geq i_2 \geq \cdots \geq i_n \geq 0$ form a minimal generating set for $L_{n,2m'-1}$. It follows that the classes

$$\omega_{1^n;\,p^n\gamma^{i_1}(m'),\ldots,p\gamma^{i_n}(m')} = \omega_{1^n;\,p^{n-1}\gamma^{i_1}(m),\ldots,\gamma^{i_n}(m)}$$

with $i_1 \geq i_2 \geq \cdots \geq i_n \geq 0$ form a minimal generating set for $L_{n,2m-1}$.

Case $m = m'p + r$, $1 \leq r \leq p-1$. By Theorem 6, there is a commutative diagram

$$\begin{tikzcd} \mathrm{Q}(\Sigma^n L_{n,2m}) \arrow[r, "\iota"] \arrow[d, "\psi_{n,2m}"] & \mathrm{Q}(\Sigma^n L_{n,2m-1}) \arrow[r, "\pi"] \arrow[dl, "\psi_{n,2m-1}"] & \mathrm{Q}\left(\Sigma^{2m(p-1)+n-1} L_{n-1,2mp-1}\right) \arrow[r] & 0, \\ \Phi\mathrm{Q}(\Sigma^n L_{n,2m'+1}) & & & \end{tikzcd}$$

where the row is exact and $\mathrm{Im}(\iota) \cong \Phi\mathrm{Q}(\Sigma^n L_{n,2m'+1})$. By inductive hypothesis for $(n-1, mp)$, the classes

$$\omega_{1^{n-1};\,p^{n-2}\gamma^{i_1}(mp),\ldots,\gamma^{i_{n-1}}(mp)} = \omega_{1^{n-1};\,p^{n-1}\gamma^{i_1}(m),\ldots,p\gamma^{i_{n-1}}(m)}$$

with $i_1 \geq i_2 \geq \cdots \geq i_{n-1} \geq 0$ form a minimal generating set for $L_{n-1,2mp-1}$. The pullback of such a class under the map π is

$$\omega_{1^n;\,p^{n-1}\gamma^{i_1}(m),\ldots,p\gamma^{i_{n-1}}(k),m}.$$

Similarly, by inductive hypothesis for $(n, m'+1)$, the classes

$$\omega_{1^n;\,p^{n-1}\gamma^{i_1}(m'+1),\ldots,\gamma^{i_n}(m'+1)}$$

with $i_1 \geq i_2 \geq \cdots \geq i_n \geq 0$ form a minimal generating set for $L_{n,2(m'+1)-1}$. The pullback of the class $\omega_{1^n;\,p^{n-1}\gamma^{i_1}(m'+1),\ldots,\gamma^{i_n}(m'+1)}$ under $\psi_{n,2m-1}$ is

$$\omega_{1^n;\,p^n\gamma^{i_1}(m'+1),\ldots,p\gamma^{i_n}(m'+1)} = \omega_{1^n;\,p^{n-1}\gamma^{i_1+1}(m),\ldots,\gamma^{i_n+1}(m)}.$$

It follows that the classes

$$\omega_{1^n;\,p^{n-1}\gamma^{i_1}(m),\ldots,\gamma^{i_n}(m)}$$

with $i_1 \geq i_2 \geq \cdots \geq i_n \geq 0$ form a minimal generating set for $L_{n,2m-1}$.

4.3 Generators for $L_{n,2m-2}$, $m \geq 1$

Combining the exactness of the sequence (Theorem 1(3))

$$0 \to \mathrm{Q}(L_{n,2m-1}) \to \mathrm{Q}(L_{n,2m-2}) \to \mathrm{Q}\left(\Sigma^{2(m-1)(p-1)} L_{n-1,2(mp-p+1)p-1}\right) \to 0$$

and the above determination of $\mathrm{Q}(L_{n,odd})$, we see that that the classes

$$\omega_{1^n;\,p^{n-1}\gamma^{i_1}(m),\ldots,p^0\gamma^{i_n}(m)}, \qquad i_1 \geq i_2 \geq \cdots \geq i_n \geq 0,$$

$$\omega_{1^{n-1},0;\,p^{n-2}\gamma^{i_1}(mp-p+1),\ldots,p^0\gamma^{i_{n-1}}(mp-p+1),k-1}, \qquad i_1 \geq i_2 \geq \cdots \geq i_n \geq 0,$$

form a minimal generating set for $L_{n,2m-2}$.

Acknowledgements This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.04-2019.10. The author is very grateful to the referees for their valuable comments and suggestions which helped to improve the quality of the manuscript.

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