Filomat 37:29 (2023), 9961–9971
https://doi.org/10.2298/FIL2329961T

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the automorphism-invariance of finitely generated ideals and formal matrix rings

Le Van Thuyet[a], Truong Cong Quynh[b,*]

[a]*Department of Mathematics, College of Education, Hue University, 34 Le Loi, Hue city, Viet Nam*
[b]*Department of Mathematics, The University of Danang - University of Science and Education, 459 Ton Duc Thang, Danang city, Vietnam*

Abstract. In this paper, we study rings having the property that every finitely generated right ideal is automorphism-invariant. Such rings are called right fa-rings. It is shown that a right fa-ring with finite Goldie dimension is a direct sum of a semisimple artinian ring and a basic semiperfect ring. Assume that R is a right fa-ring with finite Goldie dimension such that every minimal right ideal is a right annihilator, its right socle is essential in R_R, R is also indecomposable (as a ring), not simple, and R has no trivial idempotents. Then R is QF. In this case, QF-rings are the same as $q-$, $fq-$, $a-$, fa-rings. We also obtain that a right module (X, Y, f, g) over a formal matrix ring $\begin{pmatrix} R & M \\ N & S \end{pmatrix}$ with canonical isomorphisms $\tilde{f}$ and $\tilde{g}$ is automorphism-invariant if and only if X is an automorphism-invariant right R-module and Y is an automorphism-invariant right S-module.

1. Introduction

Johnson and Wong [7] proved that a module M is invariant under any endomorphism of its injective envelope if and only if any homomorphism from a submodule of M to M can be extended to an endomorphism of M. A module satisfying one of these equivalent conditions is called a *quasi-injective* module. Clearly any injective module is quasi-injective. A module M which is invariant under automorphisms of its injective envelope has been called an *automorphism-invariant* module. The class of these modules were investigated by many authors, e.g., [1], [5], [9, 10], [12], [15–20], [22]. The generalizations of quasi-injectivity were considered. Many results were obtained for a right *q-ring* (i.e., every right ideal is quasi-injective) (see [4], [6]), for a right *a-ring* (i.e., every right ideal is automorphism-invariant) (see [8]), for a right *fq-ring* (i.e., every finitely generated right ideal is quasi-injective), for a right *fa-ring* (i.e., every finitely generated right ideal is automorphism-invariant) (see [15]). In this paper, we continue to consider the structure of a fa-ring with some addition conditions, for example, the finite Goldie dimension of the ring R, or R is semiperfect,.... Besides, we also consider the automorphism-invariance of formal matrix rings.

2020 *Mathematics Subject Classification*. 16D50, 16D70, 16D80, 16S50.
Keywords. Automorphism-invariant module; fa-ring, Finite Goldie dimension; Formal matrix ring.
Received: 30 November 2022; Accepted: 23 June 2023
Communicated by Dijana Mosić
The authors acknowledge the support/partial support of the Core Research Program of Hue University, Grant No. NCM.DHH.2020.15. Parts of this paper were written during a stay of the authors in the Vietnam Institute For Advanced Study in Mathematics (VIASM)
* Corresponding author: Truong Cong Quynh
Email addresses: lvthuyet@hueuni.edu.vn (Le Van Thuyet), tcquynh@ued.udn.vn (Truong Cong Quynh)

Throughout this article all rings are associative rings with identity and all modules are right unital unless stated otherwise. For a submodule N of M, we use $N \leq M$ ($N < M$, resp.) to mean that N is a submodule of M (proper submodule, resp.), and we write $N \leq^e M$ and $N \leq^{\oplus} M$ to indicate that N is an essential submodule of M and N is a direct summand of M, respectively. We denote by $Soc(M)$ and $E(M)$, the socle and the injective envelope of M, respectively. The Jacobson radical of a ring R is denoted by $J(R)$ or J. A ring R is called *semiperfect* in case $R/J(R)$ is semisimple artinian and idempotents lift modulo $J(R)$. It is equivalent to every finitely generated right (left) R-module has a projective cover. A module is called *uniform* if the intersection of any two nonzero submodules is nonzero. A ring R is called *I-finite* if it contains no infinite orthogonal family of idempotents. A ring R is said to have *finite right Goldie dimension* if R does not contain an infinite direct sum of nonzero right ideals. A ring R is called right *pseudo-Frobenius* (briefly, right PF) if R is right self-injective, semiperfect and $\mathrm{Soc}(R_R) \leq^e R_R$. A ring R is *local* if R has a unique maximal left (right) ideal. We call an idempotent $e \in R$ *local* if $eRe \cong End_R(eR)$ is a local ring. For any term not defined here the reader is referred to [2], [11] and [21].

Our paper will be structured as follows: In Section 1, we will give the basic concepts and some known results that are used or cited throughout in this paper. Section 2 deals with rings whose finitely generated ideals are automorphism-invariant. We prove that a right fa-ring with finite Goldie dimension is a direct sum of a semisimple artinian ring and a basic semiperfect ring. Next, we consider the right fa-ring with finite Goldie dimension such that every minimal right ideal is a right annihilator and its right socle is essential in R_R. We obtain some properties of the kind of these rings. From these, we have that for this ring and moreover it is also indecomposable (as ring), not simple with non-trivial idempotents then it is QF. In this case, QF-rings are the same as $q-$, $fq-$, $a-$, fa-rings. Section 3 discusses about the invariance of formal matrix rings. Let $K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ and (X, Y, f, g) be a right K-module, $\tilde{f}$ and $\tilde{g}$ be isomorphisms. Then (X, Y, f, g) is an automorphism-invariant right K-module if and only if X is an automorphism-invariant right R-module and Y is an automorphism-invariant right S-module.

2. On fa-Rings with finite Goldie dimension

Recall that a ring R is a right fa-ring (resp., fq-ring) if every finitely generated right ideal of R is automorphism-invariant (resp., quasi-injective).

Remark 2.1. *Applying [8, Lemma 2.1] we deduce the following result:*
Let R be commutative ring. Then R is a fa-ring if and only if it is an automorphism-invariant ring.

Example 2.2. *It is clear that a-rings are fa-rings. And we have the example of a-rings but not self-injective. For example, consider the ring R consisting of all eventually constant sequences of elements from $\mathbb{F}_2$. Clearly, R is a commutative a-ring. But R is not self-injective. Thus, fa-rings are not fq-rings.*

Example 2.3. *The ring of linear transformations $R := End(V_D)$ of a vector space V infinite-dimensional over a division ring D. Then R is not a right a-ring, because V is not finite dimensional. But R is a right fa-ring, since every finitely generated ideal is a direct summand of R and R is right self-injective.*

Let R be a semiperfect ring. Then, there exists a set of orthogonal local idempotents $\{e_1, e_2, \ldots, e_m\}$ such that $1 = e_1 + e_2 + \cdots + e_m$. We may assume that $\{e_iR/e_iJ(R) | 1 \leq i \leq n\}$ is a complete set of representatives of the isomorphism classes of the simple right R-modules. In this case, $\{e_1, e_2, \ldots, e_n\}$ is called the set of *basic idempotents* for R, and if $e = e_1 + e_2 + \cdots + e_n$, the ring eRe is called the *basic ring* of R. Note that $eR \cong fR$ if and only if $eR/eJ(R) \cong fR/fJ(R)$ for idempotents e and f of R by Jacobson's Lemma (see [14, Lemma B.12]). The ring R is itself called a *basic semiperfect* ring if $m = n$, that is, if $1 = e_1 + e_2 + \cdots + e_n$, where $\{e_1, e_2, \ldots, e_n\}$ is a basic set of local idempotents.

Lemma 2.4. *If R is a right automorphism-invariant I-finite ring, then R is a semiperfect ring.*

The following result is the main result of this section.

Theorem 2.5. *Let R be a right fa-ring with finite Goldie dimension. Then R is a direct sum of a semisimple artinian ring and a basic semiperfect ring.*

Proof. By Lemma 2.4, R is a semiperfect ring, and so there exists a set of orthogonal local idempotents $\{e_1, e_2, \ldots, e_m\}$ such that $1 = e_1 + e_2 + \cdots + e_m$. Suppose that $e_iR \not\cong e_jR$ for all $i \neq j$ with $i, j \in \{1, 2, \ldots, m\}$. Then, we are done. Assume that e_i, for some $i \in \{1, 2, \ldots, m\}$, is a local idempotent of R such that there are direct summands isomorphic to e_iR in each decomposition of R_R as a direct sum of indecomposable modules. Thus, there exists an idempotent e' of R such that $e_iR \cap e'R = 0$ and $e_iR \cong e'R$. It follows, from [15, Lemma 4.2], that e_iR is a semisimple right R-module. On the other and, we have that e_iR is an idecomposable module and obtain that e_iR is simple. Let eR be the direct sum of all copies of e_iR in the decomposition of $R = e_1R \oplus e_2R \oplus \cdots \oplus e_mR$. Note that eR is a direct summand of R. We can assume that e is an idempotent of R. Then, we have a decomposition $R = eR \oplus (1 - e)R$. Next, we show that eR and $(1 - e)R$ are ideals of R. In order to show this, it is necessary to prove that $eR(1 - e) = 0$ and $(1 - e)Re = 0$.

Suppose $(1 - e)Re \neq 0$. Take $(1 - e)te \neq 0$ for some $t \in R$. Then, there are primitive idempotents e_j and e_k such that $e_jR \cong e_iR, e_kR \not\cong e_iR$ with $j, k \in \{1, 2, \ldots, m\}$, $e_j \in eR, e_k \in (1 - e)R$ and $e_kte_j \neq 0$. We consider the following map $\alpha : e_jR \to e_kR$ defined by $\alpha(e_jr) = e_kte_jr$ for all $r \in R$. One can check that α is a nonzero homomorphism. Note that e_jR is simple. Thus, α is a monomorphism. Since R is a right fa-ring, $e_jR \oplus e_kR$ is an automorphism-invariant module, and so e_jR is e_kR-injective by [12, Theorem 5]. From this, it immediately follows that α splits. We have that e_kR is simple and obtain $e_jR \cong e_kR$, a contradiction. We deduce that $(1 - e)Re = 0$, and so eR is an ideal of R.

Similarly to the above proof, suppose that $eR(1 - e) \neq 0$. Call $eu(1 - e) \neq 0$ for some $u \in R$. Then there are primitive idempotents e_p and e_q of R such that $e_pR \cong e_iR, e_qR \not\cong e_iR$ with $p, q \in \{1, 2, \ldots, m\}$, $e_p \in eR, e_q \in (1-e)R$ and $e_pue_q \neq 0$. We consider the following map $\beta : e_qR \to e_pR$ defined by $\beta(e_qr) = e_pue_qr$ for all $r \in R$. Then, β is a nonzero epimorphism by the simplicity of e_pR. Since e_pR is projective, β splits. One can check that $e_qR \cong e_pR$. This is a contradiction, and so $eR(1 - e) = 0$. We deduce that $(1 - e)R$ is an ideal of R.

Thus, eR is a semisimple artinian ring and $(1 - e)R$ is a basic semiperfect ring.
□

Next, we give some properties of minimal right and left ideals of R. Moreover, the self-injectivity of R is considered.

Lemma 2.6. *Let R be a right automorphism-invariant ring and $Soc(R_R) \leq^e R_R$ such that every minimal right ideal is a right annihilator.*

(1) *If xR is a minimal right ideal of R, then $l_Rr_R(x) = Rx$ and Rx is a minimal left ideal of R.*
(2) *If Ry is a minimal left ideal of R then yR is a minimal right ideal of R and $l_Rr_R(Ry) = Ry$. In particular, $Soc(R_R) = Soc(_RR)$ is denoted by S.*
(3) *$Soc(eR)$ and $Soc(Re)$ are simple for all local idempotents $e \in R$.*
(4) *If R is I-finite then R is a right PF-ring.*

Proof. (1) Assume that xR is a minimal right ideal of R. It is easy to see that $Rx \leq l_Rr_R(x)$. For the converse, let $t \in l_Rr_R(x)$ be a nonzero element. Then, we have $r_R(x) \leq r_R(t)$, and so $r_R(x) = r_R(t)$ by the maximality of $r_R(x)$. It follows that $Rx = Rt$ by [16, Lemma 1]. Then, $t \in Rx$ and so $l_Rr_R(x) \leq Rx$ or $l_Rr_R(x) = Rx$. On the other hand, for any nonzero element y in Rx, we have $r_R(x) \leq r_R(y)$, and so $r_R(x) = r_R(y)$ by the maximality of $r_R(x)$. It shows that $Rx = Ry$ is a minimal left ideal. We deduce that Rx is a minimal left ideal of R.

(2) Suppose that Ry is a minimal left ideal of R. Since $Soc(R_R) \leq^e R_R$, yR contains a minimal right ideal mR of R. Thus, $l_R(y) = l_R(m)$. It follows that $y \in r_Rl_R(y) = r_Rl_R(m) = mR \leq yR$ by our assumption, and so $yR = mR$. Thus, yR is a minimal right ideal of R. The rest is followed by (1).

(3) Take kR a minimal right ideal of eR. Then, Rk is a minimal left ideal of R. Therefore, $l_R(kR) \geq R(1 - e)$ and $l_R(kR) = l_R(k) \geq J(R)$. It follows that $l_R(kR) = J(R) + R(1 - e)$ because $J(R) + R(1 - e)$ is the unique maximal left ideal containing $R(1 - e)$. By our assumption we have

$$kR = r_Rl_R(kR) = r_R[J(R) + R(1 - e)] = r_R(J(R)) \cap eR = Soc(R_R) \cap eR = Soc(eR)$$

It shows that $Soc(eR)$ is a minimal right ideal of R.

Similarly, we also have $Soc(Re)$ is simple for all local idempotents $e \in R$.

(4) From the hypothesis, we have R is a semiperfect ring. We have a decomposition $R = e_1R \oplus e_2R \oplus \cdots \oplus e_mR$. By (2), we have that e_iR is uniform for any $i \in \{1, 2, \ldots, m\}$, and so R is right self-injective by [12, Corollary 15]. We deduce that R is a right PF-ring. □

Fact 2.7. *All endomorphism rings of indecomposable automorphism-invariant modules are local rings.*

Lemma 2.8. *Let R be a right fa-ring with finite Goldie dimension, e be a primitive idempotent of R. Then the following conditions hold:*

(1) *If $\alpha : eR \to R$ is a nonzero homomorphism with $eR \cap \alpha(eR) = 0$ then $\alpha(eR)$ is a simple module.*
(2) *If $(1 - e)Re \neq 0$ then $eR(1 - e) \neq 0$.*

Proof. (1) Note that eR is local. Then, $\alpha(eR)$ is indecomposable. Let U be an arbitrary essential submodule of $\alpha(eR)$, then $E(U) = E(\alpha(eR))$. Since R has finite Goldie dimension, there exists a finitely generated right ideal I with $I \leq^e U$. It follows that $I \leq^e U \leq^e \alpha(eR)$, and so $E(I) = E(U) = E(\alpha(eR))$. Since $I \oplus eR$ is a finitely generated right ideal of R, $I \oplus eR$ is automorphism-invariant. It follows that I is $eR-$injective. On the other hand, there exists a homomorphism $\bar{\alpha} : E(eR) \to E(\alpha(eR))$ such that $\bar{\alpha}|_{eR} = \alpha$. We have that $E(I) = E(\alpha(eR))$ and I is $eR-$injective and obtain that $\bar{\alpha}(eR) \leq I \leq U$. It shows that $\alpha(eR) \leq U$. We deduce that $\alpha(eR) = Soc(\alpha(eR))$, and so $\alpha(eR)$ is semisimple. We deduce that $\alpha(eR)$ is simple.

(2) Assume that $(1 - e)Re \neq 0$. Note that R is automorphism-invariant, eR is $(1 - e)R$-injective and $(1 - e)R$ is eR-injective. Call $\alpha : eR \to (1 - e)R$ a nonzero homomorphism. Now, we assume that $eR(1 - e) = 0$. Then, $eRe = eR$ is a local ring with its unique maximal ideal $eJ(R)$. If $eJ(R) = 0$ then eR is simple right R-module and so $\alpha(eR) \cong eR$. It follows that $\alpha^{-1} : \alpha(eR) \to eR$ is extended to a homomorphism from $(1 - e)R$ to eR. It means that $eR(1 - e) \neq 0$. Now, if $eJ(R)$ is nonzero, then we get a nonzero element x in $eJ(R)$. We have that eRe is local and obtain that there exists an eRe-epimorphism $\beta : xeR \to eR/eJ(R)$. On the other hand, we have $eRe = eR$ and so β is an R-homomorphism. From (1) it immediately infers that $eR/eJ(R) \cong \alpha(eR) \leq (1 - e)R$. Then, there exists a nonzero homomorphism $\gamma : eR/eJ(R) \to (1 - e)R$. It follows that composition of β and γ is a nonzero homomorphism $\gamma \circ \beta : xeR \to (1 - e)R$. Again, $(1 - e)R$ is eR-injective we have that there is a nonzero homomorphism $\theta : eR \to (1 - e)R$ such that θ is an extension of $\gamma \circ \beta$. Moreover, we have $xeR \leq eJ(R) = \mathrm{Ker}(\theta)$ (by (1)) which implies that $(\gamma \circ \beta)(xeR) = \theta(xeR) = 0$, a contradiction. Thus, $eR(1 - e) \neq 0$. □

Proposition 2.9. *An indecomposable right fa-ring with finite Goldie dimension such that every minimal right ideal is a right annihilator. Then the following conditions are equivalent:*

(1) *R has essential right socle.*
(2) $\mathrm{Soc}(R_R) = \mathrm{Soc}(_RR)$.

Proof. (1) ⇒ (2) by Lemma 2.6.

(2) ⇒ (1). Assume that $\mathrm{Soc}(R_R) = \mathrm{Soc}(_RR)$. Since R is semiperfect, $R = e_1R \oplus e_2R \oplus \cdots \oplus e_mR$ with a set of orthogonal local idempotents $\{e_1, e_2, \ldots, e_m\}$ of R. Since R is an indecomposable ring, $e_iR(1 - e_i) \neq 0$ or $(1 - e_i)Re_i \neq 0$ for all $i \in \{1, 2, \ldots, m\}$. Suppose that $(1 - e_i)Re_i \neq 0$. Then by Lemma 2.8 we have $e_iR(1 - e_i) \neq 0$. We deduce that $e_iR(1 - e_i) \neq 0$ for all $i \in \{1, 2, \ldots, m\}$. Take $\alpha_i : (1 - e_i)R \to e_iR$ a nonzero homomorphism. Then by Lemma 4.2 in [15], $\mathrm{Im}(\alpha_i)$ is semisimple. It follows that $\mathrm{Soc}(e_iR) \neq 0$ for all $i \in \{1, 2, \ldots, m\}$.

For any $i \in \{1, 2, \ldots, m\}$, take kR a minimal right ideal of e_iR. Then, Rk is a minimal left ideal of R. Therefore, $l_R(kR) \geq R(1 - e_i)$ and $l_R(kR) = l_R(k) \geq J(R)$. It follows that $l_R(kR) = J(R) + R(1 - e_i)$ because $J(R) + R(1 - e_i)$ is the unique maximal left ideal containing $R(1 - e_i)$. By our assumption we have

$$kR = r_R l_R(kR) = r_R[J(R) + R(1 - e_i)] = r_R(J(R)) \cap e_iR = Soc(R_R) \cap e_iR = Soc(e_iR)$$

It shows that $Soc(e_iR)$ is a minimal right ideal of R for all $i \in \{1, 2, \ldots, m\}$. It follows that $Soc(e_iR)$ is essential in e_iR. Thus, $Soc(R)$ is essential in R_R. □

In this section, we assume that R **is a right** fa**-ring with finite Goldie dimension such that every minimal right ideal is a right annihilator and** $\mathrm{Soc}(R_R)$ **is essential in** R_R. Moreover, R is semiperfect, and so there exists a set of orthogonal local idempotents $\{e_1, e_2, \ldots, e_m\}$ of R such that $1 = e_1 + e_2 + \cdots + e_m$. Call $\{e_1, e_2, \ldots, e_n\}$ a set of basic idempotents for R with $n \leq m$.

Lemma 2.10. *If e and f are two orthogonal idempotents of R then* $eRf \subseteq Soc(R_R)$.

Proof. Suppose that e and f are two orthogonal idempotents of R. Then, $eR \cap fR = 0$. If $eRf = 0$, we are done. Otherwise, let exf be a nonzero arbitrary element of eRf. We consider a nonzero homomorphism $\alpha : fR \to eR$ defined by $\alpha(fr) = exfr$ for all $r \in R$. By [15, Lemma 4.2], we have that $\mathrm{Im}(\alpha) = exfR$ is semisimple. It follows that $exf \in Soc(R_R)$. We deduce that $eRf \subseteq Soc(R_R)$. □

Let R be a semiperfect ring with basic idempotents $\{e_1, e_2, \ldots, e_n\}$. A permutation σ of $\{1, 2, \ldots, n\}$ is called a *Nakayama permutation* for R if $Soc(Re_{\sigma(i)}) \cong Re_i/J(R)e_i$ and $Soc(e_iR) \cong e_{\sigma(i)}R/e_{\sigma(i)}J(R)$ for each $i = \{1, 2, \ldots, n\}$. A ring R is called *quasi-Frobenius* (brief, QF) if R is one-sided artinian one-sided self-injective, see [14]. It is well-known that every QF-ring has a Nakayama permutation.

Lemma 2.11. *Let R be an indecomposable ring with non-trivial idempotents. Then, R has a Nakayama permutation* σ *of* $\{1, 2, \ldots, n\}$. *In particular,* $\sigma(i) \neq i$ *for all* $i = 1, 2, \ldots, n$ *if R is not a simple ring.*

Proof. By the hypothesis, R is indecomposable and so R is either semisimple artinian or basic semiperfect by Theorem 2.5. If R is a semisimple artinian ring then R has a Nakayama permutation. Now, we assume that R is not a simple ring. It follows that R is a basic semiperfect ring.

For any $i \in \{1, 2, \ldots, n\}$, from the simplicity of $Soc(e_iR)$, it infers that there exists $\sigma(i) \in \{1, 2, \ldots, n\}$ such that $Soc(e_iR) \cong e_{\sigma(i)}R/e_{\sigma(i)}J(R)$. This map σ is a permutation of $\{1, 2, \ldots, n\}$ because $\sigma(i) = \sigma(j)$ implies that $Soc(e_iR) \cong Soc(e_jR)$. By the injectivity of e_iR and e_jR, we infer that $e_iR \cong e_jR$, and so $i = j$ (because the e_i are basic). Let $\alpha : e_{\sigma(i)}R/e_{\sigma(i)}J(R) \to Soc(e_iR)$ be an isomorphism and $s_i = \alpha(e_{\sigma(i)} + e_{\sigma(i)}J(R))$. It follows that $s_iR = Soc(e_iR)$ is a minimal right ideal of R. One can check that $J(R) + R(1 - e_i) \leq l_R(s_i)$. But $R/[J(R) + R(1 - e_i)] \cong Re_i/J(R)e_i$ is simple, and so $l_R(s_i) = J(R) + R(1 - e_i)$. It follows that $Rs_i \cong Re_i/J(R)e_i$. Now observe that $s_i = s_ie_{\sigma(i)} \in Soc(_RR)e_{\sigma(i)} = Soc(Re_{\sigma(i)})$. We have, from Lemma 2.6, that $Soc(Re_{\sigma(i)})$ is simple and obtain that $Soc(Re_{\sigma(i)}) \cong Re_i/J(R)e_i$. Thus, R has a Nakayama permutation σ of $\{1, 2, \ldots, n\}$.

Next, we suppose that $\sigma(i) = i$ for some $i \in \{1, 2, \ldots, n\}$ or $Soc(e_iR) \cong e_iR/e_iJ(R)$. Assume that $e_iR(1-e_i) \neq 0$. Since R is a basic semiperfect ring, there would exist $j \in \{1, 2, \ldots, n\}$ with $j \neq i$ such that $e_iRe_j \neq 0$. Then, there exists a nonzero homomorphism $\beta : e_jR \to e_iR$. By [8, Lemma 4.1] and e_iR is uniform, we infer that $\mathrm{Im}(\beta)$ is simple. It follows that $\mathrm{Im}(\beta) = Soc(e_iR)$ and $\mathrm{Ker}(\beta)$ is maximal in e_jR. Then, $\mathrm{Ker}(\beta) = e_jJ(R)$ which implies that $e_jR/e_jJ(R) \cong Soc(e_iR) \cong e_iR/e_iJ(R)$. From this, it immediately infers that $e_iR \cong e_jR$, a contradiction. It is shown that $e_iR(1 - e_i) = 0$. Similarly, we have $(1 - e_i)Re_i = 0$. In fact, if $(1 - e_i)Re_i \neq 0$, then $e_kRe_i \neq 0$ for some $k \in \{1, 2, \ldots, n\}$ with $k \neq i$. By the above similar proof, we infer that $Soc(e_iR) \cong e_iR/e_iJ(R) \cong Soc(e_kR)$. By the injectivity of e_iR and e_kR, we have $e_iR \cong e_kR$ which is impossible. It is shown that e_i is central, a contradiction. We deduce that $\sigma(i) \neq i$ for all $i = 1, 2, \ldots, n$.

□

Lemma 2.12. *Let R be an indecomposable ring not simple with non-trivial idempotents. Then,* e_iRe_i *is a division ring for any* $i \in \{1, 2, \ldots, n\}$.

Proof. By the hypothesis, R is a basic semiperfect ring and $1 = e_1 + e_2 + \cdots + e_n$. For any $i \in \{1, 2, \ldots, n\}$, there exists $j \neq i$ with $j \in \{1, 2, \ldots, n\}$ such that $e_iRe_j \neq 0$ by Lemma 2.11. Suppose that $e_iR(1 - e_i) = 0$. Then, $e_iR(\sum_{k \neq i}^{n} e_k) = 0$ which implies that $e_iRe_j = 0$, a contradiction. Thus, $e_iR(1 - e_i) \neq 0$. Next, we show that $e_iJ(R)e_i = 0$. We have $e_iR(1 - e_i) \subset \mathrm{Soc}(eR)$ by Lemma 2.10, and so $e_iR(1 - e_i) = \mathrm{Soc}(e_iR)(1 - e_i)$. Now, we show that $e_iJ(R)e_i$ is a submodule of e_iR. Since R is right automorphism-invariant, $J(R) = \{a \in R : r_R(a) \leq^e R_R\}$

by [5, Proposition 1] and so $J(R)\,\mathrm{Soc}(e_iR) = 0$. Now $(e_iJ(R)e_i)\,\mathrm{Soc}(e_iR) = e_iJ(R)\,\mathrm{Soc}(e_iR) = 0$ which implies $(e_iJ(R)e_i)(e_iR(1-e_i)) = 0$. On the other hand, we have

$$e_iJ(R)e_iR = e_iJ(R)e_i(Re_i + R(1-e_i)) = e_iJ(R)e_iRe_i \subset e_iJ(R)e_i.$$

Hence $e_iJ(R)e_i$ is an R-submodule of e_iR. Since $\mathrm{Soc}(e_iR)$ is simple, we have $e_iJ(R)e_i \cap \mathrm{Soc}(e_iR) = 0$ or $\mathrm{Soc}(e_iR) \leq e_iJ(R)e_i$. Suppose $\mathrm{Soc}(e_iR) \leq e_iJ(R)e_i$. Then $e_iR(1-e_i) = \mathrm{Soc}(e_iR)(1-e_i) \leq e_iJ(R)e_i(1-e_i) = 0$, a contradiction. It follows that $e_iJ(R)e_i \cap \mathrm{Soc}(e_iR) = 0$. Thus $e_iJ(R)e_i = 0$ because $\mathrm{Soc}(e_iR)$ is essential in e_iR. Note that $e_iRe_i \cong \mathrm{End}(e_iR)$ is a local ring. We deduce that e_iRe_i is a division ring.
□

Theorem 2.13. *If R is an indecomposable (as ring) ring not simple with non-trivial idempotents, then R is a QF-ring.*

Proof. By Lemma 2.6 and the hypothesis, R is a basic semiperfect right self-injective ring and $Soc(R_R)$ is an artinian right R-module. We have a decomposition $R = e_1R \oplus e_2R \oplus \cdots \oplus e_nR$. Then

$$R = \sum_{i=1}^{n} e_iRe_i + \sum_{i \neq j}^{n} e_iRe_j$$

Note that $e_iRe_j \subseteq Soc(R_R)$ for all $i \neq j$ by Lemma 2.10. We consider the following mapping

$$\phi : R/Soc(R_R) \to \bigoplus_{i=1}^{n} e_iRe_i$$

via $\phi(\sum_{i=1}^{n} e_ir_ie_i) + Soc(R_R) = \sum_{i=1}^{n} e_ir_ie_i$ We show that ϕ is an isomorphism. If $\sum_{i=1}^{n} e_ir_ie_i \in S$, then $e_ir_ie_i \in e_iSe_i$ for all $i = 1, 2, \ldots, n$. Since $e_iJ(R)$ is the unique maximal submodule of e_iR, $e_iSoc(R_R) \leq e_iJ(R)$, and so $e_ir_ie_i \in e_iJ(R)e_i$. Note that $e_iJ(R)e_i = 0$ by Lemma 2.12. It shows that ϕ is a mapping. One can check that ϕ is a ring homomorphism. Moreover, ϕ is a bijection, and so ϕ is a ring isomorphism. It shows that $R/Soc(R_R)$ is a semisimple artinian ring. We deduce that R is a right artinian ring, and so R is QF. □

Corollary 2.14. *Let R be an indecomposable (as ring) ring not simple with non-trivial idempotents. Then, the following conditions are equivalent:*

(1) *R is a right q-ring.*
(2) *R is a right fq-ring.*
(3) *R is a right a-ring.*
(4) *R is a right fa-ring.*
(5) *$eRf \subseteq Soc(R_R)$ for each pair e, f of orthogonal idempotents of R.*
(6) *R is an QF-ring.*

Proof. (1) ⇒ (2), (3); (2) ⇒ (4) and (3) ⇒ (4) are obvious.

(4) ⇒ (5) by Lemma 2.10.

(5) ⇒ (6). By Theorem 2.13, R is a basic semiperfect QF-ring.

(6) ⇒ (1). Since R is QF, it follows that R_R is injective cogenerator. Thus, R is a right q-ring by [4, Theorem 2.9]. □

3. The automorphism-invariance of formal matrix rings

Let R and S be two rings and M be an $R-S$-bimodule and N be a $S-R$-bimodule. Take the set of matrices

$$K = \begin{pmatrix} R & M \\ N & S \end{pmatrix} = \left\{ \begin{pmatrix} r & m \\ n & s \end{pmatrix} \middle| \; r \in R, s \in S, m \in M, n \in N \right\}$$

Assume that there exist an R-homomorphism $\varphi : M \otimes_S N \to R$ and an S-homomorphism $\psi : N \otimes_R M \to S$ such that

$$\varphi(m \otimes n)m' = m\psi(n \otimes m'),\ \psi(n \otimes m)n' = n\varphi(m \otimes n')$$

for all $m, m' \in M$ and $n, n' \in N$. For convenience in using notations, we can write $\varphi(m \otimes n) := mn$, $\psi(n \otimes m) := nm$ and $MN := \varphi(M \otimes_S N)$, $NM := \psi(N \otimes_R M)$.

Then, K is a ring with the addition and multiplication as follows:

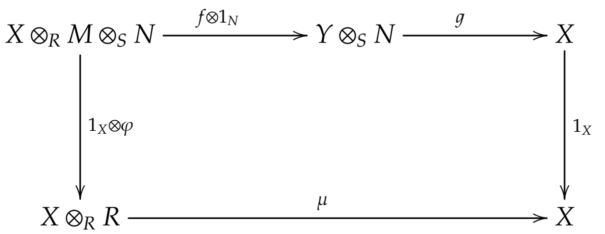
$$\begin{pmatrix} r & m \\ n & s \end{pmatrix} + \begin{pmatrix} r' & m' \\ n' & s' \end{pmatrix} = \begin{pmatrix} r + r' & m + m' \\ n + n' & s + s' \end{pmatrix}$$

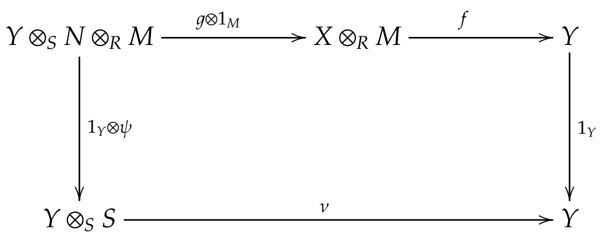
$$\begin{pmatrix} r & m \\ n & s \end{pmatrix} \begin{pmatrix} r' & m' \\ n' & s' \end{pmatrix} = \begin{pmatrix} rr' + mn' & rm' + ms' \\ nr' + sn' & nm' + ss' \end{pmatrix}$$

The ring K is called a *formal matrix ring or generalized matrix rings* (see [11] or [13]). It is well-known that the category of right K-module Mod-K is equivalent to the category $\mathcal{A}(K)$ of objects (X, Y, f, g), where X is a right R-module, Y is a right S-module, $f : X \otimes_R M \to Y$ is an S-homomorphism and $g : Y \otimes_S N \to X$ is an R-homomorphism. The right K-module (X, Y, f, g) is the additive group $X \oplus Y$ with right K-action given by

$$(x\ y) \begin{pmatrix} r & m \\ n & s \end{pmatrix} = (xr + g(y \otimes n), f(x \otimes m) + ys)$$

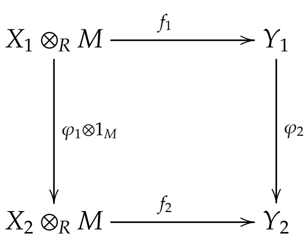
such that the following diagrams are commutative





where $\mu : X \otimes_R R \to X$ and $\nu : Y \otimes_S S \to Y$ are canonical isomorphisms.

Next, we consider homomorphisms of K-modules. Let (X_1, Y_1, f_1, g_1) and (X_2, Y_2, f_2, g_2) be right K-modules. A right K-homomorphism $\varphi : (X_1, Y_1, f_1, g_1) \to (X_2, Y_2, f_2, g_2)$ is a pair (φ_1, φ_2) where $\varphi_1 : X_1 \to X_2$ is an R-homomorphism and $\varphi_2 : Y_1 \to Y_2$ is an S-homomorphism such that the following diagrams are commutative



$$\begin{array}{ccc} Y_1 \otimes_S N & \xrightarrow{g_1} & X_1 \\ \downarrow{\scriptstyle \varphi_2 \otimes 1_N} & & \downarrow{\scriptstyle \varphi_1} \\ Y_2 \otimes_S N & \xrightarrow{g_2} & X_2 \end{array}$$

Note that a K-homomorphism $\varphi = (\varphi_1, \varphi_2) : (X_1, Y_1, f_1, g_1) \to (X_2, Y_2, f_2, g_2)$ is a monomorphism (epimorphism, resp.) if and only if φ_1 and φ_2 are monomorphisms (epimorphisms, resp.).

A submodule of a right K-module (X, Y, f, g) is a quadrupe (X_0, Y_0, f_0, g_0), where $X_0 \le X_R$, $Y_0 \le Y_S$ such that the following diagrams are commutative.

$$\begin{array}{ccc} X_0 \otimes_R M & \xrightarrow{f_0} & Y_0 \\ \downarrow{\scriptstyle \iota_1 \otimes 1_M} & & \downarrow{\scriptstyle \iota_2} \\ X \otimes_R M & \xrightarrow{f} & Y \end{array}$$

$$\begin{array}{ccc} Y_0 \otimes_S N & \xrightarrow{g_0} & X_0 \\ \downarrow{\scriptstyle \iota_2 \otimes 1_N} & & \downarrow{\scriptstyle \iota_1} \\ Y \otimes_S N & \xrightarrow{g} & X \end{array}$$

with $\iota_1 : X_0 \to X$, $\iota_2 : Y_0 \to Y$ the inclusion maps. This is equivalent $X_0 M \subseteq Y_0$ and $Y_0 N \subseteq X_0$.

Let $K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ and X be a right R-module. Denote by $H(X) = \mathrm{Hom}_R(N, X)$. We consider the following homomorphisms

$$\begin{aligned} u_X : X \otimes_R M &\longrightarrow \mathrm{Hom}_R(N, X) \\ x \otimes m &\longmapsto u(x \otimes m) : N \to X \\ &\qquad\qquad n \mapsto u(x \otimes m)(n) = x(mn) \end{aligned}$$

and

$$\begin{aligned} v_X : \mathrm{Hom}_R(N, X) \otimes_S N &\longrightarrow X \\ \alpha \otimes n &\longmapsto \alpha(n) \end{aligned}$$

One can check that $(X, H(X), u_X, v_X)$ is a right K-module. Similarly, we also have that $(H(Y), Y, v_Y, u_Y)$ is a right K-module for all right S-module Y with $H(Y) = \mathrm{Hom}_S(M, Y)$ and $v_Y : H(Y) \otimes_R M \to Y$ and $u_Y : Y \otimes_S N \to H(Y)$.

Let (X, Y, f, g) be a right K-module. Then, we have the following R-homomorphism

$$\begin{aligned} \tilde{f} : X &\longrightarrow \mathrm{Hom}_S(M, Y) = H(Y) \\ x &\longmapsto \tilde{f}(x) : M \to Y \\ &\qquad\qquad m \mapsto \tilde{f}(x)(m) = f(x \otimes m) \end{aligned}$$

and S-homomorphism

$$\begin{aligned} \tilde{g} : Y &\longrightarrow \mathrm{Hom}_S(N, X) = H(X) \\ y &\longmapsto \tilde{g}(y) : N \to X \\ & \qquad\qquad n \mapsto \tilde{g}(y)(n) = g(y \otimes n) \end{aligned}$$

Theorem 3.1. *Let* $K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ *and* (X, Y, f, g) *be a right K-module. Assume that* $\tilde{f}$ *and* $\tilde{g}$ *are isomorphisms. Then the following conditions are equivalent:*

(1) (X, Y, f, g) *is an automorphism-invariant right K-module.*
(2) (a) X *is an automorphism-invariant right R-module.*
  (b) Y *is an automorphism-invariant right S-module.*

Proof. (2) $\Rightarrow$ (1). By Lemma 2.3 in [13], there exist isomorphisms $\tilde{\mu} : E(X) \to Hom_S(M, E(Y))$ and $\tilde{\eta} : E(Y) \to Hom_R(N, E(X)$ such that $(E(X), E(Y), \mu, \eta)$ is the injective envelope of (X, Y, f, g). Let $\varphi = (\varphi_1, \varphi_2)$ be an automorphism of $(E(X), E(Y), \mu, \eta)$ then φ_1 is an R-automorphism of $E(X)$ and φ_2 is an S-automorphism of $E(Y)$. Since X is an automorphism-invariant right R-module and Y is an automorphism-invariant right S-module, it follows that (X, Y, f, g) is an automorphism-invariant right K-module.

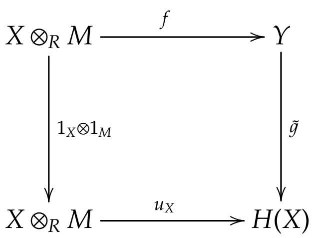
(1) $\Rightarrow$ (2) Assume that (X, Y, f, g) is an automorphism-invariant right K-module. We show that X is an automorphism-invariant right R-module. To prove this, firstly we show that $(X, Y, f, g) \cong (X, H(X), u_X, v_X)$. In fact we consider the mapping $(1_X, \tilde{g}) : (X, Y, f, g) \to (X, H(X), u_X, v_X)$. Since (X, Y, f, g) is a right K-module, $g \circ (f \otimes 1_N) = \mu \circ (1_X \otimes \varphi)$, where $\mu : X \otimes_R R \to X$ is the canonical isomorphism and $\varphi : M \otimes_S N \to R$ is the multipilication in K. Then, for all $x \in X$, $m \in M$ and $n \in M$, we have

$$(\tilde{g} \circ f)(x \otimes m)(n) = g(f(x \otimes m) \otimes n) = \mu(1_X \otimes \varphi)(x \otimes m \otimes n) = x(mn)$$

and

$$u_X(1_X \otimes 1_M)(x \otimes m)(n) = u_X(x \otimes m)(n) = x(mn)$$

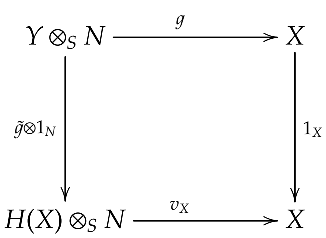
It shows that $\tilde{g} \circ f = u_X \circ (1_X \otimes 1_M)$ and so the following diagram is commutative.



On the other hand, for all $y \in Y$ and $n \in N$, we have

$$v_X(\tilde{g} \otimes 1_N)(y \otimes n) = v_X(\tilde{g}(y) \otimes n) = \tilde{g}(y)(n) = g(y \otimes n) = 1_X g(y \otimes n)$$

and so $1_X \circ g = v_X \circ (\tilde{g} \otimes 1_N)$. It means that the following diagram is commutative.



Thus, $(1_X, \tilde{g}) : (X, Y, f, g) \to (X, H(X), u_X, v_X)$ is a K-homomorphism. By our assumption, $\tilde{g}$ is an isomorphism, $(1_X, \tilde{g})$ is an isomorphism. Then,
$(X, H(X), u_X, v_X)$ is an automorphism-invariant right K-module.

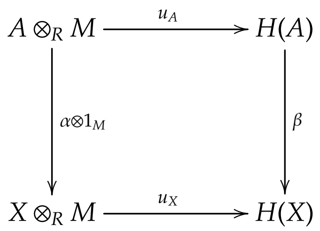
Now, we show that X is an automorphism-invariant right R-module. Let $\alpha : A \to X$ be an R-monomorphism. Then, we have that $(A, H(A), u_A, v_A)$ is a submodule of $(X, H(X), u_X, v_X)$. We consider the mapping $\beta : H(A) \to H(X)$ via by the relation $\beta(h)(n) = \alpha(v_A(h \otimes n))$. One can check that β is an S-homomorphism. For all $a \in A$, $m \in M$ and $n \in M$, we have

$$(\beta \circ u_A)(a \otimes m)(n) = \alpha(v_A(u_A(a \otimes m) \otimes n)) = \alpha(\mu(1_A \otimes \varphi)(a \otimes m \otimes n)) = \alpha(a)mn$$

and

$$u_X(\alpha \otimes 1_M)(a \otimes m)(n) = u_X(\alpha(a) \otimes m)(n) = \alpha(a)mn$$

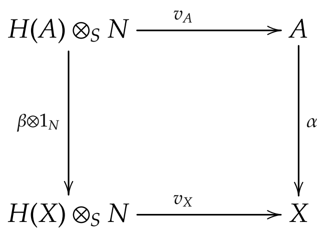
It shows that $\beta \circ u_A = u_X \circ (\alpha \otimes 1_M)$ and so the following diagram is commutative.



On the other hand, for all $h \in H(A)$ and $n \in N$, we have

$$v_X(\beta \otimes 1_N)(h \otimes n) = v_X(\beta(h) \otimes n) = \beta(h)(n) = \alpha v_A(h \otimes n)$$

and so $\alpha \circ v_A = v_X \circ (\beta \otimes 1_N)$. It means that the following diagram is commutative.



Thus, $(\alpha, \beta) : (A, H(A), u_A, v_A) \to (X, H(X), u_X, v_X)$ is a K-monomorphism. Since $(X, H(X), u_X, v_X)$ is an automorphism-invariant right K-module, there exists an endomorphism (γ, θ) of $(X, H(X), u_X, v_X)$ such that (γ, θ) is an extension of (α, β). Thus, $\gamma : X \to X$ is an extension of α. We deduce that X is an automorphism-invariant right R-module.

Similarly, we also prove that Y is an automorphism-invariant right S-module. □

By [11, Lemma 3.8.1] and Theorem 3.1, we have the following result:

Corollary 3.2. *Let* $K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ *and* (X, Y, f, g) *be a right* K*-module. Assume that* $MN = R$ *and* $NM = S$*. Then the following conditions are equivalent:*

(1) (X, Y, f, g) *is an automorphism-invariant right* K*-module.*
(2) (a) X *is an automorphism-invariant right* R*-module.*
    (b) Y *is an automorphism-invariant right* S*-module.*

Corollary 3.3. *Let e be a non-zero idempotent of a ring R, $K = \begin{pmatrix} R & Re \\ eR & eRe \end{pmatrix}$ and (X, Y, f, g) be a right K-module. Assume that $\tilde{f}$ and $\tilde{g}$ are isomorphisms. Then (X, Y, f, g) is an automorphism-invariant right K-module if and only if X is an automorphism-invariant right R-module and Y is an automorphism-invariant right eRe-module.*

If e is an idempotent of a ring R such that $ReR = R$ then $R \approx eRe$. So in this case, we have:

Corollary 3.4. *Let e be an idempotent of a ring R such that $ReR = R$ and $K = \begin{pmatrix} R & Re \\ eR & eRe \end{pmatrix}$. Assume that R is a right fa-ring and $\tilde{f}$, $\tilde{g}$ are isomorphisms. Then (eR, Re, f, g) is an automorphism-invariant right K-module.*

Acknowledgment

The authors are grateful to the referee for her/his valuable comments.

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