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Duality for asymptotic invariants of graded families

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ABSTRACT

The starting point of this paper is a duality for sequences of natural numbers which, under mild hypotheses, interchanges subadditive and superadditive sequences and inverts their asymptotic growth constants.

We are motivated to explore this sequence duality since it arises naturally in at least two important algebraic-geometric contexts. The first context is Macaulay-Matlis duality, where the sequence of initial degrees of the family of symbolic powers of a radical ideal is dual to the sequence of Castelnuovo-Mumford regularity values of a quotient by ideals generated by powers of linear forms. This philosophy is drawn from an influential paper of Emsalem and Iarrobino. We generalize this duality to differentially closed graded filtrations of ideals.

In a different direction, we establish a duality between the sequence of Castelnuovo-Mumford regularity values of the symbolic powers of certain ideals and a geometrically inspired sequence we term the jet separation sequence. We show that this duality underpins the reciprocity between two important

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geometric invariants: the multipoint Seshadri constant and the asymptotic regularity of a set of points in projective space.

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1. Introduction

As Michael Atiyah [1] points out, "Duality in mathematics is not a theorem, but a principle". Indeed, forms of duality occur in all branches of mathematics manifesting in ways specific to the subject area. In this paper we study manifestations of duality which take effect primarily in an algebraic-geometric context. More precisely, our starting point is a notion of duality for sequences of natural numbers. This prompts the question of determining the dual sequences for certain numerical sequences which occur in commutative algebra, for example, the sequence of initial degrees of a graded family of homogeneous ideals, or the sequence of Castelnuovo-Mumford regularity values of a family of ideals. Our techniques allow to relate the asymptotic growth factors of these sequences to those of the dual sequences. We explore this theme in contexts where these asymptotic growth factors carry significant meaning.

At the level of numerical sequences we single out two transformations which act on nondecreasing sequences of integers. Given a sequence $\underline{\alpha} = \{\alpha_n\}_{n\in\mathbb{N}}$, the transformed sequences are as follows:

$$\overleftarrow{\alpha}_n = \inf\{d \mid \alpha_d \geq n\},$$
$$\overrightarrow{\alpha}_n = \sup\{d \mid \alpha_d \leq n\}.$$

It turns out that these transformations are mutual inverses [36]. If furthermore $\underline{\alpha}$ is either a subadditive or superadditive sequence (see Definition 2.3) then it has a well-defined asymptotic growth factor $\widehat{\alpha} = \lim_{n\to\infty} \frac{\alpha_n}{n}$. The above transformations interchange the classes of subadditive and superadditive sequences. Moreover, under these hypotheses, we are able to derive the following reciprocation formulas for the respective asymptotic growth factors in Theorem 2.6:

$$\widehat{\overrightarrow{\alpha}} = \widehat{\alpha}^{-1} \quad \text{and} \quad \widehat{\overleftarrow{\alpha}} = \widehat{\alpha}^{-1}.$$

In fact, we generalize the above transformations as well as the reciprocation formulas in Theorem 2.5. The more technical statement of this result is relegated to section 2.

We apply the duality principle described above to several numerical sequences. Our interest in such sequences is spurred by the study of the family of symbolic powers $\{I^{(d)}\}_{d\in\mathbb{N}}$ of a homogeneous ideal I. In the case when I is the defining ideal of an algebraic variety X, this family features prominently in algebraic geometry by encoding the set of functions vanishing to higher order on X. In commutative algebra, the symbolic powers have been studied most recently by means of comparison with the family of ordinary powers $\{I^n\}_{n\in\mathbb{N}}$; see [16,31,39,4].

A sequence of interest in this area of study is given by the initial degrees for the symbolic power ideals. Its asymptotic growth factor, dubbed the *Waldschmidt constant*, is $\widehat{\alpha}(I) = \lim_{d\to\infty} \frac{\alpha(I^{(d)})}{d}$. It is well-known that this sequence is subadditive. The same is true for any sequence that results from applying a discrete valuation to a graded family of ideals (see Lemma 3.3). Taking this more general perspective leads to considering valuative sequences for any discrete valuation ν

$$\beta_n^\nu = \beta_n^\nu(I) = \sup\{d : \nu(I^{(d)}) < \nu(I^n)\}.$$

In Proposition 3.8 we apply our duality results to relate the growth factor $\widehat{\beta^\nu}$ of this sequence to those of the sequences $\{\nu(I^{(d)})\}_{d\in\mathbb{N}}$ and $\{\nu(I^n)\}_{n\in\mathbb{N}}$. This has consequences on the *containment problem* between the ordinary and symbolic powers of I. Building on [11,10], we show that there exists a valuation ν for which the asymptotic growth factor of β^ν recovers the *asymptotic resurgence* of [23].

In section 4 we study the dual of a sequence closely related to the initial degree sequence of the family of symbolic powers. In this pursuit, we are led to consider a notion of inverse systems which dates back to Macaulay [38]. Emsalem and Iarrobino determined in an influential paper [17] the inverse system for the symbolic powers of a radical ideal. We generalize their results by introducing a new notion of differentially closed graded filtrations of ideals for which the inverse systems behave particularly well. Examples of differentially closed graded filtrations include many families of powers of a homogeneous ideal (differential, ordinary, symbolic, and integral Frobenius powers) and any family obtained by intersecting these.

Theorem *(Theorem 4.9 and Theorem 4.20). Suppose $\mathcal{I} = \{I_n\}_{n\in\mathbb{N}}$ is a differentially closed graded filtration of proper ideals in $R = \mathbb{K}[x_0, \dots, x_N]$. Let $\mathcal{D} = \bigoplus_{i\geq 0} \operatorname{Hom}(R_i, \mathbb{K})$, equipped with the structure of a divided power algebra. For each $s \in \mathbb{N}$ put*

$$\mathcal{L}^s(\mathcal{I}) := \bigoplus_{d \geq s+1} \left(I_{d-s}^{\perp}\right)_d \subseteq \mathcal{D}.$$

Then $\mathcal{L}^s(\mathcal{I})$ is an ideal of $\mathcal{D}$ for each $s \in \mathbb{N}$. The sequence $\alpha_n = \alpha(I_n)$ is subadditive and $\beta_s = \sup\{d : (\mathcal{D}/\mathcal{L}^s(\mathcal{I}))_d \neq 0\}$ is superadditive. Assuming that the sequence $\{\alpha_n - n\}_{n \in \mathbb{N}}$ is not bounded above, we have

$$\widehat{\beta} = \frac{\widehat{\alpha}}{\widehat{\alpha} - 1} \text{ and } \widehat{\alpha} = \frac{\widehat{\beta}}{\widehat{\beta} - 1}.$$

Several forms of algebraic duality manifest themselves in the setup above. The degree-wise vector space duality between R and $\mathcal{D}$ manifests itself via apolarity (orthogonality). The inverse systems considered in section 4 are a form of Matlis duality. Finally the projective duality between points in $p = (p_0 : \cdots : p_N) \in \mathbb{P}^N$ and linear forms $L_p = p_0 x_0 + \cdots + p_n x_n \in R$ yields a celebrated description of $\mathcal{L}^s(\mathcal{I})$ when $\mathcal{I}$ is the family of symbolic powers for the defining ideal of a projective variety; see Example 4.15.

In contrast to the above setting where the algebraic duality is more evident while the numerical duality of asymptotic invariants is more elusive, we study a different setup where duality of asymptotic invariants has been observed before (see [37, §5.1]), but the underpinning reasons have not previously been discovered.

Theorem *(Theorem 5.5 and Corollary 5.8). Let I be the defining ideal of a set X of $r \geq 2$ points in $\mathbb{P}^N$. Set $s_d = s(X, d-1)$ to be the jet separation sequence of X (Definition 5.3) and $r_k = \operatorname{reg}(I(X)^{(k+1)})$ the sequence of Castelnuovo-Mumford regularities for the symbolic powers of $I(X)$. There is a duality between these sequences*

$$s = \overrightarrow{r} \text{ and } r = \overleftarrow{s}.$$

This duality underlies the following identity relating the Seshadri constant $\varepsilon(X)$ (see Definition 5.1) of X and the asymptotic regularity of X

$$\varepsilon(X) = \lim_{d \to \infty} \frac{s_d}{d} = \left(\lim_{k \to \infty} \frac{I(X)^{(k)}}{k} \right)^{-1} =: \widehat{\operatorname{reg}}(I(X))^{-1}.$$

In section 6 we take the opportunity to revisit the celebrated conjectures of Nagata and Iarrobino regarding linear systems of polynomials vanishing to higher order at a finite set of points in projective space. We give homological reformulations for these conjectures based on the results discussed above. This leads into further open problems presented in the final section 7.

2. Duality for numerical sequences

In this paper the set $\mathbb{N}$ of natural numbers does not include 0.

The purpose of this section is to study duality of sequences of natural numbers. To define this duality we first generalize two operations on sequences introduced in [36]. These operations are discrete analogues for the notion of pseudo-inverse functions described in [43].

Definition 2.1. Given sequences $\underline{\alpha} = \{\alpha_d\}_{d\in\mathbb{N}}$ and $\underline{\beta} = \{\beta_n\}_{n\in\mathbb{N}}$ of real numbers define new sequences $\overleftarrow{\underline{\alpha}}^{\beta}, \overrightarrow{\underline{\alpha}}^{\beta}$ associated to the pair $\underline{\alpha}, \underline{\beta}$ in the following manner, where we allow that $\overleftarrow{\alpha}_n^{\beta}, \overrightarrow{\alpha}_n^{\beta} \in \mathbb{N} \cup \{-\infty, \infty\}$ (by convention $\sup(\emptyset) = -\infty, \inf(\emptyset) = \infty$):

$$\overleftarrow{\alpha}_n^{\beta} = \inf\{d \in \mathbb{N} \mid \alpha_d \geq \beta_n\},$$
$$\overrightarrow{\alpha}_n^{\beta} = \sup\{d \in \mathbb{N} \mid \alpha_d \leq \beta_n\}.$$

Setting $\mathrm{id}_n = n$ yields the two particularly important sequences previously studied in [36], for which we use the shortened notation $\overleftarrow{\underline{\alpha}}^{\mathrm{id}} = \overleftarrow{\underline{\alpha}}$ and $\overrightarrow{\underline{\alpha}}^{\mathrm{id}} = \overrightarrow{\underline{\alpha}}$. They are given by

$$\overleftarrow{\alpha}_n = \inf\{d \mid \alpha_d \geq n\},$$
$$\overrightarrow{\alpha}_n = \sup\{d \mid \alpha_d \leq n\}.$$

In the remainder of the paper we will be interested in situations when the sequences $\underline{\alpha}, \underline{\beta}$ consist of natural numbers and for all $n \in \mathbb{N}$ they yield $\overleftarrow{\alpha}_n^{\beta} \in \mathbb{N}$ and $\overrightarrow{\alpha}_n^{\beta} \in \mathbb{N}$.

Example 2.2. If $\underline{\alpha}$ is a sequence of natural numbers there are identities $\overrightarrow{\underline{\mathrm{id}}}^{\alpha} = \overleftarrow{\underline{\mathrm{id}}}^{\alpha} = \underline{\alpha}$.

We shall be interested in applying the transformations in Definition 2.1 to subadditive and superadditive sequences respectively. We now recall these notions.

Definition 2.3. A sequence of real numbers $\underline{\alpha} = \{\alpha_n\}_{n\geq n_0}$ for some $n_0 \in \mathbb{N}$ is called

- **subadditive** if it satisfies $\alpha_{i+j} \leq \alpha_i + \alpha_j$ for all $i, j \geq n_0$.
- **superadditive** if it satisfies $\alpha_i + \alpha_j \leq \alpha_{i+j}$ for all $i, j \geq n_0$.

Fekete's lemma [19] guarantees the existence of $\widehat{\alpha} = \lim_{n\to\infty} \frac{\alpha_n}{n}$ for any subadditive or superadditive sequence of real numbers $\underline{\alpha} = \{\alpha_n\}_{n\in\mathbb{N}}$, allowing for the value of the limit to be $-\infty$ in the subadditive case and ∞ in the superadditive case respectively. In the subadditive case, the value of the limit coincides with $\inf_{n\in\mathbb{N}} \frac{\alpha_n}{n}$ and in the superadditive case with $\sup_{n\in\mathbb{N}} \frac{\alpha_n}{n}$.

Definition 2.4. Given a subadditive or superadditive sequence of real numbers $\underline{\alpha} = \{\alpha_n\}_{n\geq n_0}$, the **asymptotic growth factor** of $\underline{\alpha}$ is the value of the limit

$$\widehat{\alpha} = \lim_{n\to\infty} \frac{\alpha_n}{n} \in \mathbb{R} \cup \{-\infty, \infty\}.$$

If additionally $\underline{\alpha}$ consists of natural numbers, we have $\widehat{\alpha} \in \mathbb{R}_{\geq 0} \cup \{\infty\}$.

We now arrive at our first main result. It shows, how the transformations in Definition 2.1 interact with the classes of subadditive and superadditive sequences and how they transform the respective asymptotic growth factors. In the statement we adopt the conventions that $r/0 = \infty$ and $r/\infty = 0$ for $r \in \mathbb{R}_{>0}$ and $\infty/0 = \infty$, $0/\infty = 0$.

Theorem 2.5. *Let $\underline{\alpha}, \underline{\beta}$ be sequences of positive real numbers such that $\underline{\alpha}$ is subadditive and $\underline{\beta}$ is superadditive. Assume also that $\overrightarrow{\alpha}^{\beta}_n, \overleftarrow{\beta}^{\alpha}_n \in \mathbb{N}$ for each $n \in \mathbb{N}$. Then we have*

(1) *the sequence $\underline{\overrightarrow{\alpha}}^{\beta}$ is superadditive and satisfies $\widehat{\overrightarrow{\alpha}^{\beta}} = \widehat{\beta}/\widehat{\alpha}$.*
(2) *the sequence $\underline{\overleftarrow{\beta}}^{\alpha}$ is subadditive and satisfies $\widehat{\overleftarrow{\beta}^{\alpha}} = \widehat{\alpha}/\widehat{\beta}$.*

Proof. (1) Let $m, n \in \mathbb{N}$ and set $d = \overrightarrow{\alpha}^{\beta}_m$ and $d' = \overrightarrow{\alpha}^{\beta}_n$. By definition we have $\alpha_d \leq \beta_m$ and $\alpha_{d'} \leq \beta_n$ whence we deduce using subadditivity of $\underline{\alpha}$ and superadditivity of $\underline{\beta}$

$$\alpha_{d+d'} \leq \alpha_d + \alpha_{d'} \leq \beta_m + \beta_n \leq \beta_{m+n}.$$

It follows that $\overrightarrow{\alpha}^{\beta}_{m+n} \geq d + d' = \overrightarrow{\alpha}^{\beta}_m + \overrightarrow{\alpha}^{\beta}_n$, establishing superadditivity for $\underline{\overrightarrow{\alpha}}^{\beta}$.

Assume first that $\widehat{\alpha} \neq 0$ and $\widehat{\beta} \in \mathbb{R}$ (i.e., $\widehat{\beta} \neq \infty$). Since $\underline{\overrightarrow{\alpha}}^{\beta}$ is superadditive, we have

$$\widehat{\overrightarrow{\alpha}^{\beta}} = \sup_{n \in \mathbb{N}} \left\{ \frac{\overrightarrow{\alpha}^{\beta}_n}{n} \right\} = \sup \left\{ \frac{d}{n} \mid \alpha_d \leq \beta_n \right\}. \tag{2.1}$$

The identities $\widehat{\alpha} = \lim\limits_{d \to \infty} \left\{ \frac{\alpha_d}{d} \right\} = \inf\limits_{d \in \mathbb{N}} \left\{ \frac{\alpha_d}{d} \right\}$ and $\widehat{\beta} = \lim\limits_{n \to \infty} \left\{ \frac{\beta_n}{n} \right\} = \sup\limits_{n \in \mathbb{N}} \left\{ \frac{\beta_n}{n} \right\}$ yield

$$\frac{\widehat{\beta}}{\widehat{\alpha}} = \sup_{n, d \in \mathbb{N}} \left\{ \frac{\beta_n}{\alpha_d} \cdot \frac{d}{n} \right\},$$

whence we deduce that $\frac{\widehat{\beta}}{\widehat{\alpha}} \geq \frac{\beta_n}{\alpha_d} \cdot \frac{d}{n} \geq \frac{d}{n}$ whenever $\alpha_d \leq \beta_n$. Combining this with (2.1) we arrive to the conclusion $\frac{\widehat{\beta}}{\widehat{\alpha}} \geq \widehat{\overrightarrow{\alpha}^{\beta}}$.

To establish the converse inequality it suffices to show that for all $n, d \in \mathbb{N}$ with $\frac{d}{n} < \frac{\widehat{\beta}}{\widehat{\alpha}}$ we have $\frac{d}{n} \leq \widehat{\overrightarrow{\alpha}^{\beta}}$. Assuming that $\frac{d}{n} < \frac{\widehat{\beta}}{\widehat{\alpha}}$ or equivalently that $\frac{d}{n} \cdot \widehat{\alpha} < \widehat{\beta}$ and writing $\frac{d}{n} \cdot \widehat{\alpha} = \lim\limits_{t \to \infty} \frac{d}{n} \cdot \frac{\alpha_{dt}}{dt}$ and $\widehat{\beta} = \lim\limits_{t \to \infty} \frac{\beta_{nt}}{nt}$ allows to conclude that for $t \gg 0$ we have

$$\frac{d}{n} \cdot \frac{\alpha_{dt}}{dt} < \frac{\beta_{nt}}{nt}, \text{ that is, } \alpha_{dt} < \beta_{nt} \text{ for } t \gg 0. \tag{2.2}$$

In view of the above inequality, (2.1) yields $\widehat{\overrightarrow{\alpha}^{\beta}} \geq \frac{dt}{nt}$ for $t \gg 0$, which leads to the desired conclusion $\widehat{\overrightarrow{\alpha}^{\beta}} \geq \frac{d}{n}$. This concludes the proof of the claim $\widehat{\overrightarrow{\alpha}^{\beta}} = \widehat{\beta}/\widehat{\alpha}$.

Now we treat the cases $\widehat{\alpha} = 0$ and $\widehat{\beta} = \infty$. In both of these situations our convention yields $\widehat{\beta}/\widehat{\alpha} = \infty$. For arbitrary $d, n \in \mathbb{N}$ the inequality $\frac{d}{n} \cdot \widehat{\alpha} < \widehat{\beta}$ is satisfied, therefore the same argument as in (2.2) yields $\widehat{\overrightarrow{\alpha}^{\beta}} \geq \frac{d}{n}$ for all $d, n \in \mathbb{N}$. It follows that $\widehat{\overrightarrow{\alpha}^{\beta}} = \infty = \widehat{\beta}/\widehat{\alpha}$, as claimed.

(2) The second part is entirely analogous to the first. $\square$

Specializing the previous theorem to the case when one of the sequences involved is $\underline{\mathrm{id}}$ allows for a result that better portrays the duality of the transformations $\overrightarrow{\underline{\alpha}}$ and $\overleftarrow{\underline{\alpha}}$. To obtain a true duality theory one must restrict to the case when the input sequence $\underline{\alpha}$ is a sequence of natural numbers unbounded above. Specifically, the next result, which constituted the starting point of our project, shows that the transformations $\overrightarrow{\underline{\alpha}}, \overleftarrow{\underline{\alpha}}$ are mutual inverses and interchange the classes of subadditive and superadditive sequences, that, when restricted to these classes of sequences, the transformations $\overrightarrow{\underline{\alpha}}, \overleftarrow{\underline{\alpha}}$ reciprocate the respective asymptotic growth factors.

For the next result we utilize the convention that $0^{-1} = \infty$ and $\infty^{-1} = 0$.

Theorem 2.6. *Let $\underline{\alpha}$ be a nondecreasing sequence of natural numbers.*

(1) *There are identities $\overrightarrow{\overleftarrow{\underline{\alpha}}} = \underline{\alpha}$ and $\overleftarrow{\overrightarrow{\underline{\alpha}}} = \underline{\alpha}$.*
(2) *If $\underline{\alpha}$ is increasing, then there are identities $\overrightarrow{\overrightarrow{\underline{\alpha}}} = \underline{\alpha}$ and $\overleftarrow{\overleftarrow{\underline{\alpha}}} = \underline{\alpha}$.*
(3) *If $\underline{\alpha}$ is subadditive then $\{\overrightarrow{\alpha}_n\}_{n \geq \alpha_1}$ is nondecreasing superadditive with $\widehat{\overrightarrow{\alpha}} = \widehat{\alpha}^{-1}$.*
(4) *If $\underline{\alpha}$ is superadditive, then $\overleftarrow{\underline{\alpha}}$ is nondecreasing subadditive with $\widehat{\overleftarrow{\alpha}} = \widehat{\alpha}^{-1}$.*

Proof. Assertion (1) as well as the assertions that whenever $\underline{\alpha}$ is superadditive, $\overleftarrow{\underline{\alpha}}$ is subadditive and whenever $\underline{\alpha}$ is superadditive, then $\overleftarrow{\underline{\alpha}}$ is subadditive are shown in [36, Corollary 2.8].

For part (2), note that whenever $\underline{\alpha}$ is increasing the following hold

$$\overrightarrow{\alpha}_{\alpha_n} = \sup\{t : \alpha_t \leq \alpha_n\} = n \tag{2.3}$$

$$\overleftarrow{\alpha}_{\alpha_n} = \inf\{t : \alpha_t \geq \alpha_n\} = n \tag{2.4}$$

Given this, we obtain by applying equation (2.3) for $\underline{\alpha}, \overrightarrow{\underline{\alpha}}$ the following identity

$$\overrightarrow{\overrightarrow{\alpha}}_n = \overrightarrow{\overrightarrow{\alpha}}_{\overrightarrow{\alpha}_{\alpha_n}} = \alpha_n.$$

Similarly, applying equation (2.4) for each of the sequences $\underline{\alpha}, \overleftarrow{\underline{\alpha}}$ we obtain

$$\overleftarrow{\overleftarrow{\alpha}}_n = \overleftarrow{\overleftarrow{\alpha}}_{\overleftarrow{\alpha}_{\alpha_n}} = \alpha_n.$$

The remaining assertions of the theorem regard the asymptotic growth factors. These can be recovered from Theorem 2.5 as follows: first, observe that setting $\mathrm{id}_n = n$ in yields

$\widehat{\mathrm{id}} = 1$, $\overleftarrow{\underline{\alpha}}^{\mathrm{id}} = \overleftarrow{\underline{\alpha}}$ and $\overrightarrow{\underline{\alpha}}^{\mathrm{id}} = \overrightarrow{\underline{\alpha}}$. Note that if $\underline{\alpha}$ is nondecreasing, so are $\overrightarrow{\underline{\alpha}}$ and $\overleftarrow{\underline{\alpha}}$ by definition.

If $\underline{\alpha}$ is a superadditive sequence of natural numbers then it is unbounded above as $\alpha_n \geq n\alpha_1 \geq n$. It follows that $\overleftarrow{\underline{\alpha}}_n \in \mathbb{N}$ for all $n \in \mathbb{N}$ whenever the sequence $\underline{\alpha}$ is superadditive. If $\underline{\alpha}$ is a nondecreasing sequence it follows that $\overrightarrow{\underline{\alpha}}_n \in \mathbb{N}$ for all $n \geq \alpha_1$.

Since $\underline{\mathrm{id}}$ is both subadditive and superadditive, setting $\beta = \mathrm{id}$ in part (1) of Theorem 2.5 yields for subadditive $\underline{\alpha}$ that $\widehat{\overrightarrow{\underline{\alpha}}} = \widehat{\alpha}^{-1}$ as in part (3) of Theorem 2.6, and setting $\underline{\alpha} = \mathrm{id}$ in part (2) of Theorem 2.5 yields for superadditive $\underline{\beta}$ that $\widehat{\overleftarrow{\underline{\beta}}} = \widehat{\beta}^{-1}$ as in part (4) of Theorem 2.6. □

Example 2.7. In the absence of the hypothesis that $\underline{\alpha}$ is nondecreasing, it need not be true that $\overleftarrow{\overrightarrow{\underline{\alpha}}} = \underline{\alpha}$. Consider the sequence $\alpha_n = \begin{cases} n & \text{if } n \text{ is odd} \\ n/2 & \text{if } n \text{ is even.} \end{cases}$ Then we compute

n	1	2	3	4	5
α_n	1	1	3	2	5
$\overrightarrow{\underline{\alpha}}_n$	2	4	6	8	10
$\overleftarrow{\overrightarrow{\underline{\alpha}}}_n$	1	1	2	2	3.

Likewise, in the absence of the hypothesis that $\underline{\alpha}$ is increasing, it need not be true that $\overrightarrow{\overrightarrow{\underline{\alpha}}} = \underline{\alpha}$. Consider $\alpha_n = \lceil n/2 \rceil$. Then $\underline{\alpha}$ is subadditive and nondecreasing (but not increasing), $\overrightarrow{\underline{\alpha}}_n = 2n$ and $\overrightarrow{\overrightarrow{\underline{\alpha}}} = \underline{\alpha}_n = \lfloor n/2 \rfloor$.

Example 2.8. In the more general setting of Definition 2.1 one does not obtain a satisfactory duality theory in the sense that the operations $\overrightarrow{\underline{\alpha}}^{\beta}$, $\overleftarrow{\underline{\alpha}}^{\beta}$ need not be mutually inverse even when both sequences $\underline{\alpha}, \underline{\beta}$ are nondecreasing. Indeed, consider $\alpha_n = \lceil \frac{n}{2} \rceil$ and $\beta_n = \lfloor \frac{n}{2} \rfloor$ which yield $\underline{\alpha} \neq \overleftarrow{\overrightarrow{\underline{\alpha}}^{\beta}}^{\beta}$ and $\underline{\alpha} \neq \overleftarrow{\overrightarrow{\underline{\alpha}}^{\beta}}^{\alpha}$ according to the table below

n	1	2	3	4	5
α_n	1	1	2	2	3
β_n	0	1	1	2	2
$\overrightarrow{\underline{\alpha}}^{\beta}_n$	$-\infty$	2	2	4	4
$\overleftarrow{\overrightarrow{\underline{\alpha}}^{\beta}}^{\beta}_n$	2	2	2	2	2
$\overleftarrow{\overrightarrow{\underline{\alpha}}^{\beta}}^{\alpha}_n$	2	2	2	2	4.

We conclude by considering the transfer of the subadditive and superadditive properties from a sequence to its subsequences.

Lemma 2.9. *Let $\underline{\alpha} = \{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence. Define for $k \in \mathbb{Z}$ the subsequence $\underline{\alpha}[k] = \{\alpha_{n+k}\}_{n\in\mathbb{N}, n>-k}$, that is, the n-th member of the sequence $\underline{\alpha}[k]$ is α_{n+k}, provided $n+k > 0$.*

(1) *If $k \geq 0$ and $\underline{\alpha}$ is subadditive and nondecreasing, then $\underline{\alpha}[k]$ is subadditive.*
(2) *If $k \leq 0$ and $\underline{\alpha}$ is superadditive and nondecreasing, then $\underline{\alpha}[k]$ is superadditive.*
(3) *If $\widehat{\alpha}$ exists then $\widehat{\alpha[k]}$ exists as well and $\widehat{\alpha[k]} = \widehat{\alpha}$.*

Proof. We focus on assertion (1), the second numbered assertion being similar. Under the hypotheses of (1)

$$\alpha[k]_{a+b} = \alpha_{a+b+k} \leq \alpha_{a+b+2k} \leq \alpha_{a+k} + \alpha_{b+k} = \alpha[k]_a + \alpha[k]_b$$

follows from the nondecreasing property of $\underline{\alpha}$ for the first inequality and subadditivity of $\overrightarrow{\alpha}$ for the second. Part (3) follows from

$$\widehat{\alpha[k]} = \lim_{n\to\infty} \frac{a_{n+k}}{n} = \lim_{n\to\infty} \frac{a_{n+k}}{n+k} \cdot \lim_{n\to\infty} \frac{n+k}{n} = \widehat{\alpha} \quad \square$$

3. Subadditive and superadditive sequences from graded families

We are interested in subadditive and superadditive sequences which occur in algebraic contexts. The following considerations introduce types of sequences we shall focus our attention on in the remainder of the manuscript.

3.1. Valuations and initial degree

Recall that a discrete valuation on a field $\mathbf{K}$ is a homomorphism $\nu : \mathbf{K}^* \to \mathbb{Z}$ on the units of $\mathbf{K}$ satisfying $\nu(xy) = \nu(x) + \nu(y)$ and $\nu(x+y) \geq \min\{\nu(x), \nu(y)\}$. If $\mathbf{K}$ is the fraction field of a domain R then a valuation is determined by its values on R via $\nu(f/g) = \nu(f) - \nu(g)$, so we abuse notation by referring to valuations on R instead of its field of fractions. We furthermore restrict ourselves to valuations which are non-negative on R, which we call R-valuations.

Example 3.1. Given a maximal ideal $\mathfrak{m}$ in a regular ring R, a simple example of an R-valuation is $\alpha_{\mathfrak{m}}(f) = \max\{k : f \in \mathfrak{m}^k\}$. If $\mathfrak{m}$ is not a maximal ideal, $\alpha_{\mathfrak{m}}$ need not be a valuation; $\alpha_{\mathfrak{m}}(xy) \leq \alpha_{\mathfrak{m}}(x) + \alpha_{\mathfrak{m}}(y)$ is always true but equality may not hold [32, Section 6.7].

Definition 3.2. If ν is an R-valuation, denote the minimum value taken by ν on I by

$$\nu(I) = \min\{\nu(f) : 0 \neq f \in I\}$$

If R is a standard graded ring with homogeneous maximal ideal $\mathfrak{m}$, then the **initial degree** of I is the minimum value taken by the valuation $\alpha_{\mathfrak{m}}$ in Example 3.1 on I

$$\alpha(I) = \min\{\deg f : 0 \neq f \in I\} = \max\{k : I \subseteq \mathfrak{m}^k\}.$$

Recall that a **graded family** of ideals $\mathcal{I} = \{I_n\}_{n\geq 1}$ of a ring R is a family which satisfies $I_a I_b \subset I_{a+b}$ for all $a, b \in \mathbb{N}$.

Lemma 3.3. *Given a graded family $\mathcal{I} = \{I_n\}_{n\geq 1}$ of ideals of a domain R and an R-valuation ν the sequence $\alpha_n = \nu(I_n)$ is subadditive.*

Proof. The property $\nu(xy) = \nu(x) + \nu(y)$ implies that $\nu(I_a I_b) = \nu(I_a) + \nu(I_b)$ for all $a, b \in \mathbb{N}$. It follows from the containment $I_a I_b \subset I_{a+b}$ for all $a, b \in \mathbb{N}$ that $\alpha_{a+b} = \nu(I_{a+b}) \leq \nu(I_a I_b) = \nu(I_a) + \nu(I_b) = \alpha_a + \alpha_b$. □

One of the graded families of interest for this paper is formed by symbolic powers.

Definition 3.4. Given an ideal I of a ring R, the n^{th} **symbolic power** of I is

$$I^{(n)} = \bigcap_{P \in \operatorname{Ass}(I)} \left(I^n R_P \cap R\right).$$

We set $I^{(0)} = R$ by convention.

The growth of initial degree of the symbolic powers of an ideal is captured by the Waldschmidt constant. This invariant, was first introduced by Waldschmidt [46] in the late 70s for finite sets of points in n-dimensional space and formally defined in terms of symbolic powers in [4]. It has often been featured implicitly in the geometric literature; see section 6 for further details and connections. More generally, the asymptotic growth factor of an arbitrary valuation applied to the symbolic powers of an ideal is dubbed a skew Waldschmidt constant in [11].

Definition 3.5. The **Waldschmidt constant** of a homogeneous ideal I is the real number

$$\widehat{\alpha}(I) = \lim_{n\to\infty} \frac{\alpha(I^{(n)})}{n} = \inf_{n\in\mathbb{N}} \frac{\alpha(I^{(n)})}{n}.$$

Given a valuation $\nu : R \to \mathbb{Z}$, the **skew Waldschmidt constant** of a homogeneous ideal I is the real number

$$\widehat{\nu}(I) = \lim_{n\to\infty} \frac{\nu(I^{(n)})}{n} = \inf_{n\in\mathbb{N}} \frac{\nu(I^{(n)})}{n}.$$

Throughout the paper $\overline{J}$ denotes the integral closure of an ideal J. The valuative criterion for integral closures [32, Theorem 6.8.3] states that for a fixed ideal $I \subset R$ and every $f \in R$, $f \in \overline{I}$ if and only if $\nu(f) \geq \nu(I)$ for every R-valuation $\nu : R \to \mathbb{Z}$. From this we get the following ideal membership test: there is containment $J \subset \overline{I}$ between two ideals if and only if $\nu(J) \geq \nu(I)$ for every R-valuation $\nu : R \to \mathbb{Z}$. We now define a sequence inspired by this criterion and its applications to the containment problem between the ordinary and symbolic powers of an ideal (see section 3.2).

We say that a valuation ν is *supported* on an ideal I if $\nu(I) > 0$.

Definition 3.6. Given an ideal $I \subset R$ and an R-valuation ν supported on I, define

$$\beta_n^\nu = \beta_n^\nu(I) = \sup\{d : \nu(I^{(d)}) < \nu(I^n)\}.$$

Remark 3.7. The fact that if R is Noetherian $\beta_n^\nu \in \mathbb{N} \cup \{0\}$ for each $n \in \mathbb{N}$ follows from Swanson's theorem on linear equivalence of the symbolic and ordinary I-adic topologies. In detail, it is shown in [44] that there exists an integer ℓ (possibly dependent upon I) such that $I^{(\ell n)} \subseteq I^n$ for all $n \in \mathbb{N}$. This yields

$$\nu(I^{(\ell n)}) \geq \nu(I^n) = n\nu(I) > \nu(R) = \nu(I^{(0)}).$$

Consequently, since the sequence $\{\nu(I^{(d)})\}_{d\in\mathbb{N}}$ is nondecreasing, we have $0 \leq \beta_n^\nu < \ell n$.

We come to our first application of Theorem 2.5.

Proposition 3.8. *For any Noetherian domain R, any ideal $I \subset R$ and any R-valuation ν supported on I the sequence $\beta_n^\nu = \beta_n^\nu(I)$ is superadditive and satisfies*

$$\widehat{\beta^\nu} = \lim_{n\to\infty} \frac{\beta_n^\nu}{n} = \sup_{n\in\mathbb{N}} \left\{ \frac{\beta_n^\nu}{n} \right\} = \frac{\nu(I)}{\widehat{\nu}(I)}$$

Proof. We first give an alternate definition for β^ν. Set $\gamma_d = \nu(I^{(d)})$ and $\delta_n = \nu(I^n) - 1 = n\nu(I) - 1$ for $n, d \in \mathbb{N}$. Note that $\underline{\gamma}$ is subadditive by Lemma 3.3, $\underline{\delta}$ is superadditive by its definition, and we have $\widehat{\gamma} = \widehat{\nu}(I)$ and $\widehat{\delta} = \nu(I)$. Then Definition 3.6 can be rewritten as $\beta^\nu = \overrightarrow{\gamma}^{\delta}$. An application of Theorem 2.5 (1) yields that the sequence β^ν is superadditive and $\widehat{\beta^\nu} = \nu(I)/\widehat{\nu}(I)$. The first equality in the claim follows from superadditivity of β^ν and Fekete's lemma. $\square$

In the next subsection we interpret the asymptotic growth factor $\widehat{\beta^\nu}$ in terms of an invariant of I termed asymptotic resurgence.

3.2. Asymptotic resurgence

The various invariants defined below under the name of resurgence were introduced to study the *containment problem* which asks for pairs of natural numbers d, n for which $I^{(d)} \subseteq I^n$.

Definition 3.9. The **resurgence** of an ideal I, introduced in [4], is the quantity

$$\rho(I) = \sup \left\{ \frac{d}{n} : I^{(d)} \not\subseteq I^n \right\}.$$

Its asymptotic counterpart is the **asymptotic resurgence** of I, introduced in [23]

$$\widehat{\rho}(I) = \sup\left\{\frac{d}{n} : I^{(dt)} \not\subseteq I^{nt} \text{ for } t \gg 0\right\}.$$

Versions of these invariants using integral closures were defined in [11]. These are the **ic-resurgence**

$$\rho_{ic}(I) = \sup\left\{\frac{d}{n} : I^{(d)} \not\subseteq \overline{I^n}\right\}$$

and the **ic-asymptotic resurgence**

$$\widehat{\rho}_{ic}(I) = \sup\left\{\frac{d}{n} : I^{(dt)} \not\subseteq \overline{I^{nt}} \text{ for } t \gg 0\right\}.$$

It is shown in [11, Corollary 4.14] that $\rho_{ic}(I) = \widehat{\rho}_{ic}(I) = \widehat{\rho}(I)$. By contrast, in general we have $\widehat{\rho}(I) \neq \rho(I)$; see [13]. Another resurgence number, $\rho_{\text{int}}(I)$, introduced in [28], is given by

$$\rho_{int}(I) = \sup\left\{\frac{d}{n} : \overline{I^d} \not\subseteq I^n\right\}.$$

In this section we discuss two numerical sequences which arise in conjunction with these notions of resurgence:

$$\lambda_n = \lambda_n(I) = \max\{d : I^{(d)} \not\subseteq I^n\} \text{ and } \beta_n = \beta_n(I) = \max\{d : I^{(d)} \not\subseteq \overline{I^n}\}.$$

Notice that

$$\rho(I) = \sup_{n\in\mathbb{N}}\left\{\frac{\lambda_n}{n}\right\} \text{ and } \widehat{\rho}(I) = \sup_{n\in\mathbb{N}}\left\{\frac{\beta_n}{n}\right\}$$

follows from the definition of resurgence and asymptotic resurgence, respectively. If R is a regular ring and I is radical then [10, Remark 5.5] implies that in fact

$$\lim_{n\to\infty}\frac{\lambda_n}{n} = \lim_{n\to\infty}\frac{\beta_n}{n} = \widehat{\rho}(I).$$

The assumption that I is radical can be removed (see [10, Remark 4.23]). Thus we see that the sequence $\{\beta_n\}$ behaves like a superadditive sequence in the sense that

$$\lim_{n\to\infty}\frac{\beta_n}{n} = \sup_{n\in\mathbb{N}}\left\{\frac{\beta_n}{n}\right\}.$$

Since there are examples where $\rho(I) \neq \widehat{\rho}(I)$ (see [13]), $\{\lambda_n\}$ is not necessarily superadditive. We do not know if $\underline{\beta} = \{\beta_n\}_{n\in\mathbb{N}}$ is always a superadditive sequence. However, we are able to replace $\underline{\beta}$ by a valuative sequence of the type discussed in Definition 3.6 which is superadditive and whose asymptotic growth rate is also equal to the asymptotic resurgence.

Proposition 3.10. *Let I be an ideal in a regular ring R. For any valuation $\nu : R \to \mathbb{Z}$, we have $\beta_n^\nu \leq \beta_n \leq \lambda_n$. Moreover there is a choice of valuation ν so that*

$$\widehat{\beta^\nu} = \lim_{n\to\infty} \frac{\beta_n^\nu}{n} = \lim_{n\to\infty} \frac{\beta_n}{n} = \lim_{n\to\infty} \frac{\lambda_n}{n} = \widehat{\rho}(I).$$

Proof. The inequalities $\beta_n^\nu \leq \beta_n \leq \lambda_n$ follow from the definitions of the sequences and the valuative criterion for integral closures. The equalities

$$\lim_{n\to\infty} \frac{\beta_n}{n} = \lim_{n\to\infty} \frac{\lambda_n}{n} = \widehat{\rho}(I)$$

follow from [10, Remark 5.5], as noted above. By [11, Theorem 4.10], $\widehat{\rho}(I) = \nu(I)/\widehat{\nu}(I)$ for some choice of valuation (in fact, one of the *Rees* valuations of I will accomplish this). Hence, for this valuation, Proposition 3.8 yields that $\lim_{n\to\infty} \beta_n^\nu/n = \widehat{\rho}(I)$, completing the proof. $\square$

3.3. Castelnuovo-Mumford regularity

Definition 3.11. Suppose $R = \bigoplus_{i\geq 0} R_i$ is a graded ring with residue field $\mathbb{K} = R/R_+$ where $R_+ = \bigoplus_{i>0} R_i$. The **Castelnuovo-Mumford regularity** of a graded module M over R is

$$\operatorname{reg}(M) = \max\{j - i \mid \operatorname{Tor}_i^R(M, \mathbb{K})_j \neq 0\}.$$

When M has finite length and R is standard graded the regularity can also be expressed as $\operatorname{reg}(M) = \operatorname{end}(M) := \max\{i \mid M_i \neq 0\}$. For arbitrary graded modules M over a standard graded ring R, there is an alternate definition in terms of the local cohomology modules of M supported at the homogeneous maximal ideal $\mathfrak{m}$

$$\operatorname{reg}(M) = \sup\{\operatorname{end}\left(H^i_{\mathfrak{m}}(M)\right) + i \mid 0 \leq i \leq \dim(M)\}.$$

Keeping with the theme of our writing, we are interested in families of ideals or modules whose Castelnuovo-Mumford regularities give subadditive or superadditive sequences. The subadditive case is considered in the following lemma, while a family with superadditive regularity sequence is illustrated in Remark 4.19.

Lemma 3.12. *Given a graded family $\mathcal{I} = \{I_n\}_{n\in\mathbb{N}}$ of homogeneous ideals of a standard graded ring R so that each quotient ring R/I_n is Cohen-Macaulay of the same dimension $\dim(R/I_n) = d$. Then the sequence of Castelnuovo-Mumford regularities of the members in the family $\{\operatorname{reg}(I_n)\}_{n\in\mathbb{N}}$ is subadditive.*

Proof. We may assume that the residue field of R is infinite by tensoring with an infinite extension of the base field if necessary; this does not change the Castelnuovo-Mumford regularity of the given ideals. Thanks to the Cohen-Macaulay property one may reduce to the Artinian case. In detail, fix $a, b \in \mathbb{N}$ and choose a sequence of linear forms $\ell_1, \dots, \ell_d$ which is simultaneously a regular sequence on $R/I_a, R/I_b$ and also on R/I_{a+b}. Now set $\widetilde{R} = R/(\ell_1, \dots, \ell_d)$ and $\widetilde{I}_n = I_n + (\ell_1, \dots, \ell_d)/(\ell_1, \dots, \ell_d)$ for $n \in \{a, b, a+b\}$. This gives that $\operatorname{reg}(\widetilde{I}_n) = \operatorname{reg}(I_n)$ for $n \in \{a, b, a+b\}$. Moreover, setting $\mathfrak{m}$ to be the homogeneous maximal ideal of $\widetilde{R}$, since each of the quotients $\widetilde{R}/\widetilde{I}_n$ is Artinian we have for $n \in \{a, b, a+b\}$

$$r_n = \operatorname{reg}(I_n) = \operatorname{reg}\left(\widetilde{I}_n\right) = \min\{d : \left(\widetilde{R}/\widetilde{I}_n\right)_d = 0\} = \min\{d : \mathfrak{m}^d \subseteq \widetilde{I}_n\}.$$

It follows from the containment $I_a I_b \subset I_{a+b}$ that $\widetilde{I}_a \widetilde{I}_b \subset \widetilde{I}_{a+b}$. We deduce

$$\mathfrak{m}^{r_a + r_b} = \mathfrak{m}^{r_a}\mathfrak{m}^{r_b} \subseteq \widetilde{I}_a \widetilde{I}_b \subseteq \widetilde{I}_{a+b}$$

and thus it follows that $r_{a+b} = \operatorname{reg}\left(\widetilde{I}_{a+b}\right) \leq r_a + r_b$. □

Definition 3.13. Ideals I for which every member of the sequence of symbolic powers $\{I^{(n)}\}$ yields a Cohen-Macaulay quotient are dubbed **aspCM** ideals in [47].

The aspCM class includes complete intersection ideals, saturated ideals with $\dim R/I = 1$, that is defining ideals for finite sets of points or fat points (not necessarily reduced schemes supported at finite sets of points) in $\mathbb{P}^N$, ideals defining matroid configurations in $\mathbb{P}^N$ [21], and generic determinantal ideals.

Remark 3.14. Even under the hypotheses of Lemma 3.12, the closely related sequence $\{\operatorname{reg}(R/I_n)\}_{n\in\mathbb{N}}$ need not be subadditive. Take for example $I_n = (f^n)$ where f is a homogeneous element of degree $d > 0$ in a standard graded polynomial ring R. Then $\operatorname{reg}(R/I_n) = dn - 1$ is not subadditive.

We define an invariant which captures the asymptotic growth of the regularity for a family of ideals.

Definition 3.15. The **asymptotic regularity** of a family $\mathcal{I} = \{I_n\}_{n\in\mathbb{N}}$ of homogeneous ideals is the following limit, provided it exists,

$$\widehat{\operatorname{reg}}(\mathcal{I}) = \lim_{n\to\infty} \frac{\operatorname{reg}(I_n)}{n}.$$

By way of Fekete's lemma, Lemma 3.12 provides a set of assumptions under which the limit in Definition 3.15 exists. We shall be primarily interested in the asymptotic regularity for the family of symbolic powers $\mathcal{I} = \{I^{(n)}\}_{n\in\mathbb{N}}$ of a given ideal I, which we denote $\widehat{\operatorname{reg}}(I)$. This family does not always satisfy the conditions of Lemma 3.12, but it does so, for example, when I defines a finite set of points in projective space. In this case, the existence of $\widehat{\operatorname{reg}}(I)$ also follows from the much more general result in [7, Theorem B]. There are a few other instances where the existence of $\widehat{\operatorname{reg}}(I)$ is known, for example, when I is a monomial ideal cf. [15, Theorem 3.6.], or more generally, when the symbolic Rees algebra of I is Noetherian, which is shown in the ongoing work of the second author with Hop and Hà.

The next example points out that in general the sequence $\{\operatorname{reg}(I_n)\}_{n\in\mathbb{N}}$ need not be subadditive for a graded family of ideals $\mathcal{I} = \{I_n\}_{n\in\mathbb{N}}$ even when that family consists of symbolic powers of monomial ideals and thus $\widehat{\operatorname{reg}}(\mathcal{I})$ exists. The ideals $J(m,s)$ in the next example yields Cohen-Macaulay quotient rings, but their symbolic powers do not, thus they are not aspCM.

Example 3.16. In [15, Theorem 5.15] Dung, Hien, Nguyen, and Trung produce examples of squarefree monomial ideals $J(m,s)$ such that

$$\operatorname{reg}\left(J(m,s)^{(t)}\right) = \begin{cases} m(s+1)n & t = 2n \\ m(s+1)n + m + s - 1 & t = 2n+1 \end{cases}.$$

Combinatorially the ideals $J(m,s)$ are described as cover ideals for corona graphs obtained by adding s pendant edges to each vertex of a complete graph K_m. The ideals in this family were singled out as examples of squarefree monomial ideals for which the function $t \mapsto \operatorname{reg}(J(m,s)^{(t)})$ is not eventually linear. For these symbolic power ideals the regularity matches the largest degree of a minimal generator, which shows that if an ideal J is generated in degrees $\leq d$ one cannot conclude that $J^{(t)}$ is generated in degrees $\leq td$. This relates to a question of Huneke [33, Problem 0.4].

A necessary condition for the sequence $\{\operatorname{reg}(J(m,s)^{(t)})\}_{t\in\mathbb{N}}$ to be subadditive is

$$\operatorname{reg}(J(m,s)^{(2t_1+2t_2+2)}) \leq \operatorname{reg}(J(m,s)^{(2t_1+1)}) + \operatorname{reg}(J(m,s)^{(2t_2+1)})$$

which can be written equivalently as

$$m(s+1) \leq 2m + 2s - 2 \text{ or } (m-2)(s-1) \leq 0.$$

It is thus evident that the regularity sequence for the symbolic powers of $J(m,s)$ is not subadditive whenever $m > 2$ and $s > 1$.

4. Inverse systems of differentially closed graded filtrations

One situation in which the duality between subadditive sequences and superadditive sequences naturally arises is in the theory of inverse systems. In this section we extend a construction using inverse systems from an influential paper of Emsalem and Iarrabino [17]; two sequences naturally associated to this construction exhibit the duality of Section 2. We begin by recalling some details about contraction and differentiation, following the survey of Geramita [20, Lecture 9].

4.1. Contraction, differentiation, and inverse systems

Let $\mathbb{K}$ be a field and $R = \mathbb{K}[x_0, \dots, x_N]$. We use a standard shorthand for monomials – if $\mathbf{a} = (a_0, \dots, a_N) \in \mathbb{Z}^{N+1}_{\geq 0}$, then $x^{\mathbf{a}} = x_0^{a_0} \cdots x_N^{a_N}$ is the corresponding monomial in R. We define $\mathcal{D} = \bigoplus_{i \geq 0} \operatorname{Hom}(R_i, \mathbb{K})$, the graded $\mathbb{K}$-dual of R. If $x^{\mathbf{a}}$ is in R_d, we write $Y^{[\mathbf{a}]}$ for the functional (in $\mathcal{D}_d$) on R_d which sends $x^{\mathbf{a}}$ to 1 and all other monomials in R_d to 0. As a vector space, $\mathcal{D}$ is isomorphic to a polynomial ring in $N+1$ variables. However, as we recall shortly, $\mathcal{D}$ has the structure of a *divided power algebra*. For this reason, we call $Y^{[\mathbf{a}]}$ a *divided* monomial.

The ring R acts on $\mathcal{D}$ by *contraction*, which we denote by $\bullet$. That is, if $x^{\mathbf{a}}$ is a monomial in R and $Y^{[\mathbf{b}]}$ is a divided monomial in $\mathcal{D}$, then

$$x^{\mathbf{a}} \bullet Y^{[\mathbf{b}]} = Y^{[\mathbf{b}-\mathbf{a}]} \text{ if } \mathbf{b} \geq \mathbf{a},$$

and 0 otherwise. This action is extended linearly to all of R and $\mathcal{D}$. This action of R on $\mathcal{D}$ gives a perfect pairing of vector spaces $R_d \times \mathcal{D}_d \to \mathbb{K}$ for any degree $d \geq 0$. Suppose U is a subspace of R_d. We define

$$U^{\perp} = \{g \in \mathcal{D}_d : f \bullet g = 0 \text{ for all } f \in U\}.$$

Macaulay [38] introduced the *inverse system* of an ideal I of R to be

$$I^{-1} := \operatorname{Ann}_S(I) = \{g \in \mathcal{D} : f \bullet g = 0 \text{ for all } f \in I\}.$$

If I is a homogeneous ideal of R then the inverse system I^{-1} can be constructed degree by degree using the identification $(I^{-1})_d = I_d^{\perp}$ [20, Proposition 2.5]. In general, I^{-1} is an R-submodule of $\mathcal{D}$ which is finitely generated if and only if I is an Artinian ideal.

A priori, $\mathcal{D}$ is simply a graded R-module. However, $\mathcal{D}$ can be equipped with a multiplication which makes it into a ring, called the divided power algebra. Suppose $\mathbf{a} = (a_0, \dots, a_N), \mathbf{b} = (b_0, \dots, b_N) \in \mathbb{Z}^{N+1}_{\geq 0}$. The multiplication in $\mathcal{D}$ is defined on monomials by

$$Y^{[\mathbf{a}]}Y^{[\mathbf{b}]} = \binom{\mathbf{a}+\mathbf{b}}{\mathbf{a}} Y^{[\mathbf{a}+\mathbf{b}]}, \tag{4.1}$$

where

$$\mathbf{a}! = \prod_{i=0}^{N} a_i! \quad \text{and} \quad \binom{\mathbf{a}+\mathbf{b}}{\mathbf{a}} = \frac{(\mathbf{a}+\mathbf{b})!}{\mathbf{a}!\mathbf{b}!} = \prod_{i=0}^{N} \binom{a_i+b_i}{a_i}. \tag{4.2}$$

This multiplication is extended linearly to all of $\mathcal{D}$. Let $\mathbf{e}_i$ be the ith standard basis vector in $\mathbb{Z}^{N+1}$ and put $Y_i := Y^{[\mathbf{e_i}]}$. For a nonnegative integer n let $Y_i^n := \left(Y^{[\mathbf{e_i}]}\right)^n$ and $Y_i^{[n]} := Y^{[n\mathbf{e}_i]}$. We see from equation (4.1) that $Y_i^n = n!Y^{[n\mathbf{e_i}]} = n!Y_i^{[n]}$. More generally for $\mathbf{a} = (a_0, \dots, a_N)$ set $Y^{\mathbf{a}} = \prod_{i=0}^N Y_i^{a_i}$. Since $Y^{\mathbf{a}} = \prod_{i=0}^N Y_i^{a_i} = \prod_{i=0}^N a_i!Y^{[a_i\mathbf{e_i}]} = \mathbf{a}!\prod_{i=0}^N Y^{[a_i\mathbf{e_i}]}$, another application of (4.1) allows to deduce that

$$Y^{\mathbf{a}} = \mathbf{a}!Y^{[\mathbf{a}]}. \tag{4.3}$$

See [20, Lecture 9] for additional details. In characteristic zero, $\mathbf{a}!$ never vanishes and so $\mathcal{D}$ is generated as an algebra by $Y_0, \dots, Y_N$, just like the polynomial ring. However, in characteristic p, $\mathcal{D}$ is infinitely generated by all the divided power monomials $Y_j^{[p^k]}$ for all $j = 0, \dots, N$ and integers $k \geq 0$. The proof of this fact follows from Lucas' identity: given base p expansions $a = \sum a_i p^i$ and $b = \sum b_i p^i$ for $a, b \in \mathbb{N}$, then

$$\binom{b}{a} = \prod_{i=0}^{\infty} \binom{b_i}{a_i} \mod p.$$

In particular, suppose $\mathbf{a} = (a_0, \dots, a_N)$ where a_j has base p expansion $a_j = \sum a_{ij}p^i$ for $j = 0, \dots, N$. Then, by Lucas' identity and equation (4.1), $Y_j^{[a_j]} = \prod_i Y_j^{[a_{ij}p^i]}$ and $(Y_j^{[p^i]})^{a_{ij}} = a_{ij}!Y_j^{[a_{ij}p^i]}$. We have

$$Y^{[\mathbf{a}]} = \prod_{j=0}^N \prod_i Y_j^{[a_{ij}p^i]} = \prod_{j=0}^N \prod_i \frac{(Y_j^{[p^i]})^{a_{ij}}}{a_{ij}!},$$

where each denominator is invertible modulo p since $0 \leq a_{ij} < p$ for all i, j. Hence $\mathcal{D}$ is infinitely generated by all the divided power monomials $Y_j^{[p^k]}$ for all $j = 0, \dots, N$ and integers $k \geq 0$.

We now revisit the characteristic zero case. Suppose $\mathbb{K}$ is a field of characteristic zero and let $S = \mathbb{K}[y_0, \dots, y_N]$ be a polynomial ring. Consider the action of R on S by partial differentiation, which we represent by '$\circ$'. That is, if $\mathbf{a} = (a_0, \dots, a_N) \in \mathbb{Z}_{\geq 0}^{N+1}$, $x^{\mathbf{a}} = x_0^{a_0} \cdots x_N^{a_N}$ is a monomial in R, and $g \in S$, we write

$$x^{\mathbf{a}} \circ g = \frac{\partial^{\mathbf{a}} g}{\partial x^{\mathbf{a}}}$$

for the action of $x^{\mathbf{a}}$ on g (extended linearly to all of R). In particular, if $\mathbf{a} \leq \mathbf{b}$, then

$$x^{\mathbf{a}} \circ y^{\mathbf{b}} = \frac{\mathbf{b}!}{(\mathbf{b} - \mathbf{a})!} y^{\mathbf{b}-\mathbf{a}},$$

where we use (4.2). This action gives a perfect pairing $R_d \times S_d \to \mathbb{K}$, and, given a homogeneous ideal $I \subset R$, we define $I_d^\perp$ and I^{-1} in the same way as we do for contraction.

Since we are in characteristic zero, the map of rings $\Phi : S \to \mathcal{D}$ defined by $\Phi(y_i) = Y_i$ extends to all monomials via (4.3) to give $\Phi(y^{\mathbf{a}}) = Y^{\mathbf{a}} = \mathbf{a}!Y^{[\mathbf{a}]}$. Thus S and $\mathcal{D}$ are isomorphic. Moreover, if $F \in R$ and $g \in S$, then $\Phi(F \circ g) = F \bullet \Phi(g)$ [20, Theorem 9.5], so S and $\mathcal{D}$ are isomorphic as R-modules.

4.2. Differential operators and differentially closed filtrations

If $\mathbf{a} = (a_0, \ldots, a_N) \in \mathbb{Z}^{N+1}_{\geq 0}$, we extend our convention on monomials to differential operators, letting $\frac{\partial^{\mathbf{a}}}{\partial x^{\mathbf{a}}} = \frac{\partial^{a_0}}{\partial x_0^{a_0}} \cdots \frac{\partial^{a_N}}{\partial x_N^{a_N}}$. Independent of characteristic, the ring of $\mathbb{K}$-linear differential operators D_R, which acts on R, can be written as $D_R = \cup_{n\in\mathbb{N}} D_R^n$, where

$$D_R^n = R\left\langle \frac{1}{\mathbf{a}!}\frac{\partial^{\mathbf{a}}}{\partial x^{\mathbf{a}}} \;\middle|\; |\mathbf{a}| \leq n \right\rangle,$$

where we use the convention (4.2) for $\mathbf{a}!$. See [8, Remark 2.7]. For simplicity, we will write $D_{\mathbf{a}} = \frac{1}{\mathbf{a}!}\frac{\partial^{\mathbf{a}}}{\partial x^{\mathbf{a}}}$. The factors $\frac{1}{a_i!}$ appearing in $\frac{1}{\mathbf{a}!}$ do not represent elements in the field; $D_{\mathbf{a}}$ is a formal representation for the $\mathbb{K}$-linear operator defined by $D_{\mathbf{a}}(x^{\mathbf{b}}) = \binom{\mathbf{b}}{\mathbf{a}} x^{\mathbf{b}-\mathbf{a}}$ if $\mathbf{b} \geq \mathbf{a}$, and otherwise $D_{\mathbf{a}}(x^{\mathbf{b}}) = 0$. Note that in characteristic 0, $\mathbf{a}!D_{\mathbf{a}} = \frac{\partial^{\mathbf{a}}}{\partial x^{\mathbf{a}}}$ is the usual partial differential operator. Thus in characteristic 0, D_R is generated as an R-algebra by either $D_{\mathbf{e}_i}$ for $i = 0, \ldots, N$ (where $\mathbf{e}_i$ is the ith standard basis vector) or by $\frac{\partial}{\partial x_i}$ for $i = 0, \ldots, N$.

In characteristic p, using Lucas' identity as in the divided power ring, one can show that if $\mathbf{a} = (a_0, \ldots, a_N)$ where $a_j = \sum a_{ij} p^i$, then $D_{\mathbf{a}} = \prod_{j=0}^N \prod_i (D_{p^i\mathbf{e_j}})^{a_{ij}}/(a_{ij}!)$ where the product just means the composition of the operators. This computation shows that D_R is generated as an R-algebra by $\{D_{p^i\mathbf{e_j}} : 0 \leq j \leq N, 0 \leq i\}$. Recall that a **filtration** of ideals $\mathcal{I} = \{I_n\}_{n\geq 1}$ of a ring R is a family which satisfies $I_{a+1} \subseteq I_a$ for all $a \in \mathbb{N}$. Note that we don't require a filtration to be a graded family.

Definition 4.1. Suppose $R = \mathbb{K}[x_0, \ldots, x_N]$ and let $\mathcal{I} = \{I_n\}_{n\geq 1}$ be a filtration of ideals. We say that $\mathcal{I}$ is **differentially closed** if, for every $n \geq 0$, every $D_{\mathbf{a}} \in D_R^{n-1}$, and every $F \in I_n$, $D_{\mathbf{a}}F \in I_{n-|\mathbf{a}|}$.

The following two lemmas follow immediately from our discussion of the R-algebra generators of D_R.

Lemma 4.2. *Suppose $R = \mathbb{K}[x_0, \ldots, x_N]$, where $\mathbb{K}$ has characteristic zero, and $\mathcal{I} = \{I_n\}_{n\geq 1}$ is a filtration of ideals so that for every $n \geq 1$ and every $F \in I_{n+1}$, $\frac{\partial F}{\partial x_i} \in I_n$ for $i = 0, \ldots, N$. Then $\mathcal{I}$ is differentially closed.*

Lemma 4.3. *Suppose $R = \mathbb{K}[x_0, \ldots, x_N]$, where $\mathbb{K}$ has characteristic $p > 0$, and $\mathcal{I} = \{I_n\}_{n\geq 1}$ is a filtration of ideals so that for every $i \in \mathbb{N}$, $n \geq 1 + p^i$ and every $F \in I_n$, $D_{p^i\mathbf{e_j}}F \in I_{n-p^i}$ for $j = 0, \ldots, N$, where $\mathbf{e_j}$ is the jth standard basis vector of $\mathbb{Z}^{N+1}$. Then $\mathcal{I}$ is differentially closed.*

Example 4.4. Let $I \subset R = \mathbb{K}[x_0, \ldots, x_N]$ be a homogeneous ideal. The nth *differential power* of I is

$$I^{<n>} = \{f \in R \mid D_{\mathbf{a}}(f) \in I \text{ for all } D_{\mathbf{a}} \in D_R^{n-1}\}.$$

Every differential power of I is an ideal by [8, Proposition 2.4]. The family $\mathcal{I} = \{I^{<n>}\}_{n=1}^{\infty}$ is clearly a differentially closed graded filtration of ideals.

If $I = I(X)$ is the ideal of a projective variety $X \subset \mathbb{P}^N$ for $\mathbb{K}$ characteristic 0 or a radical ideal for $\mathbb{K}$ a perfect field, the Zariski-Nagata theorem [49,41] and its extension to perfect fields [8, Proposition 2.14] states that the symbolic powers and differential powers of I coincide, that is $I^{(n)} = I^{<n>}$ for $n \geq 1$. In either case, $\mathcal{I} = \{I^{(n)}\}_{n \geq 1}$ is a differentially closed graded filtration. We will see in Example 4.14 that, by using Zariski's main lemma on holomorphic functions [49] instead of the Zariski-Nagata theorem, we can drop the assumption that $\mathbb{K}$ is perfect.

Example 4.5. Suppose $R = \mathbb{K}[x_0, \ldots, x_N]$, $I \subset R$ is any ideal, and $\mathcal{I} = \{I^n\}_{n \geq 1}$ is the graded filtration consisting of powers of I. We prove that $\mathcal{I}$ is differentially closed.

In characteristic 0, $\mathcal{I}$ is differentially closed by Lemma 4.2 and the product rule. To prove that $\mathcal{I} = \{I^n\}$ is a differentially closed graded filtration in arbitrary characteristic, it suffices by Lemma 4.3 to prove that

$$\text{if } f \in I^n \text{ then } D_{k\mathbf{e}_i}(f) \in I^{n-k} \text{ for } k \leq n-1. \tag{4.4}$$

We prove this using the following extension of the product rule for differential operators of the form $D_{k\mathbf{e}_i}$: for any $f, g \in R$

$$D_{k\mathbf{e}_i}(fg) = \sum_{j=0}^{k} D_{j\mathbf{e}_i}(f) D_{(k-j)\mathbf{e}_i}(g). \tag{4.5}$$

We include a proof of this identity in Appendix A. From (4.5) an induction yields

$$D_{k\mathbf{e}_i}(f_1 \cdots f_n) = \sum_{j_1 + \cdots + j_n = k} D_{j_1\mathbf{e}_i}(f_1) D_{j_2\mathbf{e}_i}(f_2) \cdots D_{j_n\mathbf{e}_i}(f_n), \tag{4.6}$$

where the sum runs over non-negative integers $j_1, \ldots, j_n$. To prove (4.4) it suffices, by linearity, to prove it in the case $f = f_1 \cdots f_n$, where $f_i \in I$ for $i = 1, \ldots, n$. Since $k < n$, at least $n - k$ of the indices $j_1, \ldots, j_n$ are zero. Thus each term in (4.6) is a product that includes at least $n - k$ factors in I, and so each term is in I^{n-k}. This proves that $\mathcal{I} = \{I^n\}_{n \geq 1}$ is differentially closed in arbitrary characteristic.

Example 4.6. Suppose $I \subset \mathbb{K}[x_0, \ldots, x_N]$ is an ideal and $\mathbb{K}$ has characteristic p. If $q = p^e$ for some integer $e \geq 0$ then the qth *Frobenius power* of I is the ideal

$$I^{[q]} = \langle f^q : f \in I \rangle.$$

In [30], Hernández, Teixera, and Witt introduce *integral* Frobenius powers

$$I^{[n]} = I^{n_0} I^{n_1[p]} \cdots I^{n_s[p^s]},$$

where n has base p expansion $n = n_0 + n_1 p + \cdots + n_s p^s$ and $I^{a[q]} = (I^a)^{[q]} = (I^{[q]})^a$. Let $\mathcal{I} = \{I^{[n]}\}$. We show that $\mathcal{I}$ is a differentially closed filtration.

First we show that if $q = p^t$ (for any integer $t \geq 0$) then differential operators of order not divisible by q vanish on $I^{[q]}$. By Lemma 4.3 it suffices to show that $D_{k\mathbf{e}_i}(f) = 0$ when $q \nmid k$ and $f \in I^{[q]}$. To this end, suppose that $f = g^q$ for $q = p^t$ and k is a positive integer so that $q \nmid k$. Since f is a linear combination of qth powers of monomials, it suffices to show that $D_{k\mathbf{e}_i}(f) = 0$ when f is a monomial. So suppose that $f = x^{q\mathbf{a}}$, where $\mathbf{a} = (a_0, \ldots, a_N)$. Then $D_{k\mathbf{e}_i}(f) = D_{k\mathbf{e}_i}(x^{q\mathbf{a}}) = \binom{q\alpha_i}{k} x^{q\mathbf{a} - k\mathbf{e}_i}$. Since q does not divide k the base p expansion $k = \sum_{i \geq 0} k_i p^i$ satisfies $k_u \neq 0$ for some $u < t$. On the other hand, the base p expansion $qa_i = \sum_{j \geq 0} a_{ij} p^j$ satisfies $a_{ij} = 0$ for all $j < t$. In particular, $a_{iu} = 0$. By Lucas' identity, $\binom{qa_i}{k} = 0$ and we are done. Notice that since $I^{a[q]} = (I^a)^{[q]}$, this also shows that differential operators of order not divisible by q vanish on $I^{a[q]}$ for any $a \geq 1$.

Next we show that, if $1 \leq k \leq a$ and $f \in I^{a[q]}$, then $D_{kq\mathbf{e}_i}(f) \in I^{(a-k)[q]}$ (where we take $I^0 = R$ by convention). It suffices by linearity to consider the case $f = f_1 \cdots f_a \in I^{a[q]}$, where $f_1, \ldots, f_a \in I^{[q]}$. By (4.6),

$$D_{k\mathbf{e}_i}(f_1 \cdots f_a) = \sum_{j_1 + \cdots + j_a = kq} D_{j_1\mathbf{e}_i}(f_1) D_{j_2\mathbf{e}_i}(f_2) \cdots D_{j_a\mathbf{e}_i}(f_a),$$

and by the previous discussion we may assume that in the sum above $j_1, \ldots, j_a$ are all divisible by q. Hence in each term of the sum above there are $a - k$ factors which are in $I^{[q]}$, thus the entire sum is in $I^{(a-k)[q]}$.

Finally, we induct on the length of the base p expansion of n to show that $D_{j\mathbf{e}_i}(I^{[n]}) \subset I^{[n-j]}$ for $j < n$. If $n < p$ then integral Frobenius powers agree with regular powers and the result follows from Example 4.5. So suppose that $n \geq p$ with base p expansion $n = n_0 + n_1 p + \cdots + n_s p^s$. Put $n' = n - n_s p^s$. Clearly the base p expansion of n' has length at least one less than the base p expansion of n. By definition, $I^{[n]} = I^{[n']} I^{n_s[p^s]}$, so it suffices to show that $D_{j\mathbf{e}_i}(fg) \in I^{[n-j]}$ where $f \in I^{[n']}$ and $g \in I^{n_s[p^s]}$. Put $q = p^s$ and suppose $j = aq + r$ where $0 \leq r < q$ (we must have $a \leq n_s$ since $j < n$). By (4.5),

$$D_{j\mathbf{e}_i}(fg) = \sum_{m=0}^{j} D_{m\mathbf{e}_i}(f) D_{(j-m)\mathbf{e}_i}(g) = D_{r\mathbf{e}_i}(f) D_{aq\mathbf{e}_i}(g),$$

since all differential operators of order not divisible by q vanish on g and all differential operators of order at least q vanish on f. By induction, $D_{r\mathbf{e}_i}(f) \in I^{[n'-r]}$. By the previous argument, $D_{aq\mathbf{e}_i}(g) \in I^{(n_k - a)[q]}$. Since there are no base p carries in the addition $(n' - r) +$

$(n_s - a)q$, we have $I^{[n'-r]}I^{(n_s-a)[q]} = I^{[n'+n_sq-(aq+r)]} = I^{[n-j]}$ by [30, Proposition 3.4]. By Lemma 4.3, this completes the proof that the integral Frobenius powers of an ideal form a differentially closed filtration.

4.3. The inverse system of a differentially closed filtration

Emsalem and Iarrobino made a remarkable observation in [17]: even though the inverse system of an ideal is not finitely generated, one could put together the graded pieces of the inverse systems of successive symbolic powers of an ideal to get an ideal of S or $\mathcal{D}$, respectively. We show that this observation of Emsalem and Iarrobino can be extended to a differentially closed graded filtration of ideals, using the following definition.

Definition 4.7. Suppose that $\mathcal{I} = \{I_n\}_{n\in\mathbb{N}}$ is a filtration of homogeneous ideals. For each integer $s \geq 1$ we define

$$\mathcal{L}^s(\mathcal{I}) := \bigoplus_{d\geq s+1} \left(I_{d-s}^{-1}\right)_d = \bigoplus_{d\geq s+1} \left(I_{d-s}^{\perp}\right)_d .$$

If the graded filtration $\mathcal{I}$ is understood, we write $\mathcal{L}^s$ instead of $\mathcal{L}^s(\mathcal{I})$. If the inverse system is computed using the partial differentiation action of R on S, $\mathcal{L}^s(\mathcal{I})$ is a subspace of S, while if the inverse system is computed using the contraction action of R on $\mathcal{D}$, $\mathcal{L}^s(\mathcal{I})$ is a subspace of $\mathcal{D}$.

If $\mathcal{I} = \{I_n\}_{n\in\mathbb{N}}$ is a graded family of ideals of R, we have defined $\mathcal{L}^s(\mathcal{I})$ so that

$$\left(I_n^{-1}\right)_d = \mathcal{L}^{d-n}(\mathcal{I})_d$$

and hence

$$(I_n)_d \cong \left(\frac{S}{\mathcal{L}^{d-n}(\mathcal{I})}\right)_d$$

or equivalently

$$\left(\frac{S}{\mathcal{L}^s(\mathcal{I})}\right)_d \cong (I_{d-s})_d. \tag{4.7}$$

Example 4.8. Suppose $p = [a_0 : \cdots : a_N] \in \mathbb{P}^N$, let $\mathfrak{m}_p \subset R = \mathbb{K}[x_0, \ldots, x_N]$ be the ideal of homogeneous polynomials vanishing at p, and put $\mathcal{I} = \{\mathfrak{m}_p^n\}_{n\geq 1}$. According to Example 4.5 this is a differentially closed graded filtration.

For the action of R on S by partial differentiation, let $L_p = a_0y_0 + \ldots + a_Ny_N \in S$ be the dual linear form of the point $p \in X$. It follows from Definition 4.7 and Lemma B.1 that

$$\mathcal{L}^s(\mathcal{I}) = \bigoplus_{d\geq 1}(\mathfrak{m}_p^d)^{\perp}_{d+s} = \bigoplus_{d\geq 1}\langle L_p^{s+1}\rangle_{d+s} = \langle L_p^{s+1}\rangle.$$

For the action of R on $\mathcal{D}$ by contraction, let $L_p = a_0Y_0 + \cdots + a_NY_N \in \mathcal{D}$ be the dual linear form of p and put $L_p^{[k]} = \sum_{|\mathbf{b}|=k} a_0^{b_0}\cdots a_N^{b_N}Y^{[\mathbf{b}]}$. It follows from Definition 4.7 and Lemma B.1 that

$$\begin{aligned}\mathcal{L}^s(\mathcal{I}) &= \textstyle\bigoplus_{d\geq 1}(\mathfrak{m}_p^d)^{\perp}_{d+s}\\ &= \textstyle\bigoplus_{d\geq 1}\operatorname{span}\{Y^{[\mathbf{a}]}L_p^{[c]} : s+1\leq c\leq d+s, |\mathbf{a}| = d-c\}\\ &= \langle L_p^{[c]} : c\geq s+1\rangle.\end{aligned}$$

Note that $\mathcal{L}^s(\mathcal{I})$ is an ideal of S (respectively $\mathcal{D}$), although it is not a finitely generated ideal of $\mathcal{D}$ in positive characteristic.

Our main result in this section is that $\mathcal{L}^s$ is an ideal of $\mathcal{D}$ (or S) precisely when $\mathcal{I}$ is a differentially closed filtration of homogeneous ideals.

Theorem 4.9. *Suppose $R = \mathbb{K}[x_0,\ldots,x_N]$ and let $\mathcal{I} = \{I_n\}_{n\geq 1}$ be a filtration of homogeneous ideals. Then $\mathcal{L}^s(\mathcal{I})$ is an ideal of $\mathcal{D}$ (arbitrary characteristic) or S (characteristic 0) if and only if $\mathcal{I}$ is differentially closed.*

In the proof of Theorem 4.9, we will use the following formula which we expect is known to experts. We give a proof of this identity (and others) in Appendix A.

Lemma 4.10. *Suppose $F\in R$ is a homogeneous polynomial and $g\in\mathcal{D}$ is a homogeneous divided power polynomial. In arbitrary characteristic,*

$$F\bullet(Y_j^{[k]}g) = \sum_{i=0}^{k} Y_j^{[k-i]}(D_{i\mathbf{e_j}}(F)\bullet g)$$

for $j = 0,\ldots,N$.

Proof of Theorem 4.9. We prove the result for R acting on $\mathcal{D}$ by contraction. Put $\mathcal{L}^s = \mathcal{L}^s(\mathcal{I})$. Note that $\mathcal{L}^s$ is an ideal of $\mathcal{D}$ if and only if $Y^{[\mathbf{b}]}g\in\mathcal{L}^s$ for every algebra generator $Y^{[\mathbf{b}]}$ of $\mathcal{D}$. In any characteristic, $\mathcal{L}^s$ is an ideal if and only if $Y_j^{[k]}g\in\mathcal{L}^s$ for every $j = 0,\ldots,N$ and any $k\geq 1$ by Lemma 4.2 and Lemma 4.3. Since $\mathcal{L}^s$ is clearly graded, we may assume g is homogeneous, say of degree d. By Definition 4.7, $g\in\mathcal{L}^s_d$ if and only if $g\in\left(I_{d-s}^{-1}\right)_d$. It follows that $\mathcal{L}^s$ is an ideal if and only if $Y_j^{[k]}g\in\mathcal{L}^s_{d+k} = \left(I_{d+k-s}^{-1}\right)_{d+k}$ for all $d\geq s+1$, $0\leq j\leq N$, $k\geq 1$, and $g\in\left(I_{d-s}^{-1}\right)_d$.

Fix a degree d and an index $0\leq j\leq N$. Then $Y_j^{[k]}g\in\left(I_{d+k-s}^{-1}\right)_{d+k}$ for all $g\in\left(I_{d-s}^{-1}\right)_d$ if and only if $F\bullet(Y_j^{[k]}g) = 0$ for every $F\in(I_{d+k-s})_{d+k}$ and $g\in\left(I_{d-s}^{-1}\right)_d$. By Lemma 4.10,

$$\begin{aligned} F \bullet (Y_j^{[k]} g) &= D_{k\mathbf{e}_j}(F) \bullet g + Y_j^{[1]}(D_{(k-1)\mathbf{e}_j}(F) \bullet g) + \cdots + Y_j^{[k]}(F \bullet g) \\ &= D_{k\mathbf{e}_j}(F) \bullet g, \end{aligned}$$

where the final equality follows because $D_{t\mathbf{e}_j}(F)$ has degree at least $d+1$ for $0 \leq t \leq k-1$ and g has degree d. It follows that $Y_j^{[k]} g \in \left(I_{d+k-s}^{-1}\right)_{d+k}$ if and only if $D_{k\mathbf{e}_j}(F) \bullet g = 0$ for all $F \in (I_{d+k-s})_{d+k}$, which is to say $D_{k\mathbf{e}_j}(F) \in (I_{d-s})_d$ for all $F \in (I_{d+k-s})_{d+k}$. Thus $\mathcal{I}$ is differentially closed if and only if $\mathcal{L}^s$ is an ideal.

If R is acting on either S or $\mathcal{D}$ in characteristic 0, the proof can be simplified. The use of Lemma 4.10 can be replaced by Lemma A.2 (for S) or Lemma A.3 (for $\mathcal{D}$). $\square$

Remark 4.11. Our interest is primarily in graded filtrations of homogeneous ideals, so we have stated Definition 4.1 and Theorem 4.9 for a *filtration* of homogeneous ideals. However, Definition 4.1 and Theorem 4.9 only use the hypothesis that $\mathcal{I}$ is a family of *homogeneous* ideals.

4.4. Intersecting differentially closed graded filtrations

In this section we describe how $\mathcal{L}^s(\mathcal{I})$ behaves under intersection of filtrations. This will give us a number of additional examples of families of differentially closed graded filtrations. Suppose A is an index set and $\mathcal{I}_a = \{I_{a,n}\}_{n\in\mathbb{N}}$ is a filtration of ideals of R for each $a \in A$. We write $\cap_{a\in A}\mathcal{I}_a$ for the filtration $\{\cap_{a\in A} I_{a,n}\}_{n\in\mathbb{N}}$.

Proposition 4.12. *Suppose A is an index set and $\mathcal{I}_a$ is a differentially closed graded filtration of ideals for each $a \in A$. Then*

(1) *$\cap_{a\in A}\mathcal{I}_a$ is a differentially closed graded filtration of ideals and*
(2) *$\mathcal{L}^s(\cap_{a\in A}\mathcal{I}_a) = \sum_{a\in A} \mathcal{L}^s(\mathcal{I}_a)$*

Proof. For (1), it is clear that $\cap_{a\in A}\mathcal{I}_a$ is a filtration. We show that it is graded. Given any two positive integers m, n, suppose $f \in (\cap_{a\in A}\mathcal{I}_a)_m = \cap_{a\in A} I_{a,m}$ and $g \in (\cap_{a\in A}\mathcal{I}_a)_n = \cap_{a\in A} I_{a,n}$. Since $\mathcal{I}_a$ is a graded family for every $a \in A$, $fg \in I_{a,m+n}$ for every $a \in A$ and thus $fg \in \cap_{a\in A} I_{a,m+n} = (\cap_{a\in A}\mathcal{I}_a)_{m+n}$. Now we show that $\cap_{a\in A}\mathcal{I}_a$ is differentially closed. Suppose $f \in (\cap_{a\in A}\mathcal{I}_a)_n = \cap_{a\in A} I_{a,n}$ Since $\mathcal{I}_a$ is a differentially closed family for every $a \in A$, $D_{\mathbf{a}}(f) \in I_{a,n-|\mathbf{a}|}$ for every $a \in A$ and $D_{\mathbf{a}} \in D_R^{n-1}$. Therefore $D_{\mathbf{a}}(f) \in \cap_{a\in A} I_{a,n}$.

Now we prove (2). Since the construction of $\mathcal{L}^s$ is accomplished by putting together graded pieces, it suffices to show that

$$\mathcal{L}^s(\cap_{a\in A}\mathcal{I}_a)_d = \sum_{a\in A} \mathcal{L}^s(\mathcal{I}_a)_d,$$

for any integer $d \geq 0$. From Definition 4.7, it suffices to show that

$$(\cap_{a\in A}(I_{a,n})_d)^{\perp}=\sum_{a\in A}(I_{a,n}^{\perp})_d.$$

For any fixed $d\geq 0$, the intersection on the left hand side and the sum on the right hand side need only run over finitely many of the graded filtrations $\{\mathcal{I}_a\}_{a\in A}$ (since the intersection occurs in the finite dimensional vector space R_d and the sum occurs in the finite dimensional vector space S_d). Then the equality follows from the fact that $(U\cap V)^{\perp}=U^{\perp}+V^{\perp}$ for any vector subspaces $U,V\subset R_d$ [20, Lemma 2.7]. □

Example 4.13. Let A be an index set, I_a an ideal of R, and $\{r_{a,n}\}_{n\in\mathbb{N}}$ an increasing subadditive sequence for every $a\in A$. For an ideal $I_a\subset R$, consider the filtration $\mathcal{I}_a=\{I_a^{r_{a,n}}\}_{n\in\mathbb{N}}$. Since $\{r_{a,n}\}_{n\in\mathbb{N}}$ is increasing, this filtration is differentially closed by Example 4.5. It is graded because

$$I_a^{r_{a,i}}I_a^{r_{a,j}}=I_a^{r_{a,i}+r_{a,j}}\subset I_a^{r_{a,i+j}},$$

where the final containment follows because $r_{a,i+j}\leq r_{a,i}+r_{a,j}$. Thus $\{I_a^{r_{a,i}}\}_{n\in\mathbb{N}}$ is a differentially closed graded filtration for every $a\in A$. It follows from Proposition 4.12 that $\mathcal{I}=\cap_{a\in A}\mathcal{I}_a$ is a differentially closed graded filtration, $\mathcal{L}^s(\mathcal{I})$ is an ideal, and $\mathcal{L}^s(\mathcal{I})=\sum_{a\in A}\mathcal{L}^s(\mathcal{I}_a)$. If $\{r_{a,n}\}_{n\in\mathbb{N}}$ is simply an increasing sequence for every $a\in A$, $\mathcal{I}_a$ is a differentially closed filtration, but not necessarily a graded family. The same conclusions still follow, that is, $\mathcal{I}=\cap_{a\in A}\mathcal{I}_a$ is a differentially closed filtration, $\mathcal{L}^s(\mathcal{I})$ is an ideal, and $\mathcal{L}^s(\mathcal{I})=\sum_{a\in A}\mathcal{L}^s(\mathcal{I}_a)$, but we may lose a reciprocity for asymptotic growth factors that we explore in Section 4.5.

Example 4.14. If $\mathbb{K}$ is a field and I is a radical ideal of $R=\mathbb{K}[x_0,\ldots,x_N]$, we show that $\mathcal{I}=\{I^{(n)}\}$ is a differentially closed graded filtration. Let $\operatorname{Max}(R)$ be the collection of maximal ideals of R. According to Zariski's Main Lemma on Holomorphic Functions [49] (see also [8, Theorem 2.12]), $I^{(n)}=\bigcap_{\substack{\mathfrak{m}\in\operatorname{Max}(R)\\ I\subset\mathfrak{m}}}\mathfrak{m}^n$. The conclusion now follows from Example 4.5 and Proposition 4.12.

Example 4.15. In this example we state the main results of [17] in terms of the notation we have introduced. Let $\mathbb{K}$ be an algebraically closed field. We can build on Example 4.14 to compute $\mathcal{L}^s(\mathcal{I})$ where $\mathcal{I}=\{I(X)^{(n)}\}\subset R$ consists of the symbolic powers of the ideal of a projective variety $X\subset\mathbb{P}^N$. For a point $p\in\mathbb{P}^N$, write $\mathfrak{m}_p\subset R$ for the ideal of p. In this context, Zariski's Main Lemma on Holomorphic Functions reads

$$I(X)^{(n)}=\bigcap_{p\in X}\mathfrak{m}_p^n.$$

Put $\mathcal{L}^s(X)=\mathcal{L}^s(\mathcal{I})$. For a point $p\in X$ write $\mathcal{I}_p=\{\mathfrak{m}_p^n\}_{n\in\mathbb{N}}$ and $\mathcal{L}^s(p)=\mathcal{L}^s(\mathcal{I}_p)$. From Example 4.8, $\mathcal{L}^s(p)=\langle L_p^{s+1}\rangle\subset S$ if we consider the action of R on S and $\mathcal{L}^s(p)=\langle L_p^{[c]}:c\geq s+1\rangle$ if we consider the action of R on $\mathcal{D}$.

Proposition 4.12 yields that $\mathcal{L}^s(X) = \sum_{p \in X} \mathcal{L}^s(p)$. Thus we obtain

$$\mathcal{L}^s(X) = \langle L_p^{s+1} : p \in X \rangle$$

for the action of R on S in characteristic 0. In arbitrary characteristic, for the action of R on $\mathcal{D}$, we have

$$\mathcal{L}^s(X) = \langle L_p^{[c]} : p \in X, c \geq s+1 \rangle.$$

Example 4.16. Suppose $\{I_n\}_{n\geq 1}$ is a differentially closed graded filtration in R and $J \subset R$ is an ideal. We leave it to the reader to verify that $\{I_n : J^\infty\}_{n\geq 1}$ is also a differentially closed graded filtration. This gives yet another way to see that symbolic powers are differentially closed, since symbolic powers may be obtained by saturating ordinary powers with respect to an appropriate ideal J, and we have seen in Example 4.5 that ordinary powers form a differentially closed graded filtration.

4.5. Dual sequences for a differentially closed graded filtration

We now return to duality of sequences. One of the sequences we study is the sequence $\alpha(I_n)$ for a graded family of ideals $\{I_n\}_{n\geq 1}$. The next lemma begins our study of the interaction of this sequence with $\mathcal{L}^s(\mathcal{I})$.

Lemma 4.17. *Suppose $\mathcal{I} = \{I_n\}_{n\geq 1}$ is a differentially closed graded filtration of ideals, and put $\alpha_n = \alpha(I_n)$. The following are equivalent:*

- *$S/\mathcal{L}^s(\mathcal{I})$ (respectively $\mathcal{D}/\mathcal{L}^s(\mathcal{I})$) has finite length,*
- *$\alpha_n > n + s$ for all n large enough.*

In particular, $S/\mathcal{L}^s(\mathcal{I})$ has finite length for all $s \geq 1$ if and only if $\{\alpha_n - n\}_{n\in\mathbb{N}}$ is not bounded above.

Proof. Fix a positive integer s and put $\mathcal{L}^s = \mathcal{L}^s(\mathcal{I})$. Suppose $S/\mathcal{L}^s$ has finite length. Then

$$\left(\frac{S}{\mathcal{L}^s(\mathcal{I})}\right)_{n+s} = 0$$

for all n large enough. By (4.7), $(I_n)_{n+s} = 0$ and hence $\alpha(I_n) > n + s$ for all n large enough.

Now suppose $\alpha(I_n) > n + s$. Then $(I_n)_{n+s} = 0$, hence $(S/\mathcal{L}^s(\mathcal{I}))_n = 0$. If this holds for all n large enough, $S/\mathcal{L}^s(\mathcal{I})$ clearly has finite length. The proof for $\mathcal{D}$ is identical. $\square$

Remark 4.18. When $X \subset \mathbb{P}^N$ is a projective variety in characteristic 0, we claim that $S/\mathcal{L}^s(X)$ has finite length if and only if X is non-degenerate (meaning X is not contained in a hyperplane). To see this, note that if X is contained in a hyperplane defined by $\ell = 0$ for some linear form $\ell \in R$, then $\ell^n \in I(X)^{(n)}$ for all $n \in \mathbb{N}$. Since we cannot have $\alpha(I(X)^{(n)}) < n$, we have $\alpha(I(X)^{(n)}) = n$ and $\alpha(I(X)^{(n)}) - n = 0$ for all $n \in \mathbb{N}$. Thus $S/\mathcal{L}^s(X)$ does not have finite length by Lemma 4.17. On the other hand, suppose X is non-degenerate. Then X contains points $p_0, \dots, p_N$ which span $\mathbb{P}^N$. By Example 4.15, $\mathcal{L}^s(X)$ contains the ideal $\langle L_{p_0}^{s+1}, \dots, L_{p_N}^{s+1}\rangle$. Since these are linearly independent, we may change coordinates so that $L_{p_0} = y_0, \dots, L_{p_N} = y_N$. Since $S/\langle y_0^{s+1}, \dots, y_N^{s+1}\rangle$ has finite length, so does $S/\mathcal{L}^s(X)$. In arbitrary characteristic, we also have a similar result that $\mathcal{D}/\mathcal{L}^s(X)$ has finite length if and only if X is non-degenerate. The proof is the same as that in the case of characteristic 0. Notice that by Example 4.15, $\mathcal{L}^s(X)$ contains the ideal $\langle L_{p_0}^{[c]}, \dots, L_{p_N}^{[c]}, c \geq s+1\rangle$, and it is clear that $\mathcal{D}/\langle y_0^{[c]}, \dots, y_N^{[c]}, c \geq s+1\rangle$ has finite length.

The second sequence we will study is the largest non-zero degree of $\mathcal{D}/\mathcal{L}^s(\mathcal{I})$, which we call the *end* of $\mathcal{D}/\mathcal{L}^s(\mathcal{I})$. That is,

$$\operatorname{end}\left(\frac{\mathcal{D}}{\mathcal{L}^s(\mathcal{I})}\right) = \max\left\{d : \left(\frac{\mathcal{D}}{\mathcal{L}^s(\mathcal{I})}\right)_d \neq 0\right\}.$$

Similarly, in characteristic 0, we define $\operatorname{end}(S/\mathcal{L}^s(\mathcal{I}))$ as the largest non-zero degree of this quotient. If $\operatorname{end}(S/\mathcal{L}^s(\mathcal{I})) < \infty$ then it is well known that $\operatorname{end}(S/\mathcal{L}^s(\mathcal{I})) = \operatorname{reg}(S/\mathcal{L}^s(\mathcal{I}))$, where the latter is the Castelnuovo-Mumford regularity of $S/\mathcal{L}^s(\mathcal{I})$.

Remark 4.19. Let $\mathcal{I}$ be a graded family of ideals. The sequence $\beta_s = \operatorname{end}(S/\mathcal{L}^s(\mathcal{I}))$ (respectively $\beta_s = \operatorname{end}(\mathcal{D}/\mathcal{L}^s(\mathcal{I}))$) can be seen to be superadditive by interpreting it as

$$\beta_s = \operatorname{end}(S/\mathcal{L}^s(\mathcal{I})) = \max\{d : (I_{d-s})_d \neq 0\} \text{ by (4.7)}.$$

The containment $(I_{d-s})_d(I_{d'-t})_{d'} \subseteq (I_{d+d'-(s+t)})_{d+d'}$ thus implies $\beta_s + \beta_t \leq \beta_{s+t}$.

Below we give a more refined version of this observation.

Theorem 4.20. *Suppose $\mathcal{I} = \{I_n\}_{n\in\mathbb{N}}$ is a differentially closed graded family of proper homogeneous ideals in R. Put $\alpha_n = \alpha(I_n)$ and $\beta_s = \operatorname{end}(S/\mathcal{L}^s(\mathcal{I}))$ (respectively, $\beta_s = \operatorname{end}(\mathcal{D}/\mathcal{L}^s(\mathcal{I}))$). Assume that the sequence $\{\alpha_n - n\}$ is nondecreasing and not bounded above. Then*

(1) *$\{\alpha_n - n\}_{n\in\mathbb{N}}$ is a nondecreasing subadditive sequence.*
(2) *$\{\beta_s - s\}_{s\in\mathbb{N}}$ and $\{\beta_s\}_{s\in\mathbb{N}}$ are nondecreasing superadditive sequences.*
(3) *$\beta_s - s = (\overrightarrow{\alpha_n - n})_s$*
(4) *$\alpha_n - n = (\overleftarrow{\beta_s - s})_n$*

Write $\widehat{\alpha} = \widehat{\alpha}(\mathcal{I}) = \lim_{n\to\infty} \frac{\alpha_n}{n}$ *and* $\widehat{\beta}(\mathcal{I}) = \lim_{s\to\infty} \frac{\beta_s}{s}$. *Then*

$$\widehat{\beta} = \frac{\widehat{\alpha}}{\widehat{\alpha}-1} \text{ and } \widehat{\alpha} = \frac{\widehat{\beta}}{\widehat{\beta}-1}.$$

Remark 4.21. Under the hypotheses of Theorem 4.20 in characteristic 0, $\{\alpha_n - n\}_{n\geq 1}$ is always nondecreasing, which we can see as follows. Since $\mathcal{I}$ is differentially closed, if $f \in I_n$ is a homogeneous polynomial of degree equal to $\alpha(I_n)$ $(n > 1)$, then $\frac{\partial f}{\partial x_i} \in I_{n-1}$ for $i = 0, \ldots, N$. Thus $\alpha(I_n) < \alpha(I_{n+1})$ unless all partials of f vanish. This happens if and only if f is constant, which is impossible since $\mathcal{I}$ consists of proper ideals. Thus $\alpha(I_{n-1}) - (n-1) \leq \alpha(I_n) - n$, which shows $\{\alpha_n - n\}_{n\geq 1}$ is nondecreasing.

Proof. For (1), the sequence α_n is subadditive by Lemma 3.3. Since n is linear, $\alpha_n - n$ is also subadditive.

We next prove (3). Set $\gamma_n = \alpha_n - n$. By Definition 4.7, we have

$$\begin{aligned} \beta_s &= \operatorname{end}(S/\mathcal{L}^s) \\ &= \max\{d : \alpha(I^{(d-s)}) \leq d\} \\ &= \max\{d - s : \alpha(I^{(d-s)}) - (d-s) \leq s\} + s \\ &= \max\{t : \alpha(I^{(t)}) - t \leq s\} + s \\ &= \overrightarrow{\gamma}_s + s, \end{aligned}$$

which proves (3). Part (4) follows immediately from (3) and Theorem 2.6 (1).

For (2), since $\beta_s - s = (\overrightarrow{\alpha_n - n})_s$ by (3), the definition of the transform $\overrightarrow{\alpha_n - n}$ implies $\{\beta_s - s\}$ is also nondecreasing. That $\{\beta_s - s\}$ is superadditive follows from (3), (1), and Theorem 2.6 (3). Clearly, we have $\beta_s = (\beta_s - s) + s$. Since each of the sequences $\{\beta_s - s\}_{s\in\mathbb{N}}$ and $\mathrm{id} = \{s\}_{s\in\mathbb{N}}$ are nondecreasing and superadditive, the same is true of their sum, $\underline{\beta}$.

Finally we prove the last two equalities, which are clearly equivalent. By Theorem 2.6 (3), the desired result follows by means of the identity

$$\widehat{\beta} - 1 = \lim_{s\to\infty} \frac{\beta_s - s}{s} = \left(\lim_{n\to\infty} \frac{\alpha_n - n}{n}\right)^{-1} = \frac{1}{\widehat{\alpha}-1},$$

which is equivalent to the claims regarding $\widehat{\alpha}$ and $\widehat{\beta}$. The proof is identical for $\beta_s = \operatorname{end}(\mathcal{D}/\mathcal{L}^s(\mathcal{I}))$. $\square$

Corollary 4.22. *With the same setup as Theorem 4.20, we have inequalities*

$$\operatorname{end}\left(\frac{\mathcal{D}}{\mathcal{L}^s(\mathcal{I})}\right) \leq \frac{s\widehat{\alpha}}{\widehat{\alpha}-1} \qquad \text{and} \qquad \frac{n\widehat{\beta}}{\widehat{\beta}-1} \leq \alpha(I_n).$$

The same statements follow with $\mathcal{D}$ *replaced by* S.

Proof. Put $\alpha_n = \alpha(I_n)$ and $\beta_s = \operatorname{end}(\mathcal{D}/\mathcal{L}^s(\mathcal{I}))$. By Theorem 4.20, $\widehat{\beta} = \frac{\widehat{\alpha}}{\widehat{\alpha}-1}$. Since β_s is superadditive, $\frac{\beta_s}{s} \le \widehat{\beta} = \frac{\widehat{\alpha}}{\widehat{\alpha}-1}$. Multiplying both sides by s gives the first inequality. Likewise, by Theorem 4.20, $\widehat{\alpha} = \frac{\widehat{\beta}}{\widehat{\beta}-1}$. Since α_n is subadditive, $\frac{\alpha_n}{n} \ge \widehat{\alpha} = \frac{\widehat{\beta}}{\widehat{\beta}-1}$. Multiplying by n gives the second inequality. The proof is identical for S. □

Example 4.23. Continuing from Example 4.15, we consider the ideal $I(X)$ of a projective variety $X \subset \mathbb{P}^N$ in characteristic 0, the filtration $\mathcal{I} = \{I(X)^{(n)}\}$, and the ideal $\mathcal{L}^s(X) = \langle L_p^{s+1} : p \in X\rangle \subset S$. We assume X is non-degenerate so that $S/\mathcal{L}^s(X)$ has finite length by Example 4.18. Following Theorem 4.20, put $\alpha_n = \alpha(I(X)^{(n)})$, $\beta_s = \operatorname{reg}(S/\mathcal{L}^s(X))$. Then $\widehat{\alpha}$ is the Waldschmidt constant of $I(X)$ (see Definition 3.5). Theorem 4.20 yields that the Waldschmidt constant can be expressed in terms of $\widehat{\beta}$ (and vice-versa). Moreover, Corollary 4.22 yields the bounds

$$\frac{n\widehat{\beta}}{\widehat{\beta}-1} \le \alpha(I(X)^{(n)}) \quad \text{and} \quad \operatorname{reg}\left(\frac{S}{\mathcal{L}^s(X)}\right) \le \frac{s\widehat{\alpha}}{\widehat{\alpha}-1}.$$

The right-hand bound was observed in [12], where it was used to determine a lower bound for the dimension of certain multivariate spline spaces.

5. Asymptotic regularity and the Seshadri constant

Throughout this section we consider a finite set of distinct points $X = \{p_1, \dots, p_r\} \subset \mathbb{P}^N$ and denote by $I(X) \subseteq R = \mathbb{K}[x_0, \dots, x_N]$ the saturated ideal defining X with its reduced scheme structure.

Definition 5.1. The multipoint Seshadri constant for X is the real number

$$\varepsilon(X) = \inf_C \left\{ \frac{\deg(C)}{\sum_{i=1}^r \operatorname{mult}_{p_i} C} \right\}$$

where C is any curve with $C \cap X \neq \emptyset$, $\deg(C)$ is the multiplicity of $R/I(C)$, and $\operatorname{mult}_{p_i} C$ is the multiplicity of C at p_i, that is, the multiplicity of the local ring $(R/I(C))_{P_i}$, where $P_i = I(p_i)$. It suffices in fact to consider irreducible curves in the definition. Since we only consider Seshadri constants of varieties $X \subseteq \mathbb{P}^N$ with respect to the line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$, we suppress this information from the notation.

Seshadri constants were introduced in [9]. For nice expositions of the circle of ideas this has led to in the intervening years see [37, §5.1] or [2].

In this section we establish a limit description for the multipoint Seshadri constant $\varepsilon(X)$. This generalizes a similar result in [37, Theorem 5.1.17] for single point Seshadri constants. Moreover we establish a duality between the sequence of jet separation indices, whose limit is the multipoint Seshadri constant, and the sequence of

Castelnuovo-Mumford regularities of the symbolic powers for the ideal $I(X)$; see Theorem 5.5. We further demonstrate how this duality underpins the well known reciprocity between the Seshadri constant $\varepsilon(X)$ and the asymptotic regularity (alternately termed the s-invariant) of X; see [37, Remark 5.4.3]. Our methods recover this reciprocity relation; see Theorem 5.8 for the specific statement

A relevant sequence for our purposes requires the following definition.

Definition 5.2. Let I be a homogeneous ideal of a standard graded ring R with homogeneous maximal ideal $\mathfrak{m}$ and $d \in \mathbb{N}$. Define the *jet separation sequence* of I by

$$s(I, d) = \sup\{k \in \mathbb{N} \mid \operatorname{reg}(R/I^{(k+1)}) \leq d\}.$$

The terminology "jet separation sequence" is justified by the following notion previously developed in the literature; see [37, Definitions 5.1.15 and 5.1.16] building on the related notion of k-jet ampleness; see [3].

Definition 5.3. A finite set of points $X = \{p_1, \ldots, p_r\} \subset \mathbb{P}^N$ with defining ideals $P_1, \ldots, P_r$ is said to *separate (uniform) k-jets in degree d* if the following map obtained by canonical projection onto each direct summand is surjective

$$\mathbb{K}[x_0, \ldots, x_n]_d \to \bigoplus_{i=1}^{r} \left(\mathbb{K}[x_0, \ldots, x_n]/P_i^{k+1}\right)_d \tag{5.1}$$

We define the *jet separation index* of X in degree d to be the integer

$$s(X, d) = \sup\{k \in \mathbb{N} \mid X \text{ separates } k\text{-jets in degree } d\}.$$

The name coincidence gives an indication that the two notions defined above are related, a fact that we make precise in the next proposition.

Proposition 5.4. *Let X be a finite set of $r \geq 2$ points in $\mathbb{P}^N$ with defining ideal $I(X)$ and $N \geq 2$. Then for each $d \in \mathbb{N}$ the jet separation indices of Definition 5.2 and Definition 5.3 agree, that is, $s(X, d) = s(I(X), d)$.*

Proof. In geometric language the map (5.1) can be written as

$$H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d)/\mathfrak{m}_1^{k+1}) \oplus \cdots \oplus H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d)/\mathfrak{m}_r^{k+1}), \tag{5.2}$$

where $\mathfrak{m}_i$ is the ideal sheaf corresponding to P_i. A necessary and sufficient condition for the surjectivity of (5.2) is $H^1(\mathbb{P}^N, \mathcal{I}^{(k+1)} \otimes \mathcal{O}_{\mathbb{P}^N}(d)) = 0$, where $\mathcal{I}^{(k+1)}$ is the ideal sheaf corresponding to $I(X)^{(k+1)}$. This follows from the long exact sequence in cohomology arising from the short exact sequence of sheaves

$$0 \to \mathcal{I}^{(k+1)} \otimes \mathcal{O}_{\mathbb{P}^N}(d) \to \mathcal{O}_{\mathbb{P}^N}(d) \to \mathcal{O}_{\mathbb{P}^N}(d)/\mathfrak{m}_1^{k+1} \otimes \cdots \otimes \mathcal{O}_{\mathbb{P}^N}(d)/\mathfrak{m}_k^{k+1} \to 0$$

and the vanishing of $H^1(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d))$ due to $N \geq 2$. Expressing regularity in terms of local cohomology (see Definition 3.11) yields

$$\begin{aligned}\operatorname{reg}(R/I(X)^{(k+1)}) &= \operatorname{end} H^1_{\mathfrak{m}}(R/I(X)^{(k+1)}) + 1 = \operatorname{end} H^2_{\mathfrak{m}}(I(X)^{(k+1)}) + 1 \\ &= \min\{d \mid H^1(\mathbb{P}^n, \mathcal{I}(X)^{(k+1)} \otimes \mathcal{O}_{\mathbb{P}^N}(d)) = 0\}.\end{aligned}$$

It follows that $\operatorname{reg}(R/I(X)^{(k+1)}) \leq d$ if and only if (5.1) is surjective in degree d. Thus the claim follows by comparing Definition 5.2 and Definition 5.3. $\square$

We can now relate the jet separation sequence of an ideal with the sequence of regularities of its symbolic powers in the style of section 2.

Theorem 5.5. *Let I be a homogeneous ideal of a graded ring R, set $s_d = s(I, d-1)$ for $d \in \mathbb{N}$, and set $r_k = \operatorname{reg}(I^{(k+1)})$. Then*

(1) *the sequences $\{s_d\}_{d\in\mathbb{N}}$ and $\{r_k\}_{k\in\mathbb{N}}$ are nondecreasing and dual as follows:*

$$s_d = \overrightarrow{r}_d \text{ and } r_k = \overleftarrow{s}_k.$$

(2) *If I is aspCM, the sequence $\{r_k\}_{k\in\mathbb{N}}$ is subadditive and $\{s_d\}_{d\geq r_1}$ is superadditive.*
(3) *In particular the shifted jet separation sequence $s(X)[-1] = \{s(X, d-1)\}_{d\geq \operatorname{reg} I(X)^{(2)}}$ for a finite set of points X in $\mathbb{P}^N$ with $N \geq 2$ is superadditive.*
(4) *If I is aspCM, the asymptotic regularity of I is related to the asymptotic growth of the jet separation sequence by*

$$\lim_{d\to\infty} \frac{s(I,d)}{d} = \widehat{\operatorname{reg}}(I)^{-1}$$

Proof. We have directly from Definition 5.2 that

$$s_d = \sup\{k \mid \operatorname{reg}(R/I^{(k+1)}) \leq d-1\} = \sup\{k \mid \operatorname{reg}(I^{(k+1)}) \leq d\} = \sup\{k \mid r_k \leq d\}.$$

This establishes the first part of claim (1) as well as giving that $\{s_d\}_{d\in\mathbb{N}}$ is nondecreasing. Applying the operator $\leftarrow$ to the identity $s_d = \overrightarrow{r}_d$ and using Theorem 2.6 (1) yields $\overleftarrow{s}_k = \overleftrightarrow{r}_k = r_k$. It follows from the definition of $\overleftarrow{s}_k$ that $\{r_k\}_{k\in\mathbb{N}}$ is nondecreasing.

To establish the remaining claims, recall from Lemma 3.12 that for aspCM I the sequence $\{\operatorname{reg}(I^{(k)})\}_{k\in\mathbb{N}}$ is subadditive and the sequence $\{r_k\}_{k\in\mathbb{N}}$ is the subsequence $\{\operatorname{reg}(I^{(k)})\}_{k\in\mathbb{N}}[1]$. Since $\{r_k\}_{k\in\mathbb{N}}$ is nondecreasing the same is true of $\{\operatorname{reg}(I^{(k)})\}_{k\in\mathbb{N}}$. Thus, Lemma 2.9 (1) allows to conclude that $\{r_k\}_{k\in\mathbb{N}}$ is subadditive.

Having established that $\{r_k\}_{k\in\mathbb{N}}$ is subadditive and that $s_d = \overrightarrow{r}_d$, we deduce that $\{s_d\}_{d\geq r_1}$ is a superadditive sequence of natural numbers by Theorem 2.6 (3). Claim (3) follows from (2) by means of Proposition 5.4.

Claim (4) follows from Theorem 2.6 (4) and part (2) of the current proposition, which yield $\widehat{s} = \widehat{r}^{-1}$. Combining this with identities $\widehat{\operatorname{reg}}(I) = \widehat{r}$ and $\lim_{d\to\infty} \frac{s(I,d)}{d} = \widehat{s}$ deduced from Lemma 2.9 (3), we obtain

$$\lim_{d\to\infty} \frac{s(I,d)}{d} = \widehat{s} = \widehat{r}^{-1} = \widehat{\operatorname{reg}}(I)^{-1}. \quad \square$$

Our next goal is to relate the multipoint Seshadri constant $\varepsilon(X)$ to the asymptotic growth of the jet separation sequence for $I(X)$. For this we will need a multipoint analogue of the well-known Seshadri criterion [37, Theorem 1.4.13], which we include for lack of a suitable reference.

Proposition 5.6 *(Multipoint Seshadri criterion). Consider a finite set of points $X = \{p_1, \dots, p_r\} \subset \mathbb{P}^N$ with $N \geq 2$ and let B be the blowup of $\mathbb{P}^N$ at X with projection map $\mu : B \to \mathbb{P}^N$ and exceptional divisor $E = \sum_{i=1}^r E_i$. Let $H = \mu^*(\mathcal{O}_{\mathbb{P}^N}(1))$. Then the Seshadri constant of Definition 5.1 can be alternatively described as*

$$\begin{aligned}\varepsilon(X) &= \sup\left\{\frac{p}{q} : p, q \in \mathbb{Q}_{>0},\ qH - pE \text{ is ample}\right\}\\ &= \sup\{\lambda \in \mathbb{R} : H - \lambda E \text{ is ample}\}\end{aligned} \tag{5.3}$$

Proof. Temporarily denote $\varepsilon'(X) := \sup\left\{\frac{p}{q} : p, q \in \mathbb{Q}_{>0},\ qH - pE \text{ is ample}\right\}$. For $p, q \in \mathbb{Q}_{>0}$ set $L_{p,q} := qH - pE$ to be a $\mathbb{Q}$-divisor on B. Suppose $L_{p,q}$ is ample and hence nef. Computing the intersection product with the pullback of a curve $C \subset \mathbb{P}^N$ gives

$$L_{p,q} \cdot \mu^*(C) = (qH - pE) \cdot \mu^*(C) = q \deg(C) - p\left(\sum_{i=1}^r \operatorname{mult}_{p_i}(C)\right) \geq 0$$

and hence $\varepsilon(X) \geq \frac{p}{q}$ by Definition 5.1. We conclude that $\varepsilon(X) \geq \varepsilon'(X)$.

Conversely, suppose $\frac{p}{q} \leq \varepsilon(X)$. We show that $L_{p,q}$ is nef. If D is a curve in B, then $D = D_1 + D_2$ with D_1 contained in E and D_2 not contained in E (we allow the possibility that $D_1 = 0$ or $D_2 = 0$). We have that $E \cdot D_1 = -\deg(N_{E/B}|_{D_1}) \leq 0$ (where $N_{E/B}$ is the normal bundle of the exceptional divisor) and $\mu(D_2) = C$ is a curve in $\mathbb{P}^N$, thus by a computation similar to the above display we conclude from $\frac{p}{q} \leq \varepsilon(X)$ that $L_{p,q} \cdot D_2 \geq 0$. Therefore we have

$$L_{p,q} \cdot D = (qH - pE) \cdot D = -pE \cdot D_1 + L_{p,q} \cdot D_2 \geq 0,$$

which shows $L_{p,q}$ is nef. Suppose now that $\frac{p}{q} < \varepsilon(X)$. By [29, Section II, Proposition 7.10] the divisor $L_{1,d} = dH - E$ is ample for $d \in \mathbb{N}, d \gg 0$. Fix such a d. Since $\frac{p}{q} < \varepsilon(X)$ and the expression $\frac{p-\delta}{q-\delta d}$ is a continuous function of $\delta \in \mathbb{R}_{>0}$ one can find $\delta \in \mathbb{Q}, \delta > 0$ so that $\frac{p-\delta}{q-\delta d} \leq \varepsilon(X)$. Then the identity

$$L_{p,q} = L_{p-\delta,q-\delta d} + \delta L_{1,d}$$

shows that $L_{p,q}$ is ample since $L_{p-\delta,q-\delta d}$ is nef by the above considerations, $\delta > 0$, and $L_{1,d}$ is ample; see [37, Corollary 1.4.10]. We have obtained that $\varepsilon'(X) \le \varepsilon(X)$, hence the first equality in (5.3) is established. The second equality follows from the first noting that $L_{p.q}$ is ample if and only if $L_{1,p/q} = H - \frac{p}{q}E$ is ample and hence the last set in the display (5.3) is the closure of the first in the topology on $\mathbb{R}$. $\square$

The following is a multipoint version of [37, Theorem 5.1.17]. Our proof follows the single point case closely, however the prior knowledge that the limit in the statement exists as a consequence of Theorem 5.5 allows for slight simplifications.

Theorem 5.7. *If X is a finite set of r points in $\mathbb{P}^N$, $N \ge 2$, the limit of the jet separation index sequence exists and is equal to the multi-point Seshadri constant*

$$\varepsilon(X) = \lim_{d\to\infty} \frac{s(X,d)}{d}.$$

Proof. Set $d_0 = \operatorname{reg}(I(X)^{(2)})$. By Theorem 5.5 (3) we have that $\underline{s} = \{s(X, d-1)\}_{d\ge d_0}$ is a superadditive sequence of natural numbers. Using Lemma 2.9 (3) applied to $s[1] = \{s(X,d)\}_{d\ge d_0-1}$, we see that $\lim_{d\to\infty} \frac{s(d,X)}{d}$ exists.

Suppose $X = \{p_1, \dots, p_r\}$ with $I(p_i) = P_i$. Let C be an irreducible curve that contains at least one point $p_{i_0} \in X$. Assume $d \ge d_0 - 1$ and set $k = s(X,d)$. Take $\overline{F_i} \in P_i^k/P_i^{k+1}$, one for each $p_i \in X$, so that the image of $\overline{F_{i_0}}$ in $\frac{{P_{i_0}}^k}{P_{i_0}^{k+1}} \otimes_R \frac{R}{I(C)}$ is nonzero. This is possible since

$$\frac{{P_{i_0}}^k}{P_{i_0}^{k+1}} \otimes_R \frac{R}{I(C)} \cong \frac{\overline{P_{i_0}}^k}{\overline{P_{i_0}}^{k+1}} \neq 0, \text{ where } \overline{P_{i_0}} = \frac{P_{i_0}}{I(C)},$$

in view of $\overline{P_{i_0}}^k \neq \overline{P_{i_0}}^{k+1}$ by Krull's intersection theorem. Due to the surjectivity of the map in equation (5.1) recalled below

$$\mathbb{K}[x_0, \dots, x_n]_d \to \bigoplus_{i=1}^r \left(\mathbb{K}[x_0,\dots,x_n]/P_i^{k+1}\right)_d,$$

there exists $F \in R_d$ that maps to the tuple $(\overline{F_1}, \dots, \overline{F_r})$. Since $\overline{F_i} \in P_i^k/P_i^{k+1}$, we have $F \in \bigcap_{i=1}^r P_i^k = I(X)^{(k)}$. Since $\overline{F_{i_0}}$ is nonzero modulo $I(C)$ we have $F \in I(X)^{(k)} \setminus I(C)$. Since $I(C)$ is prime, F is regular on $R/I(C)$ and thus the associativity formula for multiplicities provides an inequality

$$\deg(C) \cdot d = e\left(\frac{R}{I(C)+F}\right) \ge \sum_{i=1}^r \operatorname{mult}_{p_i} C \cdot \operatorname{mult}_{p_i} F \ge k \left(\sum_{i=1}^r \operatorname{mult}_{p_i} C\right).$$

We have shown for each curve C with $C \cap X \neq \emptyset$ and each $d \geq d_0 - 1$ that the following inequality holds

$$\frac{\deg(C)}{\sum_{i=1}^{r} \operatorname{mult}_{p_i} C} \geq \frac{s(X,d)}{d}, \text{ thus } \varepsilon(X) \geq \frac{s(X,d)}{d}.$$

Taking limits we deduce

$$\varepsilon(X) \geq \lim_{d\to\infty} \frac{s(d,X)}{d}. \tag{5.4}$$

It remains to establish the opposite inequality to (5.4). For this, fix integers p, q with $0 < \frac{p}{q} < \varepsilon(X)$ and let $\mu : B \to \mathbb{P}^N$ be the blow up of $\mathbb{P}^N$ at X, with exceptional divisor E and $H = \mu^*(\mathcal{O}(1))$ as in Proposition 5.6. Then $L_{p,q} = qH - pE$ is ample by the aforementioned result. By asymptotic Serre vanishing [29, Chapter III, Proposition 5.3] we have that there exists $m_0 \in \mathbb{N}$ so that

$$H^1(B, \mathcal{O}_B(mL_{p,q})) = 0 \text{ for } m \geq m_0.$$

The leftmost cohomology group is in turn isomorphic to the one listed below, by [37, Lemma 4.3.16], where $\mathcal{I}^{(k+1)}$ is the ideal sheaf corresponding to $I(X)^{(k+1)}$. Its vanishing

$$H^1(\mathbb{P}^n, \mathcal{I}^{(mp)} \otimes \mathcal{O}_{\mathbb{P}^N}(mq)) = 0$$

indicates via the definition of regularity in terms of local cohomology with respect to the maximal homogeneous ideal $\mathfrak{m}$ or $R = k[\mathbb{P}^N]$ (see Definition 3.11) that

$$\begin{aligned}
\operatorname{reg}(I^{(mp)}) - 1 &= \operatorname{reg}(R/I^{(mp)}) = \operatorname{end} H^1_{\mathfrak{m}}(R/I^{(mp)}) + 1 = \operatorname{end} H^2_{\mathfrak{m}}(I^{(mp)}) + 1 \\
&= \min\{d \mid H^1(\mathbb{P}^n, \mathcal{I}^{(mp)} \otimes \mathcal{O}_{\mathbb{P}^N}(d)) = 0\} \leq mq, \\
\text{so} \qquad \frac{\operatorname{reg}(I^{(mp)})}{mp} &< \frac{mq}{mp} = \frac{q}{p} \text{ for } m \gg 0.
\end{aligned}$$

Taking the limit as $m \to \infty$ we obtain $\widehat{\operatorname{reg}}(I) \leq \frac{q}{p}$. Equivalently, by Theorem 5.5 it follows that

$$\lim_{d\to\infty} \frac{s(X,d)}{d} = \widehat{\operatorname{reg}}(I)^{-1} \geq \frac{p}{q}.$$

Replacing $\frac{p}{q}$ by a sequence of rational numbers that converges to $\varepsilon(X)$ shows that $\lim_{d\to\infty} \frac{s(X,d)}{d} \geq \varepsilon(X)$ and completes the proof. $\square$

The following corollary recovers a particular instance of the well-known reciprocity between the Seshadri constant and the asymptotic regularity noted in [7, Remark 1.3 and Theorem B]. Our main contribution here is to show that this reciprocity holds for a very precise structural reason, that is, the duality of the sequences in Theorem 5.5.

Corollary 5.8. *The asymptotic regularity of a finite set X of $r \geq 2$ points in $\mathbb{P}^N$ with $N \geq 2$ is the reciprocal of the Seshadri constant. In symbols, we have*

$$\widehat{\operatorname{reg}}(I(X)) = \varepsilon(X)^{-1}.$$

Proof. Theorem 5.5 (5) together with Proposition 5.4 and Theorem 5.7 yields the desired conclusion

$$\widehat{\operatorname{reg}}(I(X)) = \left(\lim_{d\to\infty} \frac{s(I(X),d)}{d}\right)^{-1} = \left(\lim_{d\to\infty} \frac{s(X,d)}{d}\right)^{-1} = \varepsilon(X)^{-1}. \quad \square$$

6. Homological reformulations of the Nagata–Iarrobino conjecture

In the following we refer to a *very general set of r points* in $\mathbb{P}^N$ to mean outside countably many proper subvarieties of the symmetric product $\operatorname{Sym}^r(\mathbb{P}^N)$ of $\mathbb{P}^N$.

In [41] Nagata established the upper bound $\widehat{\alpha}(I(X)) \leq \sqrt{r}$ for any set X of $r \geq 9$ very general points in $\mathbb{P}^2$. Note that this upper bound also holds true for all sets of points (see, e.g., [27, Example 1.3.7]). Nagata also proposed, in different language, the following conjecture to the effect that very general sets of points attain the maximum value of the Waldschmidt constant permitted by this inequality.

Conjecture 6.1 *(Nagata). Any set X of $r \geq 10$ very general points in $\mathbb{P}^2$ over a field of characteristic zero satisfies $\alpha(I(X)^{(n)}) > n\sqrt{r}$ for all $n \in \mathbb{N}$. Equivalently, there is an equality*

$$\widehat{\alpha}(I(X)) = \sqrt{r}.$$

This statement holds true for r a perfect square, by Nagata's work in [41], but it remains open for all other values of $r \geq 10$. We comment on the equivalence of the two claims in the above conjecture. For $\sqrt{r} \notin \mathbb{N}$ (the case that is still open), the conjectured inequality for initial degrees in Conjecture 6.1 is equivalent to $\alpha(I(X)^{(n)}) \geq n\sqrt{r}$. Utilizing the known upper bound $\widehat{\alpha}(I_X) \leq \sqrt{r}$ and the description of the Waldschmidt constant as an infimum (see Definition 3.5), we see that the two statements in Conjecture 6.1 are indeed equivalent.

Notable advances on Conjecture 6.1 have been made in [48,45,25,26], however in its full generality it currently seems out of reach. See [5] for further information and possible generalizations of this long-standing conjecture. Conjecture 6.1 can be equivalently reformulated in terms of the Seshadri constant as

$$\varepsilon(X) = \frac{1}{\sqrt{r}}. \tag{6.1}$$

The inequality $\varepsilon(X) \leq 1/\sqrt{r}$ is known to hold in $\mathbb{P}^2$; this is equivalent to the known upper bound $\widehat{\alpha}(I_X) \leq \sqrt{r}$ by the arguments in the proof of Proposition 6.5. Below we use this equivalence to give further equivalent homological formulations of Nagata's conjecture. Intuitively, in homological terms this conjecture becomes the statement that the width of the Betti table of the symbolic powers of $I(X)$ grows sub-linearly.

Conjecture 6.2. *Any set X of $r \geq 10$ very general points in $\mathbb{P}^2$ over a field of characteristic zero satisfies*

$$\widehat{\alpha}(I(X)) = \widehat{\operatorname{reg}}(I(X)), \qquad \text{equivalently} \qquad \lim_{n\to\infty} \frac{\operatorname{reg}(I(X)^{(n)}) - \alpha(I(X)^{(n)})}{n} = 0.$$

Iarrobino [34] generalized Conjecture 6.1 to projective spaces of arbitrary dimension.

Conjecture 6.3 *(Iarrobino). A set X of r very general points in the projective space $\mathbb{P}^N$ over a field of characteristic zero with $r \geq \max\{N+5, 2^N\}$ and $(r, N) \notin \{(7,2),(8,2),(9,3)\}$ satisfies $\alpha(I(X)^{(n)}) \geq n\sqrt[N]{r}$ for all $n \in \mathbb{N}$. Equivalently, apart from the given list of exceptions, there is an equality*

$$\widehat{\alpha}(I(X)) = \sqrt[N]{r}.$$

Conjecture 6.3 is known to hold only for the case $r = s^N$; see [18].

We use our results to reformulate Conjecture 6.3 in homological terms using inverse systems.

Conjecture 6.4. *Under the hypotheses of Conjecture 6.3 the following holds*

$$\lim_{s\to\infty} \frac{\operatorname{reg}(S/\mathcal{L}^s(X))}{s} = \frac{\sqrt[N]{r}}{\sqrt[N]{r}-1},$$

where $\mathcal{L}^s(X) = \langle L_{p_1}^{s+1}, \ldots, L_{p_r}^{s+1}\rangle \subset S = \mathbb{K}[y_0, \ldots, y_N]$.

Proposition 6.5. *Conjectures 6.3 and 6.4 are equivalent. Moreover, Conjectures 6.1 and 6.2 are equivalent.*

Proof. The equivalence of Conjecture 6.4 to Conjecture 6.3 follows immediately from the duality of asymptotic invariants given by Theorem 4.20.

Now we show the equivalence of Conjectures 6.1 and 6.2. From Definition 5.1 one sees that there is an inequality relating the Waldschmidt constant, and the Seshadri constant

$$\widehat{\alpha}(X) \geq r\varepsilon(X) \tag{6.2}$$

Indeed, let C be a curve in $\mathbb{P}^2$ with $\deg(C) = \alpha(I^{(n)})$ and $\operatorname{mult}_{p_i} C = n$ for each i. Then one has $\varepsilon(X) \leq \frac{\alpha(I^{(n)})}{nr}$ by the definition of $\varepsilon(X)$ and the inequality follows by passing to the limit. While equality need not hold in (6.2) in general, remarkably equality does hold for a very general set of points X; see [4, Lemma 2.3.1]; thus under the hypotheses of our conjectures we have $\widehat{\alpha}(I(X)) = r\varepsilon(X)$. This justifies the equivalence of Conjecture 6.1 asserting $\widehat{\alpha}(I(X)) = \sqrt{r}$ and the identity (6.1) mentioned above.

Rewriting the identity $\widehat{\alpha}(I(X)) = r\varepsilon(X)$ using Corollary 5.8 yields

$$\widehat{\alpha}(I(X)) \cdot \widehat{\operatorname{reg}}(I(X)) = r.$$

It follows that the claim $\widehat{\alpha}(I(X)) = \sqrt{r}$ of Conjecture 6.1 is equivalent to $\widehat{\operatorname{reg}}(I(X)) = \sqrt{r}$ and also equivalent to $\widehat{\alpha}(I(X)) = \widehat{\operatorname{reg}}(I(X))$. The second claim of Conjecture 6.2 follows from feeding the definitions of these asymptotic invariants into the equality. □

Example 6.6. Here we illustrate some of the exceptions to Conjecture 6.3 and Conjecture 6.4. By contrast to Conjecture 6.3, which predicts irrational values for the Waldschmidt constant whenever $\sqrt[N]{r} \notin \mathbb{N}$, the Waldschmidt constant and the asymptotic regularity for sets X of few general points in $\mathbb{P}^N$ are given by rational functions in the number of points, in particular they are rational numbers given by

$$\widehat{\alpha}(I_X) = \begin{cases} \frac{r}{r-1} & \text{if } \#X = r \leq N+1, \\ \frac{r}{r-2} & \text{if } \#X = r = N+2, \\ \frac{r-1}{r-3} & \text{if } \#X = r = N+3 \text{ is even}, \\ \frac{r(r-2)}{r^2-4r+2} & \text{if } \#X = r = N+3 \text{ is odd}. \end{cases}$$

See [14, Proposition B.1.1] for the last three cases and [42, Proposition 5.1] for a more general result in this direction. Utilizing the formulas in Theorem 4.20 we obtain the asymptotic growth factor for the regularity of the inverse systems $\mathcal{L}^s(Z)$

$$\lim_{s\to\infty} \frac{\operatorname{reg}(S/\mathcal{L}^s(X))}{s} = \begin{cases} r & \text{if } \#X \leq N+1, \\ r/2 & \text{if } \#X = r = N+2, \\ (r-1)/2 & \text{if } \#X = r = N+3 \text{ is even}, \\ r(r-2)/2(r-1) & \text{if } \#X = r = N+3 \text{ is odd}. \end{cases}$$

The same result can be derived from [42, Theorem 4.4 and Theorem 4.7].

7. Closing comments and invitations for future work

We close with a number of questions which arose in the process of our writing. The first two questions concern the subadditivity of sequences associated to the symbolic powers of an ideal. We saw in section 3 that if $\nu : R \to \mathbb{Z}$ is an R-valuation then the sequence $\nu(I^{(n)})$ is subadditive for any ideal $I \subset R$, and we relate sequences of this form to the *resurgence* $\rho(I)$ and *asymptotic resurgence* $\widehat{\rho}(I)$. In section 3.2 we define

$$\lambda_n(I) = \max\{d : I^{(d)} \not\subseteq I^n\} \quad \text{and} \quad \beta_n(I) = \max\{d : I^{(d)} \not\subseteq \overline{I^n}\},$$

where $\overline{I^n}$ is the integral closure of I^n. We have examples of ideals I where $\lambda_n(I)$ is not superadditive since $\rho(I) = \lim_{n\to\infty} \frac{\lambda_n}{n} \neq \sup\{\lambda_n/n\} = \widehat{\rho}(I)$. However, the sequence β_n necessarily satisfies $\lim_{n\to\infty} \frac{\beta_n}{n} = \sup\{\beta_n/n\} = \widehat{\rho}(I)$ by Proposition 3.10, which is one of the properties of a superadditive sequence. Thus it seems natural to ask if $\beta_n(I)$ is a superadditive sequence.

Question 7.1. *For an ideal I in a regular ring R, is the sequence $\beta_n(I) = \max\{d : I^{(d)} \not\subseteq \overline{I^n}\}$ a superadditive sequence?*

If I is an ideal so that Question 7.1 has a negative answer, then the failure of containment $I^{(d)} \not\subseteq \overline{I^n}$ is necessarily detected by *different* valuations as n increases, which is an interesting behavior.

Our next question concerns the (Castelnuovo-Mumford) regularity of symbolic powers. If all symbolic powers of an ideal I are Cohen-Macaulay, Lemma 3.12 shows that $\operatorname{reg}(I^{(n)})$ is a subadditive sequence, while Example 3.16 shows that this sequence may not be subadditive even if I is a squarefree monomial ideal. This example is not so far from being subadditive, however, which leads us to the following question.

Question 7.2. *For a radical ideal I in a polynomial ring, is the sequence $\operatorname{reg}(I^{(n)}) + K$ a subadditive sequence for some appropriate integer K? In particular, is this true if K is the number of variables in the polynomial ring?*

In Example 3.16, a calculation shows that $\operatorname{reg}(J(m,s)^{(t)}) + K$ is subadditive for any $K \geq (m-2)(s-1)$; the number of variables in the ambient polynomial ring is $m(s+1)$.

Our next question concerns the differentially closed graded filtrations of ideals introduced in section 4. If $\mathcal{I} = \{I_n\}_{n\geq 1}$ is a differentially closed graded filtration of ideals in R, we found in Theorem 4.20 a duality between the sequences $\alpha_n = \alpha(I_n)$ and $\beta_r = \operatorname{end}(\mathcal{D}/\mathcal{L}^r(\mathcal{I}))$ (with the contraction operation) or $\beta_r = \operatorname{reg}(S/\mathcal{L}^r)$ (with the differentiation action). This duality of sequences arose from Macaulay-Matlis duality. Following the discussion of section 3, we note that $\alpha(I_n)$ is a special case of the sequence $\nu(I_n)$ for an R-valuation $\nu : R \to \mathbb{Z}$. In general, $\nu(I_n)$ is subadditive and, in case $I_n = I^{(n)}$ for a fixed ideal I, its asymptotic growth factor can be used to bound or find the asymptotic resurgence of I (Proposition 3.8 and Proposition 3.10). With this in mind, we ask the following open-ended question.

Question 7.3. *Suppose $\mathcal{I} = \{I_n\}_{n\geq 1}$ is a differentially closed graded family. Does Macaulay-Matlis duality give a meaningful algebraic interpretation for the sequence $\overrightarrow{\nu(I_n)}$ for an arbitrary valuation $\nu : R \to \mathbb{Z}$, extending Theorem 4.20? If not, can the valuation ν be used to twist Macaulay-Matlis duality in a way that does give a meaningful interpretation of $\overrightarrow{\nu(I^{(n)})}$? As in Theorem 4.20, we likely need to shift the sequence $\nu(I_n)$ appropriately.*

If $\mathcal{I} = \{I_n\}$ consists of the symbolic powers of a radical ideal over an algebraically closed field, then Emsalem and Iarrobino [17] give a concrete description for the ideal $\mathcal{L}^s(\mathcal{I})$. Inspired by their description, we pose the following question.

Question 7.4. *If $\mathcal{I} = \{I_n\}_{n\geq 1}$ is a differentially closed graded filtration, under what conditions can we give a concrete description of the generators of $\mathcal{L}^s(\mathcal{I})$? Under what conditions do the generators have a geometric interpretation?*

From the end of section 4, we have a large pool of differentially closed graded filtrations for which we can ask Question 7.4.

If $\mathcal{I} = \{I_n\}_{n\geq 1}$ is a graded family of monomial ideals, then one may associate to $\mathcal{I}$ its *Newton-Okounkov body* [24]. For instance, if $\mathcal{I}$ consists of the symbolic powers of a monomial ideal I, the Newton-Okounkov body of $\mathcal{I}$ is the *symbolic polyhedron* introduced in [6]. It is natural to ask if there is an appropriate *dual body* for the family $\mathcal{L}^s(\mathcal{I})$. We plan to address aspects of the following question in an upcoming paper.

Question 7.5. *If $\mathcal{I}$ is a differentially closed graded family of monomial ideals, is there an associated convex body which encodes the monomials not in $\mathcal{L}^s(\mathcal{I})$? If so, when do these convex bodies limit to a polyhedron (like the symbolic polyhedron)? In what situations can we determine the bounding inequalities?*

In section 6, we saw a number of reformulations of the Nagata conjecture concerning the Waldschmidt constant of at least 10 very general points in $\mathbb{P}^2$. Conjecture 6.2 rephrases this conjecture as an equality of the Waldschmidt constant with the asymptotic regularity. We ask which varieties X satisfy sub-linear growth for the width of the Betti table of $I(X)^{(n)}$.

Question 7.6. *What varieties X can Conjecture 6.2 be extended to? That is, for what varieties X do we have the equality $\widehat{\alpha}(I(X)) = \widehat{\operatorname{reg}}(I(X))$?*

Let $\omega(I^{(n)})$ be the largest degree of a generator of $I^{(n)}$. We always have $\alpha(I^{(n)}) \leq \omega(I^{(n)}) \leq \operatorname{reg}(I^{(n)})$. If $I = I(X)$ is the ideal of a variety answering Question 7.6 positively, then $\omega(I^{(n)}) - \alpha(I^{(n)})$ must also grow sub-linearly. There are many ideals for which it is known that $\operatorname{reg}(I^{(n)})$ differs from $\omega(I^{(n)})$ by a constant independent of n - for instance *star configurations of hypersurfaces* [40]. However, in the case of star configurations of hypersurfaces, $\omega(I^{(n)}) - \alpha(I^{(n)})$ does *not* grow in a sublinear fashion, hence star configurations of hypersurfaces do not satisfy $\widehat{\alpha}(I) = \widehat{\operatorname{reg}}(I)$.

If I is a monomial ideal, then [6] shows that $\widehat{\alpha}(I)$ is the minimum sum of the coordinates of a vertex of the symbolic polyhedron of I, while [15, Theorem 1.3] shows that $\widehat{\operatorname{reg}}(I)$ is the maximum sum of the coordinates of a vertex of the symbolic polyhedron of I. Thus Question 7.6 has a positive answer for a monomial ideal precisely when all vertices of the symbolic polyhedron have the same coordinate sum. More concretely, Question 7.6 has a positive answer for any monomial ideal $I = \langle x^{\alpha_1}, \ldots, x^{\alpha_n}\rangle$ whose symbolic polyhedron has a unique maximal bounded face (under inclusion) which can be described as both:

- The convex hull of the vertices of the symbolic polyhedron or
- The intersection of the symbolic polyhedron with a hyperplane of the form $|\alpha| = c$ for some rational number c.

For instance, both bullet points are satisfied if I is the edge ideal of a bipartite graph (in this case it is known that the ordinary and symbolic powers coincide [22]). More generally both bullet points are satisfied if I is a monomial ideal generated in a single degree and $I^{(n)} = \overline{I^n}$ for all $n \geq 1$ (for squarefree monomial ideals this is also related to the *packing problem* [8]). We are not aware of an algebraic characterization for those monomial ideals which have a symbolic polyhedron whose vertices all have the same coordinate sum.

Remark 7.7. If I is a squarefree monomial ideal which satisfies the two bullet points above, then we can show that I is generated in a single degree and that the number c in the second bullet point above is precisely the generating degree of I. To prove this, we need only show that for a squarefree ideal I there is at least one generator of I whose exponent vector is a vertex of the symbolic polyhedron $\mathrm{SP}(I)$.

Recall that if I is squarefree then there are monomial prime ideals $P_0, \ldots, P_k \subset R = \mathbb{K}[x_0, \ldots, x_N]$ such that $P_i \not\subset P_j$ for any $1 \leq i, j \leq k$, and $I = P_0 \cap \cdots \cap P_k$. Take a generator of I which has minimal support; re-indexing the variables if necessary we may suppose that $M = x_0 \ldots x_t$ is the product of the first $t+1$ variables of R. Since M has minimal support among generators of I, the monomial $M_i = M/x_i$ is not in I for any $i = 0, \ldots, t$. Re-ordering the primes $P_0, \ldots, P_k$ if necessary, we may assume that $M_i \notin P_i$ for $i = 0, \ldots, t$. This implies that P_i is generated by x_i and some subset of the variables $\{x_{t+1}, \ldots, x_N\}$ for $i = 0, \ldots, t$. Recall that the defining inequalities of $\mathrm{SP}(I)$ are given by $\sum_{x_i \in P_j} a_i \geq 1$ for $j = 1, \ldots, k$ and $a_i \geq 0$ for $i = 0, \ldots, N$. Consider the system of equations given by $a_{t+1} = \ldots = a_N = 0$ and $\displaystyle\sum_{x_j \in P_i} a_j = 1$, $i = 0, \ldots, t$. Since $x_i \in P_i$ and P_i is generated by a subset of $\{x_i, x_{t+1}, \ldots, x_N\}$, this system has a unique solution $a_i = 1$, $i = 0, \ldots, t$ and $a_{t+1} = \ldots = a_N = 0$. This is a vertex of $\mathrm{SP}(I)$ and is clearly the exponent vector of the monomial M, completing the proof.

In this paper we have explored the sequence duality of Definition 2.1 in the context of initial degree and regularity of symbolic powers (and more generally, differentially closed filtrations). We close with the following invitation to the reader.

Question 7.8. *In what other algebraic-geometric contexts do subadditive and superadditive sequences naturally appear? For each such sequence, is there a meaningful algebraic interpretation for the dual sequence of Definition 2.1?*

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Appendix A. Formulas involving differentiation and contraction

In this appendix we collect, for the convenience of the reader, proofs of some of the formulas that we use in Section 4. Let $R = \mathbb{K}[x_0, \ldots, x_N]$ be a polynomial ring, D_R the ring of $\mathbb{K}$-linear differential operators on R, $\mathcal{D}$ the divided power algebra on the divided power monomials $Y^{[\mathbf{a}]}$, with $\mathbf{a} \in \mathbb{Z}_{\geq 0}^{N+1}$, and S the polynomial ring $\mathbb{K}[y_0, \ldots, y_N]$. We have the action of R on $\mathcal{D}$ by contraction, written $\bullet$, and R on S by partial differentiation, written $\circ$. First we prove the higher order product rule (4.5).

Lemma A.1. *Let $f, g \in R$, i be an integer between 0 and N, and $k \geq 1$ an integer. In characteristic 0 we have*

$$\frac{\partial^k (fg)}{\partial x_i^k} = \sum_{j=0}^{k} \binom{k}{j} \frac{\partial^j f}{\partial x_i^j} \frac{\partial^{k-j} g}{\partial x_i^{k-j}}$$

In arbitrary characteristic we have

$$D_{k\mathbf{e}_i}(fg) = \sum_{j=0}^{k} D_{j\mathbf{e}_i}(f) D_{(k-j)\mathbf{e}_i}(g).$$

Proof. The first formula follows from induction and the ordinary product rule. It also follows from the second via the identification $D_{\mathbf{a}} = \frac{1}{\mathbf{a}!}\frac{\partial^{\mathbf{a}}}{\partial x^{\mathbf{a}}}$, so we prove the second. We start by proving the formula for $f = x_i^m$ and $g = x_i^n$ where m, n are integers. Then $D_{k\mathbf{e}_i}(x^{m+n}) = \binom{m+n}{k} x_i^{m+n-k}$ and

$$\begin{aligned}
\sum_{j=0}^{k} D_{j\mathbf{e}_i}(x_i^m) D_{(k-j)\mathbf{e}_i}(x_i^n) &= \sum_{j=0}^{k} \binom{m}{j} x_i^{m-j} \binom{n}{k-j} x_i^{n-k+j} \\
&= \left(\sum_{j=0}^{k} \binom{m}{j} \binom{n}{k-j} \right) x_i^{m+n-k} \\
&= \binom{m+n}{k} x_i^{m+n-k},
\end{aligned}$$

where in the identity above, if either $j > m$ or $k - j > n$, we interpret $x_i^{m-j} = 0$ or $x_i^{n-k+j} = 0$, respectively. The binomial coefficients are also interpreted in this way.

Now suppose that $f = x^{\mathbf{a}} = x_i^m x^{\mathbf{a}'}$ and $g = x^{\mathbf{b}} = x_i^n x^{\mathbf{b}'}$ for $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_{\geq 0}^{N+1}$, where $x^{\mathbf{a}'}$ and $x^{\mathbf{b}'}$ are not divisible by x_i. Then $D_{k\mathbf{e}_i}(fg) = x^{\mathbf{a}'+\mathbf{b}'} D_{k\mathbf{e}_i}(x_i^{m+n})$ and likewise $D_{j\mathbf{e}_i}(x^{\mathbf{a}}) D_{(k-j)\mathbf{e}_i}(x^{\mathbf{b}}) = x^{\mathbf{a}'+\mathbf{b}'} D_{j\mathbf{e}_i}(x_i^m) D_{(k-j)\mathbf{e}_i}(x_i^n)$ for $j = 0, \dots, k$. Since the same factor of $x^{\mathbf{a}'+\mathbf{b}'}$ pulls out of both sides of the identity, it reduces to what we have already shown. To get the result where f is an arbitrary polynomial and g is a monomial, we use linearity of the differential operators in f. Finally, to get the full result we use linearity in g. $\square$

Lemma A.2. *Suppose $g \in S$ is a homogeneous polynomial. Let $F \in R$ be homogeneous of degree $d \geq 1$. In characteristic 0, we have*

$$F \circ (y_j g) = \frac{\partial F}{\partial x_j} \circ g + y_j\,(F \circ g)$$

for every $j = 0, \dots, N$.

Proof. Suppose F is a monomial. We induct on the exponent of x_j in F. First suppose that the exponent of x_j in F is 0. In this case, y_j acts as a constant as far as differentiation by F is concerned and thus $F \circ (y_j g) = y_j(F \circ g)$. Since we also have $(\partial F)/(\partial x_j) = 0$, this proves the lemma when the exponent of x_j in F is 0. Now suppose that the exponent on x_j is positive. Then we can write $F = x_j F_0$ for some monomial F_0. We have

$$F \circ (y_j g) = (F_0 x_j) \circ (y_j g) = F_0 \circ (x_j \circ (y_j g)) = F_0 \circ g + F_0 \circ (y_j (x_j \circ g)), \tag{A.1}$$

where the last equality follows from the product rule. Since the exponent of x_j in F_0 is one less than the exponent of x_j in F, our induction hypothesis yields

$$F_0 \circ (y_j(x_j \circ g)) = \frac{\partial F_0}{\partial x_j} \circ (x_j \circ g) + y_j(F_0 \circ (x_j \circ g)) = \frac{\partial F_0}{\partial x_j} \circ (x_j \circ g) + y_j(F \circ g).$$

Substituting this in to the last equality in (A.1) yields

$$\begin{aligned} F \circ (y_j g) &= F_0 \circ g + F_0 \circ (y_j(x_j \circ g)) \\ &= F_0 \circ g + \frac{\partial F_0}{\partial x_j} \circ (x_j \circ g) + y_j(F \circ g) \\ &= \left(F_0 + x_j \frac{\partial F_0}{\partial x_j}\right) \circ g + y_j(F \circ g) \\ &= \frac{\partial F}{\partial x_j} \circ g + y_j(F \circ g). \end{aligned}$$

This proves the lemma when F is a monomial. The general result follows from linearity of the derivative. $\square$

Lemma A.3. *Suppose $g \in \mathcal{D}$ is a divided power homogeneous polynomial (that is, all of it divided power monomials have the same degree). Let $F \in R$ be homogeneous. In arbitrary characteristic, we have*

$$F \bullet (Y_j g) = D_{\mathbf{e}_j}(F) \bullet g + Y_j\,(F \bullet g)$$

for every $j = 0, \ldots, N$.

Proof. First, we show that the formula holds when $F = x_j^m$ and $g = Y_j^{[n]}$. In fact, both sides are 0 if $n \leq m-2$, and if $n = m-1$, $F \bullet (Y_j g) = x_j^m \bullet ((n+1)Y_j^{[n+1]}) = m = m + 0 = m x_j^{m-1} \bullet (Y_j^{[m-1]}) + 0 = D_{\mathbf{e}_j}(F) \bullet g + Y_j(F \bullet g)$. Otherwise,

$$F \bullet (Y_j g) = (n+1)Y_j^{[n+1-m]} = m x_j^{m-1} \bullet (Y_j^{[n]}) + Y_j Y_j^{[n-m]} = D_{\mathbf{e}_j}(F) \bullet g + Y_j(F \bullet g).$$

Now suppose $F = x_j^m$ and $g = Y^{[\mathbf{b}]} = Y^{[\mathbf{b}']}Y_j^{[n]}$, where $Y^{[\mathbf{b}']}$ is not divisible by Y_j. Then

$$\begin{aligned} F \bullet (Y_j g) &= Y^{[\mathbf{b}']}(F \bullet (Y_j Y_j^{[n]})) \\ &= Y^{[\mathbf{b}']}(D_{\mathbf{e}_j}(F) \bullet Y_j^{[n]}) + Y^{[\mathbf{b}']}Y_j(F \bullet Y_j^{[n]}) = D_{\mathbf{e}_j}(F) \bullet Y^{[\mathbf{b}]} + Y_j(F \bullet Y^{[\mathbf{b}]}), \end{aligned}$$

since we have proved the identity for $g = Y_j^{[n]}$ and we can pull $Y^{[\mathbf{b}']}$ in and out of the contraction with F because $Y^{[\mathbf{b}']}$ acts like a constant under contraction with F. Now suppose $F = x^{\mathbf{a}} = x^{\mathbf{a}'}x_j^m$, where $x^{\mathbf{a}'}$ is not divisible by x_j, and $g = Y^{[\mathbf{b}]}$. Then

$$\begin{aligned} F \bullet (Y_j g) &= x^{[\mathbf{b}']} \bullet (x_j^m \bullet (Y_j g)) \\ &= x^{[\mathbf{b}']} \bullet (D_{\mathbf{e}_j}(x_j^m) \bullet g) + x^{\mathbf{b}'} \bullet (Y_j(x_j^m \bullet g) = D_{\mathbf{e}_j}(x^{\mathbf{b}}) \bullet g + Y_j(x^{\mathbf{b}} \bullet g), \end{aligned}$$

since we have proved the identity for any divided power monomial $g = Y^{[\mathbf{b}]}$, contraction is linear, differentiation with respect to x_j commutes with $x^{\mathbf{b}'}$, and contraction by $x^{\mathbf{b}'}$ commutes with Y_j because $x^{\mathbf{b}'}$ is not divisible by x_j. Thus the desired equality holds if F is a monomial (if $m = 0$ we interpret x_j^{m-1} as 0, not x_j^{-1}) and g is a monomial. The full result follows from linearity of the contraction. $\square$

Lemma A.4. *Suppose $F \in R$ and $g \in S$ are both homogeneous. In characteristic* 0*,*

$$F \circ (y_j^k g) = \sum_{i=0}^{k} \binom{k}{i} y_j^{k-i} \left(\frac{\partial^i F}{\partial x_j^i} \circ g \right)$$

for every $j = 0, \ldots, N$ and every $k \in \mathbb{N}$. If $g \in \mathcal{D}$ is homogeneous then we have, in arbitrary characteristic,

$$F \bullet (Y_j^{[k]} g) = \sum_{i=0}^{k} Y_j^{[k-i]} (D_{i\mathbf{e_j}}(F) \bullet g)$$

for every $j = 0, \ldots, N$ and every $k \in \mathbb{N}$.

Proof. In characteristic 0, both statements can be proven by induction on k, where the base case is Lemma A.2 for S and Lemma A.3 for $\mathcal{D}$. We leave the details to the interested reader. The strongest statement is the second in arbitrary characteristic, and we prove this one. (Note that the first statement also follows from the second in characteristic 0 by the identification $D_{i\mathbf{e_j}}(F) = \frac{1}{i!}\frac{\partial^i F}{\partial x_j^i}$ and the R-module isomorphism between S and $\mathcal{D}$.)

We start by proving the second statement when $F = x_j^m$ and $g = Y_j^{[n]}$ for non-negative integers m and n. On the one hand, we have

$$F \bullet (Y_j^{[k]} g) = \binom{k+n}{k} x_j^m \bullet Y_j^{[k+n]} = \binom{k+n}{k} Y_j^{[k+n-m]}. \tag{A.2}$$

On the other hand,

$$\begin{aligned} \sum_{i=0}^{k} Y^{[k-i]} (D_{i\mathbf{e_j}}(F) \bullet g) &= \sum_{i=0}^{k} Y^{[k-i]} \left(\binom{m}{i} x_j^{m-i} \circ Y_j^{[n]} \right) \\ &= \sum_{i=0}^{k} \binom{m}{i} Y^{[k-i]} Y_j^{[n-m+i]} \qquad (\text{A.3}) \\ &= \sum_{i=0}^{k} \binom{m}{i} \binom{k+n-m}{k-i} Y_j^{[k+n-m]}. \end{aligned}$$

In the above sum, the terms when $i > m$ (corresponding to $D_{i\mathbf{e_j}}(F) = 0$) or when $k - i > k + n - m$ (corresponding to $n < m - i$, hence $x_j^{m-i} \bullet y_j^{[n]} = 0$) are 0. The lemma holds from the combinatorial identity

$$\sum_{i=0}^{k} \binom{m}{i} \binom{k+n-m}{k-i} = \binom{k+n}{k}.$$

Suppose $F = x_j^m$ and $g = Y^{[\mathbf{b}]} = Y^{[\mathbf{b}']} Y_j^{[n]}$. Then the factor $Y^{[\mathbf{b}']}$ will pull out of (A.2) and of every summand in (A.3). Thus the result follows from what has been shown. Now suppose $F = x^{\mathbf{a}} = x^{\mathbf{a}'} x_j^m$, where $x^{\mathbf{a}'}$ is not divisible by x_j, and $g = Y^{[\mathbf{b}]}$. Then $F \bullet (Y_j^{[k]}) = x^{\mathbf{a}'} \bullet (x_j^m \bullet (Y_j^{[k]} g)) = x^{\mathbf{a}'} \bullet \left(\sum_{i=0}^{k} Y_j^{[k-i]} (D_{i\mathbf{e_j}}(x_j^m) \bullet g) \right)$. Now

$$x^{\mathbf{a}'} \bullet \left(\sum_{i=0}^{k} Y_j^{[k-i]} (D_{i\mathbf{e_j}}(x_j^m) \bullet g) \right) = \sum_{i=0}^{k} x^{\mathbf{a}'} \bullet \left(Y_j^{[k-i]} (D_{i\mathbf{e_j}}(x_j^m) \bullet g) \right)$$

$$= \sum_{i=0}^{k} Y_j^{[k-i]} \left(x^{\mathbf{a}'} \bullet (D_{i\mathbf{e_j}}(x_j^m) \bullet g) \right)$$
$$= \sum_{i=0}^{k} Y_j^{[k-i]} \left((x^{\mathbf{a}'} D_{i\mathbf{e_j}}(x_j^m)) \bullet g \right)$$
$$= \sum_{i=0}^{k} Y_j^{[k-i]} \left(D_{i\mathbf{e_j}}(x^{\mathbf{a}'} x_j^m) \bullet g \right)$$
$$= \sum_{i=0}^{k} Y_j^{[k-i]} \left(D_{i\mathbf{e_j}}(x^{\mathbf{a}}) \bullet g \right),$$

where the first equality follows by linearity of contraction, the second because $x^{\mathbf{a}'}$ does not involve the variable x_j, the third by the definition of contraction, and the fourth also because $x^{\mathbf{a}'}$ does not involve the variable x_j. This proves the identity when F and g are monomials. The identity now follows when F and g are polynomials by linearity. □

Appendix B. The inverse system of powers of the ideal of a point

Emsalem and Iarrobino show in [17] that the fundamental computation when finding the inverse system of the symbolic powers of a variety is finding the inverse system of the symbolic powers of the ideal of a single point. We revisit this computation using Lemma A.4. Let $p = [b_0 : b_1 : \ldots : b_N] \in \mathbb{P}^N$ and

$$\mathfrak{m}_p = \langle b_1 x_0 - b_0 x_1, \ldots, b_N x_0 - b_0 x_N \rangle \subset R = \mathbb{K}[x_0, \ldots, x_N].$$

be the ideal of homogeneous polynomials vanishing on p. We write $L_p = b_0 y_0 + \ldots + b_N y_N \in S$ for the dual linear form. An important observation in [17] is that, if $F \in R$ is homogeneous of degree $d \leq k$, then

$$F \circ L_p^k = \frac{k!}{(k-d)!} L_p^{k-d} F(p), \tag{B.1}$$

where $F(p)$ is the evaluation of F at p.

In arbitrary characteristic, we also let L_p denote the dual linear form $b_0 Y_0 + \cdots + b_N Y_N \in \mathcal{D}$, relying on context to differentiate between $L_p \in S$ and $L_p \in \mathcal{D}$. In $\mathcal{D}$, we define the *divided power* of L_p by $L_p^{[k]} = \sum_{|\mathbf{a}|=k} b_0^{a_0} \cdots b_N^{a_N} Y^{[\mathbf{a}]}$.

The definition of $L_p^{[k]}$ is made precisely so that the analog of (B.1) holds. Namely, if $F \in R$ is homogeneous of degree $d \leq k$, then

$$F \bullet L_p^{[k]} = L_p^{[k-d]} F(p), \tag{B.2}$$

where again $F(p)$ is the evaluation of F at p. Both (B.1) and (B.2) follow from a direct computation. The following result is shown in [17] (see also [20]).

Lemma B.1. *In characteristic* 0,

$$(\mathfrak{m}_p^n)_d^\perp = \begin{cases} S_d & \text{if } d < n \\ \langle L_p^{d-n+1} \rangle_d & \text{if } d \geq n \end{cases}.$$

In arbitrary characteristic,

$$(\mathfrak{m}_p^n)_d^\perp = \begin{cases} \mathcal{D}_d & \text{if } d < n \\ \operatorname{span}\{Y^{[\mathbf{a}]} L_p^{[c]} : d-n+1 \leq c \leq d, |\mathbf{a}| = d-c\} & \text{if } d \geq n \end{cases}.$$

Proof. We prove the formula for the action of R on $\mathcal{D}$. The case when $d < n$ is clear, so we assume that $d \geq n$. It is straightforward to show that, when $d \geq n$,

$$\dim(\mathfrak{m}_p^n)_d = \binom{d+N+1}{N+1} - \binom{n+N}{N+1} \quad \text{and hence} \quad \dim(\mathfrak{m}_p^n)_d^\perp = \binom{n+N}{N+1}.$$

Examining the terms of $Y^{[\mathbf{a}]} L_p^{[c]}$, we see that, for some $0 \leq i \leq N$, the divided monomials of the form

$$\{Y_i^{[c]} Y^{[\mathbf{a}']} : Y_i \text{ does not appear in } Y^{[\mathbf{a}']}, d-n+1 \leq c \leq d, \text{ and } c + |\mathbf{a}'| = d\}$$

all appear as a term in some $Y^{[\mathbf{a}]} L^{[c]}$ on the right hand side. There are $\binom{n+N}{N+1}$ of these monomials, thus the dimension of the right hand side is at least the dimension of $\dim(\mathfrak{m}_p^n)_d^\perp$. Thus it suffices to show that $Y^{[\mathbf{a}]} L_p^{[c]} \in (\mathfrak{m}_p^n)_d^\perp$ for $d-n+1 \leq c \leq d$ and $|\mathbf{a}| = d-c$. For this we take a form $F \in (\mathfrak{m}_p^n)_d$ and show that $F \bullet (Y^{[\mathbf{a}]} L_p^{[c]}) = 0$.

We induct on n and $|\mathbf{a}|$. If $n = 1$ or $|\mathbf{a}| = 0$ then $c = d$ and $F \bullet L_p^{[c]} = F \bullet L_p^{[d]} = F(p)$ by (B.2). Since $F \in \mathfrak{m}_p, F(p) = 0$ and we are done. Now suppose $n > 1$ and $|\mathbf{a}| > 0$. Then, for some $0 \leq i \leq N+1$, we can write $Y^{[\mathbf{a}]} = Y_i^{[k]} Y^{[\mathbf{a}']}$ where $0 < k \leq d-c$ and Y_i does not appear in $Y^{[\mathbf{a}']}$. By Lemma A.4,

$$F \bullet (Y^{[\mathbf{a}]} L_p^{[c]}) = \sum_{j=0}^{k} Y_i^{[k-j]} (D_{j\mathbf{e}_i}(F) \bullet Y^{[\mathbf{a}']} L_p^{[c]}). \tag{B.3}$$

Note that if $j = 0$ then $D_{0\mathbf{e}_i}(F) = F$ and $F \bullet Y^{[\mathbf{a}']} L_p^{[c]} = 0$ by induction on $|\mathbf{a}|$. If $1 < j \leq k$ then $D_{j\mathbf{e}_i}(F) \in \mathfrak{m}_p^{n-j}$ by Example 4.5 and thus $D_{j\mathbf{e}_i}(F) \bullet Y^{[\mathbf{a}']} L_p^{[c]} = 0$ by induction on n. So all terms in (B.3) vanish and we are done.

An identical strategy can be used to show the formula for $(\mathfrak{m}_p^n)_d^\perp$ for the action of R on S; the proof can be simplified a little using Lemma A.2 instead of Lemma A.4. □

Remark B.2. A different proof of Lemma B.1 relies on the GL_{N+1}-equivariance of the differentiation and contraction actions (see [35, Proposition A.3]), under which we may assume that $p = [1 : 0 : \cdots : 0]$.

References

[1] M. Atiyah, Duality in Mathematics and Physics, Lecture notes from the Institut de Matematica de la Universitat de Barcelona , 2007.
[2] T. Bauer, S. Di Rocco, B. Harbourne MichałKapustka, A. Knutsen, W. Syzdek, T. Szemberg, A primer on Seshadri constants, in: Interactions of Classical and Numerical Algebraic Geometry, in: Contemp. Math., vol. 496, Amer. Math. Soc., Providence, RI, 2009, pp. 33–70.
[3] M.C. Beltrametti, A.J. Sommese, On k-jet ampleness, in: Complex Analysis and Geometry, in: Univ. Ser. Math., Plenum, New York, 1993, pp. 355–376.
[4] C. Bocci, B. Harbourne, Comparing powers and symbolic powers of ideals, J. Algebraic Geom. 19 (3) (2010) 399–417.
[5] C. Ciliberto, B. Harbourne, R. Miranda, J. Roé, Variations of Nagata's conjecture, in: A Celebration of Algebraic Geometry, in: Clay Math. Proc., vol. 18, Amer. Math. Soc., Providence, RI, 2013, pp. 185–203.
[6] S.M. Cooper, R.J.D. Embree, H. Tài Hà, A.H. Hoefel, Symbolic powers of monomial ideals, Proc. Edinb. Math. Soc. (2) 60 (1) (2017) 39–55.
[7] S.D. Cutkosky, L. Ein, R. Lazarsfeld, Positivity and complexity of ideal sheaves, Math. Ann. 321 (2) (2001) 213–234.
[8] H. Dao, A. De Stefani, E. Grifo, C. Huneke, L.N. Betancourt, Symbolic powers of ideals, in: Singularities and Foliations. Geometry, Topology and Applications, in: Springer Proc. Math. Stat., vol. 222, Springer, Cham, 2018, pp. 387–432.
[9] J.-P. Demailly, Singular Hermitian metrics on positive line bundles, in: Complex Algebraic Varieties, Bayreuth, 1990, in: Lecture Notes in Math., vol. 1507, Springer, Berlin, 1992, pp. 87–104.
[10] M. DiPasquale, B. Drabkin, On resurgence via asymptotic resurgence, J. Algebra 587 (2021) 64–84.
[11] M. DiPasquale, C.A. Francisco, J. Mermin, J. Schweig, Asymptotic resurgence via integral closures, Trans. Am. Math. Soc. 372 (9) (2019) 6655–6676.
[12] M. DiPasquale, N. Villamizar, A lower bound for splines on tetrahedral vertex stars, SIAM J. Appl. Algebra Geom. 5 (2) (2021) 250–277.
[13] M. Dumnicki, B. Harbourne, U. Nagel, A. Seceleanu, T. Szemberg, H. Tutaj-Gasińska, Resurgences for ideals of special point configurations in $\mathbf{P}^N$ coming from hyperplane arrangements, J. Algebra 443 (2015) 383–394.
[14] M. Dumnicki, B. Harbourne, T. Szemberg, H. Tutaj-Gasińska, Linear subspaces, symbolic powers and Nagata type conjectures, Adv. Math. 252 (2014) 471–491.
[15] L.X. Dung, T.T. Hien, H.D. Nguyen, T.N. Trung, Regularity and Koszul property of symbolic powers of monomial ideals, Math. Z. 298 (3–4) (2021) 1487–1522.
[16] L. Ein, R. Lazarsfeld, K.E. Smith, Uniform bounds and symbolic powers on smooth varieties, Invent. Math. 144 (2) (2001) 241–252.
[17] J. Emsalem, A. Iarrobino, Inverse system of a symbolic powers. I, J. Algebra 174 (3) (1995) 1080–1090.
[18] L. Evain, On the postulation of s^d fat points in $\mathbb{P}^d$, J. Algebra 285 (2) (2005) 516–530.
[19] M. Fekete, Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten, Math. Z. 17 (1) (1923) 228–249.
[20] A.V. Geramita, Inverse systems of fat points: Waring's problem, secant varieties of Veronese varieties and parameter spaces for Gorenstein ideals, in: The Curves Seminar at Queen's, Vol. X, Kingston, ON, 1995, in: Queen's Papers in Pure and Appl. Math., vol. 102, Queen's Univ., Kingston, ON, 1996, pp. 2–114.
[21] A.V. Geramita, B. Harbourne, J. Migliore, U. Nagel, Matroid configurations and symbolic powers of their ideals, Trans. Am. Math. Soc. 369 (10) (2017) 7049–7066.
[22] I. Gitler, C. Valencia, R.H. Villarreal, A note on the Rees algebra of a bipartite graph, J. Pure Appl. Algebra 201 (1–3) (2005) 17–24.
[23] E. Guardo, B. Harbourne, A. Van Tuyl, Asymptotic resurgences for ideals of positive dimensional subschemes of projective space, Adv. Math. 246 (2013) 114–127.
[24] H.T. Hà, T.T. Nguyễn, Newton-Okounkov body, Rees algebra, and analytic spread of graded families of monomial ideals, arXiv e-prints, arXiv:2111.00681, October 2021.
[25] B. Harbourne, On Nagata's conjecture, J. Algebra 236 (2) (2001) 692–702.
[26] B. Harbourne, J. Roé, Discrete behavior of Seshadri constants on surfaces, J. Pure Appl. Algebra 212 (3) (2008) 616–627.

[27] B. Harbourne, Asymptotics of Linear Systems, with Connections to Line Arrangements. Phenomenological Approach to Algebraic Geometry, Banach Center Publ., vol. 116, Polish Acad. Sci. Inst. Math., Warsaw, 2018.
[28] B. Harbourne, J. Kettinger, F. Zimmitti, Extreme values of the resurgence for homogeneous ideals in polynomial rings, J. Pure Appl. Algebra 226 (16) (2022) 106811.
[29] R. Hartshorne, Algebraic Geometry, Graduate Texts in Mathematics, vol. 52, Springer-Verlag, New York-Heidelberg, 1977.
[30] D.J. Hernández, P. Teixeira, E.E. Witt, Frobenius powers, Math. Z. 296 (1–2) (2020) 541–572.
[31] M. Hochster, C. Huneke, Comparison of symbolic and ordinary powers of ideals, Invent. Math. 147 (2) (2002) 349–369.
[32] C. Huneke, I. Swanson, Integral Closure of Ideals, Rings, and Modules, London Mathematical Society Lecture Note Series, vol. 336, Cambridge University Press, Cambridge, 2006.
[33] C. Huneke, Open problems on powers of ideals, https://www.aimath.org/WWN/integralclosure/Huneke.pdf, 2006.
[34] A. Iarrobino, Inverse system of a symbolic power. III. Thin algebras and fat points, Compos. Math. 108 (3) (1997) 319–356.
[35] A. Iarrobino, V. Kanev, Power Sums, Gorenstein Algebras, and Determinantal Loci, Lecture Notes in Mathematics, vol. 1721, Springer-Verlag, Berlin, 1999, Appendix C by Iarrobino and Steven L. Kleiman.
[36] K. Johnson, Super-additive sequences and algebras of polynomials, Proc. Am. Math. Soc. 139 (10) (2011) 3431–3443.
[37] R. Lazarsfeld, Positivity in Algebraic Geometry. II: Positivity for Vector Bundles, and Multiplier Ideals, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge (A Series of Modern Surveys in Mathematics. Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics), vol. 49, Springer-Verlag, Berlin, 2004.
[38] F.S. Macaulay, The Algebraic Theory of Modular Systems, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1994, Revised reprint of the 1916 original, With an introduction by Paul Roberts.
[39] L. Ma, K. Schwede, Perfectoid multiplier/test ideals in regular rings and bounds on symbolic powers, Invent. Math. 214 (2) (2018) 913–955.
[40] P. Mantero, The structure and free resolutions of the symbolic powers of star configurations of hypersurfaces, Trans. Am. Math. Soc. 373 (12) (2020) 8785–8835.
[41] M. Nagata, On the 14-th problem of Hilbert, Am. J. Math. 81 (1959) 766–772.
[42] U. Nagel, B. Trok, Interpolation and the weak Lefschetz property, Trans. Am. Math. Soc. 372 (12) (2019) 8849–8870.
[43] L.P. Østerdal, Subadditive functions and their (pseudo-)inverses, J. Math. Anal. Appl. 317 (2) (2006) 724–731.
[44] I. Swanson, Powers of ideals. Primary decompositions, Artin-Rees lemma and regularity, Math. Ann. 307 (2) (1997) 299–313.
[45] H. Tutaj-Gasińska, A bound for Seshadri constants on $\mathbb{P}^2$, Math. Nachr. 257 (2003) 108–116.
[46] M. Waldschmidt, Propriétés arithmétiques de fonctions de plusieurs variables. II, in: Séminaire Pierre Lelong (Analyse) (année 1975/76); Journées sur les Fonctions Analytiques, Toulouse, 1976, in: Lecture Notes in Math., vol. 578, Springer, Berlin, 1977, pp. 108–135.
[47] Robert M. Walker, The symbolic generic initial system of an a.s.p. ideal, preprint 2021.
[48] G. Xu, Curves in $\mathbf{P}^2$ and symplectic packings, Math. Ann. 299 (4) (1994) 609–613.
[49] Z. Oscar, A fundamental lemma from the theory of holomorphic functions on an algebraic variety, Ann. Mat. Pura Appl. (4) 29 (1949) 187–198.