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INITIAL DEGREE OF SYMBOLIC POWERS OF IDEALS OF FERMAT CONFIGURATIONS OF POINTS

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Let $n \geq 2$ be an integer and consider the defining ideal of the Fermat configuration of points in $\mathbb{P}^2$: $I_n = \big(x(y^n - z^n), y(z^n - x^n), z(x^n - y^n)\big) \subset R = \mathbb{C}[x, y, z]$. We explicitly compute the least degree of generators (the initial degree) of its symbolic powers in all unknown cases. As direct applications, we verify Chudnovsky's conjecture, Demailly's conjecture and the Harbourne–Huneke containment problem, as well as calculate the Waldschmidt constant and (asymptotic) resurgence number.

1. Introduction

Let $n \geq 2$ be an integer and consider the *Fermat ideal*

$$I_n = \big(x(y^n - z^n), y(z^n - x^n), z(x^n - y^n)\big) \subset R = \mathbb{C}[x, y, z].$$

This ideal corresponds to a Fermat arrangement of lines (or a Ceva arrangement in some literature) in $\mathbb{P}^2$. More precisely, the variety of I_n is a reduced set of $n^2 + 3$ points in $\mathbb{P}^2$ [25], where n^2 of these points form the intersection locus of the pencil of curves spanned by $x^n - y^n$ and $x^n - z^n$, while the other 3 are the coordinate points $[1:0:0]$, $[0:1:0]$ and $[0:0:1]$. This set of points is said to be the Fermat configuration of points, justifying the terminology *Fermat ideal*. Fermat ideals have attracted a lot of attention recently in commutative algebra research since they appeared as the first example of the noncontainment between the third symbolic power and the second ordinary power of a defining ideal of a set of points in $\mathbb{P}^2$ in the work of Dumnicki, Szemberg and Tutaj-Gasińska [14] (when $n = 3$) and were generalized by Harbourne and Seceleanu [25]. It is worth emphasizing that this is a quite surprising relation between ordinary and symbolic powers of an ideal of points in $\mathbb{P}^2$, since I^2 always contains $I^{(3)}$, where I is the ideal of a general set of points [5].

Since then, much has become known about Fermat ideals for $n \geq 3$. Fermat ideals can also be thought of as the ideals determining the singular loci of the arrangements of lines given by the monomial groups $G(n, n, 3)$, see [11] or [35]. The *Waldschmidt constant* and (*asymptotic*) *resurgence number* of Fermat ideals have been computed in [15] for $n \geq 3$. Nagel and Seceleanu [32] studied Rees algebras and symbolic Rees algebras of Fermat ideals and their minimal generators, as well as the minimal free resolutions of all their ordinary powers and many symbolic powers. Specifically, it was shown that the symbolic Rees algebra of I_n is Noetherian; the Castelnuovo–Mumford regularity of powers of I_n and their reduction

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ideals were provided. In term of minimal generators, when $n \geq 3$, they provided the minimal generators for all ordinary powers, as well all multiple of n symbolic powers. All known results are:

- For all $k \geq 1$, $\alpha(I_n^{(3k)}) = 3nk$ by [15, Theorem 2.1].
- For all $k \geq 1$, $\alpha(I_n^{(nk)}) = n^2 k$ by [32, Theorem 3.6].
- For all $m \geq 0$, $\alpha(I_3^{(3m+2)}) = 9m + 8$ by [30, Example 4.4].

In this paper, we will compute explicitly the least degree of generators (the initial degree) of symbolic powers of I_n for *all* remaining cases to provide a more complete picture of Fermat ideals. We summarize our results and results known in the literature in the following theorem:

Theorem 1.1. *Let $n \geq 2$ be an integer and $I_n = \big(x(y^n - z^n), y(z^n - x^n), z(x^n - y^n)\big)$ in $\mathbb{C}[x, y, z]$ be a Fermat ideal. Then*:

(1) *For all $n \geq 2$, $\alpha(I_n^{(2)}) = \alpha(I_n^2) = 2(n+1)$; see Theorem 3.4 and Theorem 4.1.*
(2) *For all $k \geq 2$, $\alpha(I_2^{(2k)}) = 5k$; see Theorem 4.1.*
(3) *For all $k \geq 0$, $\alpha(I_2^{(2k+1)}) = 5k + 3$; see Theorem 4.1.*
(4) *For all $k \geq 1$, $\alpha(I_3^{(3k)}) = 9k$; see* [15, Theorem 2.1].
(5) *For all $k \geq 1$, $\alpha(I_3^{(3k+1)}) = 9k + 4$; see Theorem 3.5.*
(6) *For all $k \geq 1$, $\alpha(I_3^{(3k+2)}) = 9k + 8$; see* [30, Example 4.4].
(7) *For all $m \geq 3$ and $m \neq 5$, $\alpha(I_4^{(m)}) = 4m$, and $\alpha(I_4^{(5)}) = 21$; see Theorem 3.2.*
(8) *For all $n \geq 5$ and $m \geq 3$, $\alpha(I_n^{(m)}) = mn$; see Theorem 3.1.*

Theorem 1.1 give a complete answer to the question of the least generating degree for all symbolic powers of I_n, see Table 1. This includes the calculation for the ideal I_2, which is less considered in the aforementioned works. Note that the ideal I_2 is the ideal determining the singular locus of the arrangement of lines given by the pseudoreflection group D_3, see [11]. It is worth pointing out the irregular value of $\alpha(I_4^{(5)}) = 21$.

The *containment problem* for an ideal I is the problem of determining the set of pairs (m, r) for which $I^{(m)} \subseteq I^r$. The deep results in [16; 26; 27] show that $I^{(m)} \subseteq I^r$ whenever $m \geq Nr$. In order to characterize the pairs (r, m) numerically, the *resurgence* $\rho(I)$ was introduced [5], and then the *asymptotic resurgence* $\hat{\rho}(I)$ was introduced [23]. It is known that for $n \geq 3$, $\rho(I_n) = \frac{3}{2}$, $\hat{\rho}(I_n) = (n+1)/n$ and $\hat{\alpha}(I_n) = n$ by [15, Theorem 2.1], where $\hat{\alpha}(I)$ is the *Waldschmidt constant* of I. We will compute the Waldschmidt constant and (asymptotic) resurgence number of I_2.

n	2			3				4			$n \geq 5$	
m	2	$2k$	$2k+1$	2	$3k$	$3k+1$	$3k+2$	2	5	$\geq 3, \neq 5$	2	≥ 3
$\alpha(I_n^{(m)})$	6	$5k$	$5k+3$	8	$9k$	$9k+4$	$9k+8$	10	21	$4m$	$2(n+1)$	nm
$\hat{\alpha}(I_n)$	$\frac{5}{2}$			3				4			n	
$\hat{\rho}(I_n)$	$\frac{6}{5}$			$(n+1)/n$								
$\rho(I_n)$	$\frac{6}{5}$			$\frac{3}{2}$								

Table 1. The initial degrees and other invariants related to symbolic powers of I_n

Theorem 1.2. *For the ideal I_2, the Waldschmidt constant is $\hat{\alpha}(I_2) = \frac{5}{2}$, and the resurgence number and asymptotic resurgence number are $\rho(I_2) = \hat{\rho}(I_2) = \frac{6}{5}$.*

In an effort to improve the containment $I^{(m)} \subseteq I^r$ for $m \geq Nr$, as well as to deduce Chudnovsky's conjecture, Harbourne and Huneke [24] conjectured that the defining ideal I for any set of points in $\mathbb{P}^N$ satisfies some stronger containment, namely, $I^{(Nm)} \subseteq \mathfrak{m}^{m(N-1)} I^m$ and $I^{(Nm-N+1)} \subseteq \mathfrak{m}^{(m-1)(N-1)} I^m$, for all $m \geqslant 1$. An interesting fact about the Fermat ideals is the verification and failure of the containment can be checked purely from their numerical invariants including the least degree of generators of symbolic powers, the regularity, or the resurgence number and the maximal degree of generators. As a consequence of the above computations, we will easily deduce that all Fermat ideals verify the *Harbourne–Huneke containment*, the *stable Harbourne containment* and some stronger containments. In [3, Example 3.5], we showed a stronger containment (which implies both containments given in Corollary 1.3) $I_n^{(2r-2)} \subseteq \mathfrak{m}^r I_n^r$ for $n \geq 3$ and for all $r \gg 0$. Here, we show the containment for all possible cases.

Corollary 1.3. *For every $n \geq 2$, Fermat configuration ideals verify the following containments*:

(1) *Harbourne–Huneke containment*, *see* [24, Conjecture 2.1],

$$I_n^{(2r)} \subseteq m^r I_n^r, \quad \forall r \geq 1.$$

(2) *Harbourne–Huneke containment*, *see* [24, Conjecture 2.1],

$$I_n^{(2r-1)} \subseteq m^{r-1} I_n^r,$$

for all $r \geq 3$ if $n \geq 3$ and for all $r \geq 1$ if $n = 2$.

(3) *A stronger containment $I_n^{(2r-2)} \subseteq m^r I_n^r$, for all $r \geq 5$.*

This work can also be thought of as a modest contribution to the theory of Hermite interpolation. Specifically, given a set of points $Z = \{P_1, \dots, P_s\}$ in a projective space $\mathbb{P}^n$ and positive integers $m_1, \dots, m_s$, a fundamental problem in the theory of Hermite interpolation is to determine the least degree of a homogeneous polynomial that vanishes to order m_i at the point P_i for every $i = 1, \dots, s$. This is a very hard problem even when $m_1 = \cdots = m_s$. In this case when all m_i are the same, thanks to the Zariski–Nagata theorem, the above interpolation problem is the same as asking for the least degree of a nonzero homogeneous polynomial in the k-th symbolic power of the defining ideal of Z, where $k = m_1 = \cdots = m_s$. Hence, this work, combined with many previous works, provides a complete answer to the above question for Z to be a Fermat configuration of points in $\mathbb{P}^2$.

In order to understand the generating degrees of symbolic powers of I_n, we discuss the maximal degree of a set of minimal generators of symbolic powers of I_n, denoted by $\omega(I_n^{(m)})$. From the description of generating sets in [32], it can be seen that for $n \geq 3$, we have $\omega(I_n^{(m)}) = m(n+1)$ for $m = kn$ or $m = n-1$. By relating this with another invariant $\beta(I_n^{(m)})$ that is defined in [24, Definition 2.2], we show that $\beta(J) \leq \omega(J)$ for an ideal of points J, in general, and use this to show that for $n \geq 3$, we have $\omega(I_n^{(m)}) = m(n+1)$ for $m \geq n^2 - 3n + 2$.

It is worth remarking that Malara and Szpond [28; 29] also studied the generalization of Fermat configurations in higher dimensions to provide more counterexamples to the containment $I^{(3)} \subseteq I^2$. It turns out that these ideals share some similar properties to Fermat ideals. We will investigate these ideals in 3-dimensional space with the same questions in the continuation paper [34] in order to keep this paper more concise and focused. We work over the field of complex numbers, but our results hold over any algebraically closed field of characteristic 0.

2. Preliminaries

Let $R = \mathbb{C}[x_0, \dots, x_N]$ be the homogeneous coordinate ring of $\mathbb{P}^N$, and let $\mathfrak{m}$ be its maximal homogeneous ideal. For a homogeneous ideal $I \subseteq R$, let $\alpha(I)$ denote the least degree of a nonzero homogeneous polynomial in I, and let

$$I^{(m)} := \bigcap_{\mathfrak{p} \in \operatorname{Ass}(R/I)} I^m R_{\mathfrak{p}} \cap R$$

denote its m-th *symbolic power*.

Geometrically, given a set of distinct points $\mathbb{X} \subseteq \mathbb{P}^N$ and an integer $m \geqslant 1$, by the Zariski–Nagata theorem [17; 31; 36] (see [9, Proposition 2.14]), the least degree of a nonzero homogeneous polynomial in the homogeneous coordinate ring $\mathbb{C}[x_0, \dots, x_N]$ that vanishes at each point in $\mathbb{X}$ of order at least m is $\alpha(I_{\mathbb{X}}^{(m)})$, where $I_{\mathbb{X}} \subseteq \mathbb{C}[x_0, \dots, x_N]$ is the defining ideal of $\mathbb{X}$.

The *Waldschmidt constant* of I is defined to be the limit and turned out to be the infimum

$$\hat{\alpha}(I) := \lim_{m \to \infty} \frac{\alpha(I^{(m)})}{m} = \inf_{m \to \infty} \frac{\alpha(I^{(m)})}{m}.$$

There is a tight connection between the Waldschmidt constant and an algebraic manifestation of the Seshadri constant, especially for a set of very general points, see [1, Section 8].

In studying the lower bound for the least degree of a homogeneous polynomial vanishing at a given set of points in $\mathbb{P}^N$ with a prescribed order, Chudnovsky [8] made the following conjecture:

Conjecture 2.1 (Chudnovsky). *Let I be the defining ideal of a set of points $\mathbb{X} \subseteq \mathbb{P}^N_{\mathbb{C}}$. Then, for all $n \geqslant 1$,*

$$\frac{\alpha(I^{(n)})}{n} \geqslant \frac{\alpha(I) + N - 1}{N}.$$

This conjecture has been investigated extensively, for example, in [3; 5; 12; 13; 18; 20; 21; 24]. Recently, the conjecture was proved for a *very general* set of points [13; 20], for a *general* set of sufficiently many points [3] and recently, for any number of general points [2]. The conjecture was also generalized by Demailly [10].

Conjecture 2.2 (Demailly). *Let I be the defining ideal of a set of points $\mathbb{X} \subseteq \mathbb{P}^N_{\mathbb{C}}$, and let $m \in \mathbb{N}$ be any integer. Then, for all $n \geqslant 1$,*

$$\frac{\alpha(I^{(n)})}{n} \geqslant \frac{\alpha(I^{(m)}) + N - 1}{m + N - 1}.$$

Demailly's conjecture for $N = 2$ was proved by Esnault and Viehweg [18]. Recent work of Malara, Szemberg and Szpond [30], and of Chang and Jow [7], showed that for a fixed integer m, Demailly's conjecture holds for a *very general* set of sufficiently many points and for a *general* set of k^N points. In [4], the results were extended to a *general* set of sufficiently many points.

The *containment problem* for an ideal I is to determine the set S_I of pairs (r, m) for which $I^{(m)} \subseteq I^r$. The deep results in [16; 26; 27] show that $I^{(m)} \subseteq I^r$ whenever $m \geq Nr$; hence, $\{(m, r) : m \geq Nr\} \subseteq S_I$. In order to characterize S_I numerically, the *resurgence number* $\rho(I)$ is introduced in [5] as

$$\rho(I) := \sup\left\{\frac{m}{r} : I^{(m)} \not\subseteq I^r\right\},$$

and the *asymptotic resurgence number* $\hat{\rho}(I)$ is introduced in [23] as

$$\hat{\rho}(I) := \sup\left\{\frac{m}{r} : I^{(mt)} \not\subseteq I^{rt} \text{ for } t \gg 0\right\}.$$

There are only a few cases for which S_I is known completely or the resurgence number has been determined. In general, $\rho(I)$ and $\hat{\rho}(I)$ are different.

In an effort to improve the containment $I^{(m)} \subseteq I^r$ for $m \geq Nr$, Harbourne and Huneke [24] conjectured that the defining ideal I for any set of points in $\mathbb{P}^N$ satisfies some stronger containment, namely that $I^{(Nm)} \subseteq \mathfrak{m}^{m(N-1)} I^m$ and $I^{(Nm-N+1)} \subseteq \mathfrak{m}^{(m-1)(N-1)} I^m$, for all $m \geqslant 1$. Knowing these containment clearly helps us to know the set S_I and the numbers $\rho(I)$ and $\hat{\rho}(I)$. Conversely, knowledge about S_I, $\rho(I)$ and $\hat{\rho}(I)$ can be helpful to prove the containment. One result that we will use is the following: If $m/r > \rho(I)$, then by definition $I^{(m)} \subseteq I^r$; and suppose in addition, $\alpha(I^{(m)}) \geq a + \omega(I^r)$ for some integer a, where $\omega(I)$ is the maximum degree of generators in a set of minimal generators of I, then $I^{(m)} \subseteq \mathfrak{m}^a I^r$. We refer interested readers to [6] for more information about the Waldschmidt constant, the resurgence number and the containment between symbolic and ordinary powers of ideals.

3. Fermat ideals for $n \geq 3$

In this section, we focus on the Fermat ideals $I_n = \big(x(y^n - z^n), y(z^n - x^n), z(x^n - y^n)\big)$ for $n \geq 3$. Let us first recall some known results about the degrees of generators, Waldschmidt constants and (asymptotic) resurgence numbers of I_n:

- For all $k \geq 1$, $\alpha(I_n^{(3k)}) = 3nk$ [15, Theorem 2.1].
- For all $k \geq 1$, $\alpha(I_n^{(nk)}) = n^2 k$ [32, Theorem 3.6].
- For all $m \geq 0$, $\alpha(I_3^{(3m+2)}) = 9m + 8$ [30, Example 4.4].
- $\rho(I_n) = \frac{3}{2}$ and $\hat{\rho}(I_n) = (n+1)/n$ [15, Theorem 2.1].

Let $f_n = y^n - z^n$, $g_n = z^n - x^n$ and $h_n = x^n - y^n$. Then

$$I_n = (xf_n, yg_n, zh_n) = (f_n, g_n) \cap (x, y) \cap (y, z) \cap (z, x).$$

It is well known that, since f_n and g_n form a regular sequence, for any $m \geq 1$, we have

$$I_n^{(m)} = (f_n, g_n)^m \cap (x, y)^m \cap (y, z)^m \cap (z, x)^m.$$

Geometrically, recall that the Fermat configuration consists of $n^2 + 3$ points which are all points having each coordinate equal to an n-th root of 1 and the points $[0:0:1]$, $[0:1:0]$ and $[1:0:0]$. Furthermore, these points are intersections of $3n$ lines L_j which have equations:

$$x - \epsilon^k y = 0, \quad y - \epsilon^k z = 0, \quad z - \epsilon^k x = 0$$

for $k = 0, 1, \ldots, n-1$. Each of these lines contains exactly $n+1$ points P_i (one coordinate point and n other points), and each of the points $[0:0:1]$, $[0:1:0]$ and $[1:0:0]$ is on exactly n lines, while each of the other n^2 points is on exactly 3 lines.

We will use these descriptions to explicitly compute the least degree of generators of $I_n^{(m)}$ in all remaining unknown cases, provided the knowledge about the Waldschmidt constant of I_n. Our main strategy in this paper and the continuation paper [34] is to study a subsequence of $\alpha(I^{(m)})$, which gives us information about $\hat{\alpha}(I)$, and then use this to calculate other $\alpha(I^{(m)})$.

The following four theorems provide us with all remaining unknown initial degrees of symbolic powers of I_n:

Theorem 3.1. *For $n \geq 5$, we have*

$$\alpha(I_n^{(m)}) = nm$$

for all $m \geq 3$.

Proof. First, for $3 \leq m \leq n$, we observe that

$$f_n g_n h_n (f_n, g_n)^{m-3} \subseteq I_n^{(m)} = (f_n, g_n)^m \cap (x, y)^m \cap (y, z)^m \cap (z, x)^m.$$

In fact, since $h_n = -(f_n + g_n)$, we have that $f_n g_n h_n (f_n, g_n)^{m-3} \subseteq (f_n, g_n)^m$. Also, it is clear that the sum of degrees with respect to x and degrees with respect to y of any monomials of $f_n g_n h_n$ is at least $n \geq m$ so $f_n g_n h_n \in (x, y)^m$. Similarly, $f_n g_n h_n \in (y, z)^m \cap (z, x)^m$.

Thus, for $3 \leq m \leq n$, we have $\alpha(I_n^{(m)}) \leq nm$. Since $\hat{\alpha}(I_n) = n$, we have $\alpha(I_n^{(m)}) \geq nm$. Therefore, $\alpha(I_n^{(m)}) = nm$.

Now for any $k \geq 2$, we claim that for $0 \leq a \leq n-1$,

$$(f_n g_n h_n)^k (f_n, g_n)^{k(n-3)-a} \subseteq I^{(kn-a)}.$$

The argument is identical to that of the above, the only thing to note here is that for $n \geq 5$,

$$k(n-3) \geq 2(n-3) \geq n-1 \geq a.$$

Thus, for all $k \geq 2$ and $0 \leq a \leq n-1$,

$$\alpha(I_n^{(kn-a)}) \leq (kn-a)n.$$

Hence, $\alpha(I_n^{(kn-a)}) = (kn-a)n$. Since for $m > n$, there are unique $k \geq 2$ and $0 \leq a \leq n-1$ such that $m = kn - a$, we have that $\alpha(I_n^{(m)}) = mn$ for all $m > n$. □

Theorem 3.2. *When $n = 4$, we have $\alpha(I_4^{(5)}) = 21$, and for all $m \geq 3$ except $m \neq 5$,*

$$\alpha(I_4^{(m)}) = 4m.$$

Proof. With the same argument we have that:

Case 1: For $3 \leq m \leq 4$, we have $f_4 g_4 h_4 (f_4, g_4)^{m-3} \subseteq I_4^{(m)}$. The statement is true for $m = 3, 4$.

For $k = 2$ and $0 \leq a \leq 2$,

$$(f_4 g_4 h_4)^k (f_4, g_4)^{k(n-3)-a} \subseteq I_4^{(kn-a)},$$

i.e,

$$(f_4 g_4 h_4)^2 (f_4, g_4)^{2-a} \subseteq I_4^{(8-a)}$$

for $0 \leq a \leq 2$. Thus, the statement is true for $m = 6, 7, 8$.

Case 2: For $k \geq 3$, we have $k(4-3) \geq 3$; so for $0 \leq a \leq 3$,

$$(f_4g_4h_4)^k (f_4, g_4)^{k(n-3)-a} \subseteq I_4^{(kn-a)}.$$

Therefore, the statement is true for all $m \geq 9$.

In Case 2, the argument does not work for $m = 5$ (since a would be 3). However, by the same argument, we can check that $zf_4^2g_4h_4^2 \in I_4^{(5)}$. Now suppose that $\alpha(I_4^{(5)}) \leq 20$. Then there is a divisor D of degree 20 vanishing to order at least 5 at every P_i in the Fermat configuration. Since the intersection of D and any line L_j in the Fermat line arrangement consists of five points to order at least 5, by Bézout's theorem, each L_j is a component of D because $\deg(D) \cdot \deg(L_j) = 20 < 5 \cdot 5$. Moreover, in the Fermat configuration, each of the coordinate points is on exactly four lines and each of the other points is on exactly three lines. Hence, the divisor $D' = D - \sum_{j=1}^{12} L_j$ of degree 8 vanishes to order at least 1 at each coordinate point and to order at least 2 along the other 16 points. Now intersecting D' with any of the lines L_j, again, since each of the lines L_j contains exactly one coordinate point and four other points, and $8 < 1 \cdot 1 + 4 \cdot 2$, we conclude by Bézout's theorem that each L_j is a component of D'. This is a contradiction, since the number of lines is 12. Therefore, $\alpha(I_4^{(5)}) = 21$. □

Remark 3.3. From Theorems 3.1 and 3.2, when $n \geq 4$, we have $\alpha(I_n^{(m)}) = nm$ for all $m \geq 3$, except for $n = 4$ and $m = 5$. This can be predicted by means of Bézout's theorem. More precisely, suppose that D is a divisor of degree nm that vanishes to order at least m along $n^2 + 3$ points of the Fermat configuration. By a similar argument using Bézout's theorem, since $nm < m(n+1)$, each line L_j is a component of D. Hence, the divisor $D_1 = D - \sum_{j=1}^{3n} L_j$ is of degree $n(m-3)$ and vanishes to order at least $m-n$ (assuming $m > n$) at each coordinate point and to order at least $m-3$ along the others n^2 points. If the number of lines is at most the degree of D', that is $3n \leq (m-3)n$, then Bézout's theorem would not yield a contradiction. Note that in this case, by Bézout's theorem again, since $n(m-3) < 1 \cdot (m-n) + n(m-3)$, each L_j is again a component of D_1. Repeating this argument k times whenever possible, the divisor $D_k = D - k\sum_{j=1}^{3n} L_j$ is of degree $n(m-3k)$ and vanishes to order at least $m-kn$ at each coordinate point and to order at least $m-3k$ along the others n^2 points. Bézout's theorem would yield a contradiction if $n(m-3k) < 1 \cdot (m-kn) + n(m-3k)$ (hence, each L_j is a component of D_k) and $n(m-3k) < 3n$ (the degree of D_k is less than the number of L_j). These two inequalities are equivalent to $m - kn \geq 1$ and $m - 3k \leq 2$. By combining them, we have $k(n-3) \leq 1$, which only happens when $n = 4$ and $k = 1$ (thus, $m = 5$), as we saw earlier.

Theorem 3.4. *For all $n \geq 3$, we have $\alpha(I_n^{(2)}) = 2(n+1)$.*

Proof. We know that $\alpha(I_n^{(2)}) \leq \alpha(I_n^2) = 2(n+1)$. Now suppose that $\alpha(I_n^{(2)}) \leq 2n+1$. Then there is a divisor D of degree $2n+1$ vanishing to order at least 2 at every P_i in the Fermat configuration. Since the intersection of D and any line L_j in the Fermat line arrangement consists of $n+1$ points to order at least 2, by Bézout's theorem, each L_j is a component of D because

$$\deg(D) \cdot \deg(L_j) = 2n + 1 < 2(n+1).$$

This is a contradiction since there are $3n$ lines and $3n > 2n + 1 = \deg(D)$ when $n \geq 3$. Therefore, $\alpha(I_n^{(2)}) = 2(n+1)$ for all $n \geq 3$. □

It is known that $\alpha(I_3^{(3m+2)}) = 9m + 8$ for all $m \geq 0$ from [30, Example 4.4] and $\alpha(I_3^{(3m)}) = 9m$ for all $m \geq 1$ from [15, Theorem 2.1] when $n = 3$. Now we compute the remaining case $\alpha(I_3^{(3m+1)})$.

Theorem 3.5. *For all $m \geq 0$, we have $\alpha(I_3^{(3m+1)}) = 9m+4$.*

Proof. Proceed by the argument in [30, Example 4.4], suppose there is $m \geq 1$ such that $\alpha(I_3^{(3m+1)}) \leq 9m+3$. Then there is a divisor D of degree $9m+3$ vanishing to order at least $3m+1$ at every point of 12 points P_i in the Fermat configuration. Intersecting D with any of the nine lines L_j, since each of the lines L_j contains exactly four points and $9m+3 < 4(3m+1)$, we conclude by Bézout's theorem that each L_j is a component of D. Hence, there exists a divisor $D' = D - \sum_{j=1}^{9} L_j$ of degree $9(m-1)+3$ vanishing to order at least $3(m-1)+1$ at every point of P_i. Repeating this argument m times, we get a contradiction with $\alpha(I_3) = 4$. Thus, $\alpha(I_3^{(3m+1)}) \geq 9m+4$ for all m.

On the other hand, by a degree argument,

$$f_3^m g_3^m h_3^{m+1} z \in (f_3, g_3)^m \cap (x, y)^m \cap (y, z)^m \cap (z, x)^m = I_3^{(3m+1)},$$

so we have $\alpha(I_3^{(3m+1)}) \leq 9m+4$ for all m. Therefore, for all m, $\alpha(I_3^{(3m+1)}) = 9m+4$. □

Example 3.6. It is worth pointing out that the first immediate application of the above calculations combined with the already known cases is the verification of Chudnovsky's conjecture and Demailly's conjecture, although the general case is already known from [18]. For any $n \geq 3$, Fermat ideals verify:

(1) Chudnovsky's conjecture:

$$\hat{\alpha}(I_n) \geq \frac{\alpha(I_n)+1}{2}.$$

(2) Demailly's conjecture:

$$\hat{\alpha}(I_n) \geq \frac{\alpha(I_n^{(m)})+1}{m+1}, \quad \forall m \geq 1.$$

Proof. Directly from the formulae of $\hat{\alpha}(I_n)$ and $\alpha(I_n^{(m)})$. □

The following containments are also direct consequences of the above calculations about $\alpha(I_n^{(m)})$. Note that these containments (and in fact, the stable version of them, i.e, the containment for $r \gg 0$) imply Chudnovsky's conjecture. First, in [3, Example 3.5], we showed the stronger containment (which implies both containments given in Corollary 1.3),

$$I_n^{(2r-2)} \subseteq \mathfrak{m}^r I_n^r$$

for $r = 6$, and thus for all $r \gg 0$. In particular, from the proof of [3, Theorem 3.1], the containments hold for $r \geq 12^2 = 144$. Here we show that the containment hold for all $r \geq 5$. Notice that for $r \leq 4$, since the resurgence number $\rho(I_n) = \frac{3}{2}$, we know that $I_n^{(2r-2)} \not\subseteq I_n^r$.

Corollary 3.7. *For every $n \geq 3$, the Fermat configuration ideal verifies the following containment:*

$$I_n^{(2r-2)} \subseteq \mathfrak{m}^r I_n^r, \quad \forall r \geq 5.$$

Proof. As before, since for all $r \geq 5$, we have $I_n^{(2r-2)} \subseteq I_n^r$, it suffices to check the inequalities

$$\alpha(I_n^{(2r-2)}) \geq r + \omega(I_n^r)$$

case-by-case:

Case 1: For $n = 3$, we have

$$r+\omega(I_n^r) = 5r \quad \text{and} \quad \alpha(I_n^{(2r-2)}) = \begin{cases} 9m, & \text{if } 2r-2 = 3m, \\ 9m+4, & \text{if } 2r-2 = 3m+1, \\ 9m+8, & \text{if } 2r-2 = 3m+2. \end{cases}$$

(a) If $2r-2 = 3m$, the inequality becomes $9m \geq \frac{5}{2}(3m+2)$, which is equivalent to $3m \geq 10$. Since $r \geq 5$, we have $3m \geq 8$. Moreover, $3m = 2r-2$ can't be 8 or 9.

(b) If $2r-2 = 3m+1$, the inequality becomes $9m+4 \geq \frac{5}{2}(3m+3)$, which is equivalent to $3m \geq 7$ (which is true because $3m = 2r-3 \geq 7$).

(c) If $2r-2 = 3m+2$, the inequality becomes $9m+8 \geq \frac{5}{2}(3m+4)$, which is equivalent to $3m \geq 4$.

Case 2: For $n \geq 4$, since $r \geq 5$, it follows that $2r-2 \geq 8$. Thus, $\alpha(I_n^{(2r-2)}) = (2r-2)n$. Furthermore, we have $(2r-2)n \geq r(n+1)+r \iff (r-2)n \geq 2r$, which is true since we have $(r-2)n \geq 4(r-2) \geq 2r$ for all $r \geq 5$. □

Although the above containment implies the Harbourne–Huneke containment for $r \geq 5$, we can check easily that the Harbourne–Huneke containment holds for all possible r by our computations.

Corollary 3.8. *For every $n \geq 3$, the Fermat configuration ideal verifies the Harbourne–Huneke containment* (*see* [24, Conjecture 2.1]):

$$I_n^{(2r)} \subseteq \mathfrak{m}^r I_n^r, \quad \forall r \geq 1.$$

Proof. Since $I_n^{(2r)} \subseteq I_n^r$, for all $r \geq 1$ the above containment comes from the fact that

$$\alpha(I_n^{(2r)}) \geq r+\omega(I_n^r)$$

for all $n \geq 3$ and $r \geq 1$ (the case $r = 0$ is trivial). Indeed, we check case-by-case:

Case 1: For $n \geq 3$ and $r = 1$, we have $2(n+1) \geq n+1+1$.

Case 2: For $n = 3$, we have

$$r+\omega(I_n^r) = 5r \quad \text{and} \quad \alpha(I_n^{(2r)}) = \begin{cases} 9m, & \text{if } 2r = 3m, \\ 9m+4, & \text{if } 2r = 3m+1, \\ 9m+8, & \text{if } 2r = 3m+2. \end{cases}$$

Case 3: For $n \geq 4$ and $r \geq 2$, we have $r+\omega(I_n^r) = r(n+2)$ and $\alpha(I_n^{(2r)}) = 2rn$. □

Corollary 3.9. *For every $n \geq 3$, the Fermat configuration ideal verifies the Harbourne–Huneke containment* (*see* [24, Conjecture 4.1.5]):

$$I_n^{(2r-1)} \subseteq \mathfrak{m}^{r-1} I_n^r, \quad \forall r \geq 3.$$

Proof. Since $\rho(I_n) = \frac{3}{2}$, for all $r \geq 3$, we have $I_n^{(2r-1)} \subseteq I_n^r$. The above containment comes from the fact that

$$\alpha(I_n^{(2r-1)}) \geq r-1+\omega(I_n^r)$$

for all $n \geq 3$ and $r \geq 3$. Notice that when $r = 1, 2$, the containment $I_n^{(2r-1)} \subseteq I_n^r$ fails. We check case-by-case:

Case 1: For $n = 3$, we have

$$r - 1 + \omega(I_n^r) = 5r - 1 \quad \text{and} \quad \alpha(I_n^{(2r-1)}) = \begin{cases} 9m, & \text{if } 2r-1 = 3m, \\ 9m+4, & \text{if } 2r-1 = 3m+1, \\ 9m+8, & \text{if } 2r-1 = 3m+2. \end{cases}$$

Case 2: For $n = 4$ and $r = 3$, we have $\alpha(I_n^{(5)}) = 21 > 3 + 5 \cdot 3 = r + \omega(I_n^3)$.

Case 3: For $n \geq 4$ and $r \neq 3$, we have $\alpha(I_n^{(2r-1)}) = (2r-1)n$. Furthermore, we have that the inequalities $(2r-1)n \geq r(n+1) + r - 1 \iff (r-1)n \geq 2r-1$, which is true since $(r-1)n \geq 4(r-1) \geq 2r-1$ for all $r \geq 3$. □

We end this section by calculating another invariant related to generating degrees of symbolic powers of I_n. As introduced in [24, Definition 2.2], for a homogeneous ideal $J \subset \mathbb{C}[\mathbb{P}^N]$, define $\beta(J)$ to be the smallest integer t such that $[J]_t$ contains a regular sequence of length two where $[J]_t$ is the graded component of degree t of J. It turns out that when $N = 2$ and J is a defining ideal of a finite set of (fat) points, $\beta(J)$ is in fact the least degree t such that the zero locus of $[J]_t$ is 0-dimensional, since the condition that $[J]_t$ contains a regular sequence of length two is equivalent to the condition that all elements of $[J]_t$ does not have a nonconstant common factor, see [25]. This invariant is related to the maximum degree of generators in a set of minimal generators of J.

Proposition 3.10. *Let $J \subset \mathbb{C}[x, y, z]$ be a defining ideal of a set of finite (fat) points. Then $\beta(J) \leq \omega(J)$.*

Proof. Suppose that $J = \langle g_1, \dots, g_k \rangle$ and $\omega(J) < \beta(J)$, then the zero locus of $[J]_{\omega(J)}$ is not 0-dimensional, by the definition of $\beta(J)$. For $j = 1, \dots, k$, consider the set A consisting of all forms $g_j x^{d_j}$, $g_j y^{d_j}$ and $g_j z^{d_j}$, where $d_j = \omega(J) - \deg(g_j)$. Then $A \subset [J]_{\omega(J)}$, hence the zero locus of A is not 0-dimensional. This is a contradiction, since the zero locus of A is also the zero locus of J. □

In the following result, we calculate $\beta(I_n^{(m)})$ and get an immediate bound for $\omega(I_n^{(m)})$:

Proposition 3.11. *For $n \geq 3$ and $m \geq 1$, we have $\beta(I_n^{(m)}) = m(n+1)$.*

Proof. First, for any $n \geq 3$, note that I_n^m is generated by all generators of the same degree $m(n+1)$. Hence, $\beta(I_n^m) = m(n+1)$, as $[I_n^m]_{m(n+1)}$ contains a regular sequence of length two. Since $I_n^m \subseteq I_n^{(m)}$, we have $\beta(I_n^{(m)}) \leq m(n+1)$ for all $m \geq 1$. On the other hand, for any $m \geq 1$, let $f \in [I_n^{(m)}]_t$ be any element where $t < m(n+1)$. Recall that each line L_j in the Fermat configuration contains exactly $n+1$ points of the configuration. Intersecting any line L_j with the variety defined by f, since f vanishes at every point in the configuration to order at least m, by Bézout's theorem, since $\deg(f)\deg(L_j) < m(n+1)$, L_j is a component of the variety of f. Therefore, for any $t < m(n+1)$, L_j is a component of the zero locus of $[I_n^{(m)}]_t$, i.e, $\beta(I_n^{(m)}) \geq m(n+1)$. □

Corollary 3.12. *For $n \geq 3$ and $m \geq 1$, we have $\omega(I_n^{(m)}) \geq m(n+1)$. Moreover, for each $n \geq 3$, $\omega(I_n^{(m)}) = m(n+1)$ for all $m \geq n^2 - 3n + 2$.*

Proof. For all m, we have that $\omega(I_n^{(m)}) \geq m(n+1)$. On the other hand, by [32, Theorem 3.10, Remark 3.11], $\operatorname{reg}(I_n^{(m)}) = m(n+1)$ for all $n \geq 3$ and $m \geq n^2 - 3n + 2$. Thus, $\omega(I_n^{(m)}) \leq \operatorname{reg}(I_n^{(m)}) = m(n+1)$ for all $n \geq 3$ and $m \geq n^2 - 3n + 2$. □

Remark 3.13. It is known that for any k and $n \geq 3$, $\omega(I_n^{(kn)}) = \beta(I_n^{(kn)}) = kn(n+1)$, since any generators of $I_n^{(kn)}$ with degree less than $kn(n+1)$ must be divisible by fgh and hence, no two elements in degree less than $kn(n+1)$ of it form a regular sequence (because they always share the common factor fgh), see [32, Theorem 3.7]. In the same paper, it is also known that $\omega(I_n^{(n-1)}) = \beta(I_n^{(n-1)}) = (n-1)(n+1)$, with the same reasoning. As in the above corollary, $\omega(I_n^{(m)}) = \beta(I_n^{(m)}) = m(n+1)$ for $m \gg 0$. It is reasonable to ask if this is the case for all m. It is suggested by Macaulay2 [22] that, in fact, $\omega(I_n^{(m)}) = m(n+1)$ for all $m \geq 1$.

4. Fermat ideal I_2

In this section, we will deal with the ideal $I_2 = \big(x(y^2 - z^2), y(z^2 - x^2), z(x^2 - y^2)\big)$. Unlike the ideals I_n for $n \geq 3$, this ideal satisfies the Harbourne containment $I_2^{(3)} \subseteq I_2^2$. This is probably one reason why it is less considered in the literature. We will see later that this ideal is very different to Fermat ideals when $n \geq 3$ in terms of the generating degree of symbolic powers and hence, in terms of the Waldschmidt constant and the (asymptotic) resurgence number.

In the following, we will compute the least degree of the generators of symbolic powers of I_2, as well as its Waldschmidt constant, and the (asymptotic) resurgence number in order to complete the picture of Fermat ideals. This ideal I_2 is also known to be the ideal determining the singular locus of the arrangement of lines given by the pseudoreflection group D_3. In general, suppose that $G \subseteq \mathrm{GL}_{n+1}(\mathbb{C})$ is a finite group generated by pseudoreflections, where a pseudoreflection is a nonidentity linear transformation that fixes a hyperplane pointwise and has finite order. Geometrically, we can view the generators of G as a hyperplane arrangement where the hyperplanes are pointwise fixed by the pseudoreflections of G. It is shown in [11, Proposition 3.9] that the singular locus (the Jacobian ideal) of the arrangement of lines correspond to $G(2, 2, 3) = D_3$ is given by I_2.

First, recall that, geometrically, I_2 is the defining ideal of the singular locus of the line arrangement in $\mathbb{P}^2$ that consists of six lines L_j whose equations are

$$x = \pm y, \quad y = \pm z, \quad z = \pm x$$

These six lines intersect at seven points P_i, which are $[1:0:0]$, $[0:1:0]$, $[0:0:1]$, $[1:1:-1]$, $[1:-1:1]$, $[1:-1:-1]$ and $[1:1:1]$ such that the first three points lie on two lines each and the rest lie on three lines each; and each line contains exactly three points.

Similar to I_n for $n \geq 3$, we can write

$$I_2 = (x^2 - y^2, y^2 - z^2) \cap (x, y) \cap (y, z) \cap (z, x),$$

so that

$$I_2^{(m)} = (x^2 - y^2, y^2 - z^2)^m \cap (x, y)^m \cap (y, z)^m \cap (z, x)^m, \quad \forall m \geq 1.$$

Theorem 4.1. *For the ideal I_2, we have*:

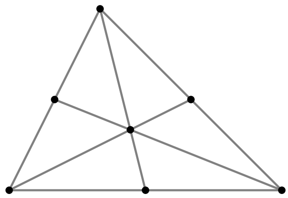
(1) $\hat{\alpha}(I_2) = \frac{5}{2}$.

(2) *For all $k \geq 2$, we have $\alpha(I_2^{(2k)}) = 5k$.*

(3) *For all $k \geq 0$, we have $\alpha(I_2^{(2k+1)}) = 5k + 3$.*

(4) $\alpha(I_2^{(2)}) = 6$.

Proof. By [19, Theorem 2.3], we can check that $\hat{\alpha}(I_2) \geq \frac{5}{2}$.



In particular, we have that $\alpha(I_2^{(2k)}) \geq 5k$ and $\alpha(I_2^{(2k+1)}) \geq 5k+3$, for all $k \geq 1$. We will show the reverse by showing there exists some element with the desired degree in the symbolic powers.

Indeed, we proceed case-by-case. Setting $K = (x^2 - y^2, y^2 - z^2)$, we have

Case 1: Assume $m = 4k$. Consider $F = (x^2 - y^2)^{2k}(y^2 - z^2)^k(z^2 - x^2)^k z^{2k}$ that has degree $10k$. Since $(x^2 - y^2)^{2k}(y^2 - z^2)^k(z^2 - x^2)^k \in K^{4k}$, we have $(x^2 - y^2)^{2k} \in (x, y)^{4k}$, $z^{2k}(y^2 - z^2)^k \in (y, z)^{4k}$ and $z^{2k}(z^2 - x^2)^k \in (z, x)^{4k}$. It follows that

$$F \in K^{4k} \cap (x, y)^{4k} \cap (y, z)^{4k} \cap (z, x)^{4k} = I_2^{(4k)}, \quad \forall k \geq 1.$$

By similar arguments, we handle the remaining cases.

Case 2: Assume $m = 4k+2$. Then $F = (x^2 - y^2)^{2k}(y^2 - z^2)^{k+1}(z^2 - x^2)^{k+1} xyz^{2k-1}$ has degree $10k+5$ and

$$F \in K^{4k+2} \cap (x, y)^{4k+2} \cap (y, z)^{4k+2} \cap (z, x)^{4k+2} = I_2^{(4k+2)}, \quad \forall k \geq 1.$$

Case 3: Assume $m = 4k+1$. Then $F = (x^2 - y^2)^{2k+1}(y^2 - z^2)^k(z^2 - x^2)^k z^{2k+1}$ has degree $10k+3$ and

$$F \in K^{4k+1} \cap (x, y)^{4k+1} \cap (y, z)^{4k+1} \cap (z, x)^{4k+1} = I_2^{(4k+1)}, \quad \forall k \geq 0.$$

Case 4: Assume $m = 4k+3$. Then $F = (x^2 - y^2)^{2k+1}(y^2 - z^2)^{k+1}(z^2 - x^2)^{k+1} xyz^{2k}$ has degree $10k+8$ and

$$F \in K^{4k+3} \cap (x, y)^{4k+3} \cap (y, z)^{4k+3} \cap (z, x)^{4k+3} = I_2^{(4k+3)}, \quad \forall k \geq 0.$$

Thus, $\alpha(I_2^{(2k)}) \leq 5k$, for all $k \geq 2$ and $\alpha(I_2^{(2k+1)}) \leq 5k+3$, for all $k \geq 0$. It follows that statements (2) and (3) are true, and since the Waldschmidt constant is the infimum of the initial degrees, $\hat{\alpha}(I_2) \leq \alpha(I_2^{(2k)})/(2k) = \frac{5}{2}$. Hence, (1) follows as well. Statement (4) can be checked directly by Macaulay2 or by Bézout's theorem argument as follows: We know that $\alpha(I_2^{(2)}) \leq \alpha(I_2^2) = 6$. Now suppose that $\alpha(I_2^{(2)}) \leq 5$. Then there is a divisor D of degree 5 vanishing to order at least 2 at every point P_i. Since the intersection of D and any line L_j consists of 3 points to order at least 2, we get a contradiction to Bézout's theorem because $\deg(D) \cdot \deg(L_j) = 5 < 2 \cdot 3$. □

For I_2, the asymptotic resurgence number and the resurgence number turn out to be the same.

Theorem 4.2. *The resurgence number and asymptotic resurgence number of I_2 are*

$$\rho(I_2) = \hat{\rho}(I_2) = \frac{6}{5}.$$

Proof. The asymptotic resurgence number $\hat{\rho}(I_2) = \frac{6}{5}$ follows from [23, Theorem 1.2] that

$$\frac{6}{5} = \frac{\alpha(I_2)}{\hat{\alpha}(I_2)} \leq \hat{\rho}(I_2) \leq \frac{\omega(I_2)}{\hat{\alpha}(I_2)} = \frac{6}{5}.$$

By [32, Theorem 2.5], since I_2 is a strict almost complete intersection ideal with minimal generators of degree 3 and its module syzygy is generated in degrees 1 and 2, the minimal free resolution of I_2^r is

$$0 \to R(-3(r+1))^{\binom{r}{2}} \xrightarrow{\psi} \begin{matrix} R(-3r-1)^{\binom{r+1}{2}} \\ \oplus \\ R(-3r-2)^{\binom{r+1}{2}} \end{matrix} \xrightarrow{\varphi} R(-3r)^{\binom{r+2}{2}} \to I_2^r \to 0$$

for any $r \geq 2$. In particular, $\operatorname{reg}(I_2^r) = 3r+1$ for all $r \geq 2$, where $\operatorname{reg}(I)$ denotes the Castelnuovo–Mumford regularity of I. We also have $\operatorname{reg}(I_2) = 4$. By [23, Theorem 1.2] again, we have $\rho(I_2) \geq \alpha(I_2)/\hat{\alpha}(I_2) = \frac{6}{5}$. Conversely, for any $\frac{m}{r} > \frac{6}{5}$, we have:

- If $m = 2k$, then $10k > 6r$ implies that $10k \geq 6r+2$ or $5k \geq 3r+1$ since both numbers are even. It follows that $\alpha(I_2^{(2k)}) \geq 5k \geq 3r+1 = \operatorname{reg}(I_2^r)$ and hence, $I_2^{(2k)} \subseteq I_2^r$.
- If $m = 2k+1$, then $10k+2 > 6r$ implies that $10k+6 \geq 6r+2$ or $5k+3 \geq 3r+1$. It follows that $\alpha(I_2^{(2k+1)}) = 5k+3 \geq 3r+1 = \operatorname{reg}(I_2^r)$ and hence, $I_2^{(2k+1)} \subseteq I_2^r$.

Thus, for any $\frac{m}{r} > \frac{6}{5}$, we have $I_2^{(m)} \subseteq I_2^r$, i.e, $\rho(I_2) = \frac{6}{5}$. □

Example 4.3. It is worth pointing out that the first direct consequence of the above calculation is the verification of I_2 to Chudnovsky's conjecture and Demailly's conjecture, although the general case is already known from [18]. Ideal I_2 verifies

(1) Chudnovsky's conjecture:

$$\hat{\alpha}(I_2) \geq \frac{\alpha(I_2)+1}{2}.$$

(2) Demailly's conjecture:

$$\hat{\alpha}(I_2) \geq \frac{\alpha(I_2^{(m)})+1}{m+1}, \quad \forall m \geq 1.$$

Proof. Follows directly from the formulae of $\hat{\alpha}(I_2)$ and $\alpha(I_2^{(m)})$. □

Example 4.4. Another difference between I_2 and I_n when $n \geq 3$ is that while $I_n^{(nk)} = (I_n^{(n)})^k$ for all k and $n \geq 3$ by [32], we have $I_2^{(4)} \neq (I_2^{(2)})^2$. In fact, as in the proof of Theorem 4.1,

$$F = (x^2-y^2)^2(y^2-z^2)(z^2-x^2)z^2 \in I_2^{(4)},$$

whereas, we can check that $F \notin (I_2^{(2)})^2$. Moreover, we can also check that $I_2^{(6)} \neq (I_2^{(3)})^2$, since

$$G = (x^2-y^2)^2(y^2-z^2)^2(z^2-x^2)^2xyz \in I_2^{(6)} \setminus (I_2^{(3)})^2.$$

It is suggested by Macaulay2 [22] that $I_2^{(8)} = (I_2^{(4)})^2$. It would be interesting to know if $I_2^{(4k)} = (I_2^{(4)})^k$ for all k.

As with all other Fermat ideals, I_2 also satisfies the following containment. In [3, Example 3.7], we showed the stronger containment (which implies both Harbourne–Huneke) containment

$$I_2^{(2r-2)} \subseteq \mathfrak{m}^r I_2^r,$$

for $r = 5$ (by Macaulay2), and thus for all $r \gg 0$ by our method. In particular, from the proof of [3, Theorem 3.1], the containment holds for $r \geq 10^2 = 100$. Here, we show that the containment holds for all $r \geq 5$.

Corollary 4.5. *For every $n \geq 3$, the ideal I_2 verifies the following stronger containment*:

$$I_2^{(2r-2)} \subseteq \mathfrak{m}^r I_2^r, \quad \forall r \geq 5.$$

Proof. As before, since $\rho(I_2) = \frac{6}{5}$, we know that $I_2^{(2r-2)} \subseteq I_2^r$ for $r \geq 3$. Hence, the containment follows from the inequality

$$\alpha(I_2^{(2r-2)}) = 5r - 5 \geq r + 3r = r + \omega(I_2^r), \quad \forall r \geq 5. \qquad \square$$

We can also detect the failure of the containment in the remaining cases by only using formulae for $\alpha(I_2^{(m)})$.

Remark 4.6. For $r \leq 4$, the above containment fails. In fact, notice that for $r \leq 2$, since $\rho(I_2) = \frac{6}{5}$, we know that $I_2^{(2r-2)} \not\subseteq I_2^r$. When $r = 3$, since $\alpha(I_2^{(4)}) = 10 < 3 + 9 = 3 + \alpha(I_2^3)$, we see that the containment $I_2^{(4)} \subseteq \mathfrak{m}^3 I_2^3$ fails. Similarly, for $r = 4$, since $\alpha(I_2^{(6)}) = 15 < 4 + 12 = 4 + \alpha(I_2^4)$, we see that the containment $I_2^{(6)} \subseteq \mathfrak{m}^4 I_2^4$ also fails.

Although the above containment implies the Harbourne–Huneke containment for $r \geq 5$, we can check easily that the Harbourne–Huneke containment holds for all possible r by our computations.

Corollary 4.7 (see [24, Conjecture 2.1]). *Ideal I_2 verifies the Harbourne–Huneke containment*

$$I_2^{(2r)} \subseteq \mathfrak{m}^r I_2^r, \quad \forall r \geq 1.$$

Proof. Since for all r, we have $I_2^{(2r)} \subseteq I_2^r$, the containment follows from the fact that

$$\alpha(I_2^{(2r)}) \geq 5r \geq r + 3r = r + \omega(I_2^r), \quad \forall r. \qquad \square$$

Corollary 4.8 (see [24, Conjecture 4.1.5]). *Ideal I_2 verifies the Harbourne–Huneke containment*

$$I_2^{(2r-1)} \subseteq \mathfrak{m}^{r-1} I_2^r, \quad \forall r \geq 1.$$

Proof. Since $\rho(I_2) = \frac{6}{5}$, for all $r \geq 2$, we have $I_2^{(2r-1)} \subseteq I_2^r$. The above containment comes from the fact that

$$\alpha(I_2^{(2r-1)}) = 5r - 2 \geq r - 1 + 3r = r - 1 + \omega(I_2^r), \quad \forall r \geq 2.$$

The case $r = 1$ is obvious. $\square$

Remark 4.9. The above corollary gives a proof for the case D_3 in [11, Proposition 6.3].

We end this section by calculating $\beta(I_2^{(m)})$.

Proposition 4.10. *For all $m \geq 1$, we have $\beta(I_2^{(m)}) = 3m$ and $\omega(I_2^{(m)}) \geq 3m$.*

Proof. The proof is the same as that of the case where $n \geq 3$. First, since $I_2^m \subseteq I_2^{(m)}$, we have that $\beta(I_2^{(m)}) \leq 3m$ for all $m \geq 1$. On the other hand, recall that each line L_j in the configuration contains exactly three points of the configuration. Thus, for any $m \geq 1$ and for any $f \in [I_n^{(m)}]_t$ where $t < 3m$, intersecting any line L_j with the variety defined by f, by Bézout's theorem, since $\deg(f)\deg(L_j) < 3m$, L_j is a component of the variety of f. Therefore, for any $t < 3m$, it follows that L_j is a component of the zero locus of $[I_2^{(m)}]_t$, i.e, $\beta(I_2^{(m)}) \geq 3m$. $\square$

Remark 4.11. As in the case where $n \geq 3$, Macaulay2 [22] suggests that $\omega(I_2^{(m)}) = 3m$. It would be interesting to know if $\omega(J^{(m)}) = \beta(J^{(m)})$ holds for any radical ideal of points J in general. As suggested by the referee, the answer is no, in general. Consider eight general points in $\mathbb{P}^2$ with defining ideal I. Then there are two cubics among the generators of I, which form a regular sequence, i.e., intersect in nine points. By the Cayley–Bacharach theorem, since any cubic containing eight points also contains the ninth point, one must use a form of degree at least 4 (and, in fact, exactly 4) to exclude the ninth point from the defining ideal of eight points. Thus, $\beta(I) = 3$ but $\omega(I) \geq 4$.

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