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A NEW PROOF OF STANLEY'S THEOREM ON THE STRONG LEFSCHETZ PROPERTY

BY

HO V. N. PHUONG and QUANG HOA TRAN (Hue City)

Abstract. A standard graded artinian monomial complete intersection algebra $A = \Bbbk[x_1, \dots, x_n]/(x_1^{a_1}, \dots, x_n^{a_n})$, with $\Bbbk$ a field of characteristic zero, has the strong Lefschetz property defined by Stanley in 1980. In this paper, we give a new proof for this result by using only the basic linear algebra. Furthermore, our proof is still valid in the case where the characteristic of $\Bbbk$ is greater than the socle degree of A, namely $a_1 + \dots + a_n - n$.

1. Introduction. Let $\Bbbk$ be a field and $R = \Bbbk[x_1, \dots, x_n]$ be a standard graded polynomial ring over $\Bbbk$ in n variables. A graded artinian $\Bbbk$-algebra $A = R/I$ is said to have the *strong Lefschetz property* if there is a linear form $\ell \in A_1$ such that the multiplication map

$$\times \ell^s : A_i \to A_{i+s}$$

has maximal rank for all s and all i, i.e., $\times \ell^s$ is either injective or surjective, for all s and all i. Such a linear form ℓ is called a *strong Lefschetz element* of A.

Three papers represent the beginning of the study of what is presently called *Lefschetz properties*: by R. P. Stanley [S80], J. Watanabe [W87], and L. Reid, L. G. Roberts and M. Roitman [RRR91]. These papers proved essentially the same result:

THEOREM (Stanley's theorem). *If $\Bbbk$ is a field of characteristic zero and $R = \Bbbk[x_1, \dots, x_n]$, then every artinian monomial complete intersection algebra*

$$A = R/(x_1^{a_1}, \dots, x_n^{a_n})$$

has the strong Lefschetz property with $x_1 + \dots + x_n$ as a strong Lefschetz element.

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In the case $\Bbbk = \mathbb{C}$, Stanley used the fact that A is isomorphic to the cohomology ring of a product of projective spaces and he applied the Hard Lefschetz Theorem to conclude that the divisor lattice of monomials in A has the Sperner property. Watanabe proved the result by using the theory of modules over the special linear Lie algebra $\mathfrak{sl}_2$, and Reid, Roberts and Roitman used Hilbert function techniques from commutative algebra. The conclusion as well as the techniques of proof in these three papers are of interest in algebraic geometry, combinatorics, representation theory, and commutative algebra.

In this paper, we will give a new proof of this theorem by using only the basic linear algebra. Furthermore, our proof is valid not only when the characteristic of $\Bbbk$ is zero, but also when the characteristic is large enough; see Theorem 3.1. More precisely, first let $I = (x_1^2, \ldots, x_n^2)$ be an artinian quadratic monomial complete intersection ideal of R. We see that the set of all square-free monomials of degree i forms a $\Bbbk$-basis of the $\Bbbk$-vector space B_i for all i, where $B = R/I$. Based on these bases, we construct the matrix of the multiplications $\times(x_1 + \cdots + x_n)^t : B_i \to B_{i+t}$ for all $0 \le i \le n$ and $0 \le t \le n - i$ and show that these multiplications have maximal rank. The main result is the following.

Theorem (Theorem 2.7). *Assume $\Bbbk$ is of characteristic zero or greater than n. Then the algebra $B = R/(x_1^2, \ldots, x_n^2)$ has the strong Lefschetz property with $x_1 + \cdots + x_n$ as a strong Lefschetz element.*

Then we show that any artinian monomial complete intersection A can be viewed as a subalgebra of an artinian quadratic monomial complete intersection B such that A and B have the same socle degree. By applying Theorem 2.7, we deduce that A also has the strong Lefschetz property, that is, Stanley's theorem is proved. Furthermore, our proof is still valid in the case where the characteristic of $\Bbbk$ is greater than the socle degree of A, namely $a_1 + \cdots + a_n - n$; see Theorem 3.1.

2. Artinian quadratic monomial complete intersection algebras

Definition 2.1. For any standard graded artinian $\Bbbk$-algebra $A = R/I = \bigoplus_{i=0}^{d} A_i$, the *Hilbert function* of A is the function

$$h_A : \mathbb{N} \to \mathbb{N}$$

defined by $h_A(t) = \dim_{\Bbbk}(A_t)$. As A is artinian, its Hilbert function is equal to its *h-vector* that one can express as a finite sequence

$$\underline{h}_A = (h_0, h_1, \ldots, h_d),$$

where $h_i = h_A(i) > 0$ and d is the last index with this property. The integer d is called the *socle degree* of A.

The h-vector $\underline{h}_A$ is said to be *unimodal* if there exists an integer $t \geq 1$ such that $h_0 \leq h_1 \leq \cdots \leq h_t \geq h_{t+1} \geq \cdots \geq h_d$. The h-vector $\underline{h}_A$ is said to be *symmetric* if $h_{d-i} = h_i$ for every $i = 0, 1, \ldots, \lfloor d/2 \rfloor$.

In this section, we consider the case where $I = (x_1^2, \ldots, x_n^2)$ is an artinian quadratic monomial complete intersection ideal in $R = \Bbbk[x_1, \ldots, x_n]$. Now set $A = R/I = \bigoplus_{j=0}^n A_j$. Hence $A = R/I$ is an artinian complete intersection of socle degree n, namely $h_A(j) = 0$ for all $j > n$ and moreover

$$h_A(j) = h_A(n-j) = \binom{n}{j}$$

for all $j = 0, 1, \ldots, n$. The h-vector of A is

$$\left(1, n, \binom{n}{2}, \binom{n}{3}, \ldots, \binom{n}{3}, \binom{n}{2}, n, 1\right).$$

In particular, the h-vector of A is unimodal and symmetric. Furthermore, we have the following.

Lemma 2.2. *The set of all square-free monomials forms a $\Bbbk$-basis of A.*

We denote by $\mathfrak{B}$ the set of all square-free monomials in R and by $\mathfrak{B}_t$ the set of all square-free monomials of degree t in R. By Lemma 2.2, $\mathfrak{B}$ is a $\Bbbk$-basis of A, and $\mathfrak{B}_t$ is a $\Bbbk$-basis of A_t. We will list the monomials of $\mathfrak{B}_t$ in decreasing order with respect to the reverse lexicographic order with $x_1 > \cdots > x_n$, i.e.,

$$\mathfrak{B}_t = \{x_{i_1} \ldots x_{i_t} \mid 1 \leq i_1 < \cdots < i_t \leq n\}.$$

For example,

$$\begin{aligned}
\mathfrak{B}_1 &= \{x_1, x_2, \ldots, x_n\}, \\
\mathfrak{B}_2 &= \{x_1x_2, x_1x_3, x_2x_3, \ldots, x_1x_n, x_2x_n, x_3x_n, \ldots, x_{n-1}x_n\}, \\
\mathfrak{B}_3 &= \{x_1x_2x_3, x_1x_2x_4, \ldots, x_{n-3}x_{n-2}x_{n-1}, x_1x_2x_n, x_1x_3x_n, \ldots, \\
&\qquad x_{n-2}x_{n-1}x_n\}.
\end{aligned}$$

Denote by $M_{m\times n}(\Bbbk)$ the set of all $m \times n$ matrices with entries in the field $\Bbbk$. For any $M \in M_{m\times n}(\Bbbk)$, it is known that $\operatorname{rank}(M) \leq \min\{m, n\}$. We say that M has *maximal rank* if $\operatorname{rank}(M) = \min\{m, n\}$.

Now we fix a general linear form $\ell = a_1x_1 + \cdots + a_nx_n$ of R. Consider the multiplication $\times\ell^t : A_i \to A_{i+t}$ for some $0 \leq i \leq n$ and $0 \leq t \leq n-i$. Let M_i^t be the matrix of $\times\ell^t$ with respect to the bases $\mathfrak{B}_i$ and $\mathfrak{B}_{i+t}$. Thus $\times\ell^t : A_i \to A_{i+t}$ has maximal rank if and only if M_i^t has maximal rank. When $t = 1$, we will write M_i instead of M_i^1. Note that M_i^t is the identity matrix, when $t = 0$.

Proposition 2.3. *With the above notation the following assertions are equivalent:*

(i) *A has the strong Lefschetz property.*
(ii) *M_i^t has maximal rank for all $0 \leq i \leq n$ and $0 \leq t \leq n-i$.*
(iii) *M_i^{n-2i} has maximal rank for all $0 \leq i < n/2$.*

Proof. Clearly, (i) is equivalent to (ii) by the definition. The equivalence of (ii) and (iii) follows from the basic properties of compositions of linear applications and the fact that $\dim_k(A_i) = \dim_k(A_{n-i})$ for all $0 \leq i < n/2$. ∎

Now, set $\overline{R} := R/(x_n) \cong \mathbb{k}[x_1, \ldots, x_{n-1}]$ and denote by $\overline{I}$ the image of I in $\overline{R}$. Therefore, $\overline{I} = (x_1^2, \ldots, x_{n-1}^2)$ and

$$\overline{A} := \overline{R}/\overline{I} = \frac{\mathbb{k}[x_1, \ldots, x_{n-1}]}{(x_1^2, \ldots, x_{n-1}^2)}$$

is also an artinian quadratic monomial complete intersection algebra. Denote by $\overline{\mathfrak{B}}_t$ the set of all square-free monomials of degree t in $\overline{R}$. By Lemma 2.2,

$$\overline{\mathfrak{B}}_t = \{x_{i_1} \ldots x_{i_t} \mid 1 \leq i_1 < \cdots < i_t \leq n-1\}$$

is a $\mathbb{k}$-basis of $\overline{A}_t$. Note that

$$\mathfrak{B}_t = \overline{\mathfrak{B}}_t \sqcup \mathfrak{B}'_t, \tag{2.1}$$

where

$$\mathfrak{B}'_t = \{x_{i_1} \ldots x_{i_{t-1}} x_n \mid 1 \leq i_1 < \cdots < i_{t-1} \leq n-1\}.$$

We identify $\mathfrak{B}'_t$ with the set

$$\overline{\mathfrak{B}}_{t-1} = \{x_{i_1} \ldots x_{i_{t-1}} \mid 1 \leq i_1 < \cdots < i_{t-1} \leq n-1\}.$$

Let $\ell = a_1 x_1 + \cdots + a_n x_n$ be a linear form in R and let $\overline{\ell}$ be the image of ℓ in $\overline{R}$. We denote by $\overline{M}_i^t$ the matrix of $\times\overline{\ell}^t : \overline{A}_i \to \overline{A}_{i+t}$ with respect to the bases $\overline{\mathfrak{B}}_i$ and $\overline{\mathfrak{B}}_{i+t}$.

Lemma 2.4. *For any $1 \leq i \leq n-1$ and $1 \leq t \leq n-i$, the matrix M_i^t of $\times \ell^t : A_i \to A_{i+t}$ with respect to the bases $\mathfrak{B}_i$ and $\mathfrak{B}_{i+t}$ is a 2×2 block matrix of the form*

$$M_i^t = \begin{bmatrix} \overline{M}_i^t & 0 \\ a_n t \overline{M}_i^{t-1} & \overline{M}_{i-1}^t \end{bmatrix},$$

where 0 is the zero matrix.

Proof. We see immediately that

$$\ell^t = \sum_{j=0}^{t} \binom{t}{j} \overline{\ell}^{t-j} (a_n x_n)^j = \overline{\ell}^t + a_n t \overline{\ell}^{t-1} x_n$$

in A. The result follows from the definition of the matrix M_i^t and the decomposition of the bases $\mathfrak{B}_i$ and $\mathfrak{B}_{i+t}$ as in (2.1) where we identify

$$\mathfrak{B}'_i \equiv \overline{\mathfrak{B}}_{i-1} \quad \text{and} \quad \mathfrak{B}'_{i+t} \equiv \overline{\mathfrak{B}}_{i+t-1}. \blacksquare$$

EXAMPLE 2.5. Assume the characteristic of $\Bbbk$ is zero or greater than 4. Consider $R = \Bbbk[x_1, x_2, x_3, x_4]$, $A = R/(x_1^2, x_2^2, x_3^2, x_4^2)$ and the linear form $\ell = x_1 + x_2 + x_3 + x_4$. Then the matrix of the multiplication $\times\ell^2 : A_1 \to A_3$ with respect to the bases $\mathfrak{B}_1$ and $\mathfrak{B}_3$ is

$$M_1^2 = \left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ \hline 2 & 2 & 0 & 2 \\ 2 & 0 & 2 & 2 \\ 0 & 2 & 2 & 2 \end{array}\right] = \begin{bmatrix} \overline{M}_1^2 & 0 \\ 2\overline{M}_1 & \overline{M}_0^2 \end{bmatrix}.$$

A straightforward computation shows that $\det(M_1^2) = -2^4 \cdot 3^2 \neq 0$, hence the map $\times\ell^2 : A_1 \to A_3$ is an isomorphism.

The following lemma is useful to determine the rank of block matrices.

LEMMA 2.6. *Let $A \in M_{m\times n}(\Bbbk)$, $B \in M_{n\times p}(\Bbbk)$, $P \in M_{n\times n}(\Bbbk)$ be such that P is non-singular. Assume that M is an $(m+n)\times(n+p)$ matrix which can be written in the form of a 2×2 block matrix as follows:*

$$M = \begin{bmatrix} AP & 0 \\ P & PB \end{bmatrix}.$$

Then

$$\operatorname{rank}(M) = n + \operatorname{rank}(APB).$$

Proof. We observe that

$$\begin{bmatrix} I_m & -A \\ 0 & P^{-1} \end{bmatrix} \times \begin{bmatrix} AP & 0 \\ P & PB \end{bmatrix} \begin{bmatrix} I_n & B \\ 0 & -I_p \end{bmatrix} = \begin{bmatrix} 0 & APB \\ I_n & 0 \end{bmatrix},$$

and conclude immediately. ∎

Now we prove the main result in this section.

THEOREM 2.7. *Assume that the characteristic of $\Bbbk$ is zero or greater than n. Then the algebra $A = R/(x_1^2, \dots, x_n^2)$ has the strong Lefschetz property with $\ell = x_1 + \cdots + x_n$ as a strong Lefschetz element.*

Proof. Note first that since $\dim_\Bbbk(A_0) = \dim_\Bbbk(A_n) = 1$, the matrix of the map $\times\ell^n : A_0 \to A_n$ is $M_0^n = (n!)$. Therefore, $\det(M_0^n) = n! \neq 0$ since the characteristic of $\Bbbk$ is zero or greater than n.

Now to prove the theorem, by Proposition 2.3, it is enough to show that the matrix M_i^{n-2i} of $\times\ell^{n-2i} : A_i \to A_{n-i}$ has maximal rank for all $0 \le i < n/2$. We show the assertion by induction on n.

First, let $n = 1, 2$. We only have the case $i = 0$. However, in this case the assertion holds as we have remarked at the beginning of the proof. Now we assume that the assertion holds for any artinian quadratic monomial complete intersection algebra in the polynomial ring in $< n$ variables. For integers $n \geq 3$ and i satisfying $0 < i < n/2$, we have to show that the matrix M_i^{n-2i} of $\times \ell^{n-2i} : A_i \to A_{n-i}$ has maximal rank. By Lemma 2.4, M_i^{n-2i} can be written in the form of a 2×2 block matrix as follows:

$$M_i^{n-2i} = \begin{bmatrix} \overline{M}_i^{n-2i} & 0 \\ (n-2i)\overline{M}_i^{n-2i-1} & \overline{M}_{i-1}^{n-2i} \end{bmatrix}.$$

Note that $0 < n - 2i < n$, hence $n - 2i$ is an invertible element of $\Bbbk$. Observe that M_i^{n-2i} and $\overline{M}_i^{n-2i-1}$ are square matrices of size $\binom{n}{i} \times \binom{n}{i}$ and $\binom{n-1}{i} \times \binom{n-1}{i}$, respectively. Moreover,

$$\overline{M}_i^{n-2i} = \overline{M}_{n-i-1}\overline{M}_i^{n-2i-1} \quad \text{and} \quad \overline{M}_{i-1}^{n-2i} = \overline{M}_i^{n-2i-1}\overline{M}_{i-1}.$$

As $\overline{A} = \overline{R}/\overline{I}$ has the strong Lefschetz property by the inductive hypothesis, $\overline{M}_i^{n-2i-1}$ is a non-singular matrix. It follows from Lemma 2.6 that

$$\operatorname{rank}(M_i^{n-2i}) = \binom{n-1}{i} + \operatorname{rank}(\overline{M}_{n-i-1}\overline{M}_i^{n-2i-1}\overline{M}_{i-1}).$$

We observe that $\overline{M}_{n-i-1}\overline{M}_i^{n-2i-1}\overline{M}_{i-1}$ is the matrix of

$$\times\overline{\ell}^{n-2i+1} : \overline{A}_{i-1} \to \overline{A}_{n-i}.$$

It is an isomorphism since $\overline{A}$ has the strong Lefschetz property. Hence

$$\operatorname{rank}(\overline{M}_{n-i-1}\overline{M}_i^{n-2i-1}\overline{M}_{i-1}) = \binom{n-1}{i-1}.$$

It follows that

$$\operatorname{rank}(M_i^{n-2i}) = \binom{n-1}{i} + \binom{n-1}{i-1} = \binom{n}{i}.$$

This implies that $\ell^{n-2i} : A_i \to A_{n-i}$ is an isomorphism, which finishes the induction and the proof. ∎

3. Proof of Stanley's theorem. Finally, we show that Stanley's theorem follows from Theorem 2.7.

THEOREM 3.1. *Let $\Bbbk$ be a field and $R = \Bbbk[x_1, \ldots, x_n]$. Then*

$$R/(x_1^{d_1}, \ldots, x_n^{d_n})$$

has the strong Lefschetz property if the characteristic of $\Bbbk$ is zero or greater than $d_1 + \cdots + d_n - n$.

Proof. For simplicity, denote

$$A = R/(x_1^{a_1+1}, \ldots, x_n^{a_n+1})$$

with $a_1, \ldots, a_n$ being positive integers. Assume that the characteristic of $\Bbbk$ is zero or greater than $a_1 + \cdots + a_n$. We need to show that A has the strong Lefschetz property.

Note that A is an artinian monomial complete intersection algebra with socle degree $m = a_1 + \cdots + a_n$. Set $\alpha_i = \sum_{j=1}^{i} a_j$ for $i = 1, \ldots, n$. Now we let $B = \Bbbk[x_1, \ldots, x_m]/(x_1^2, \ldots, x_m^2)$. By Theorem 2.7, the algebra B has the strong Lefschetz property with $\ell = x_1 + \cdots + x_m$ as a strong Lefschetz element. Set $S := \Bbbk[y_1, \ldots, y_n]$ and consider the algebra homomorphism $\phi : S \to B$ given by

$$\begin{aligned} y_1 &\mapsto x_1 + \cdots + x_{\alpha_1}, \\ y_2 &\mapsto x_{\alpha_1+1} + \cdots + x_{\alpha_2}, \\ &\cdots \\ y_n &\mapsto x_{\alpha_{n-1}+1} + \cdots + x_{\alpha_n}. \end{aligned}$$

Set $J = (y_1^{a_1+1}, \ldots, y_n^{a_n+1})$. We have the following assertion.

Claim. $\operatorname{Ker}(\phi) = J$.

Proof of Claim. First, for each $j = 1, \ldots, n$, we see that

$$\phi(y_j^{a_j+1}) = (\underbrace{x_{\alpha_{j-1}+1} + \cdots + x_{\alpha_j}}_{a_j \text{ variables}})^{a_j+1} = 0$$

in B. In other words, $y_j^{a_j+1} \in \operatorname{Ker}(\phi)$, so $J \subset \operatorname{Ker}(\phi)$. Assume the contrary that $\operatorname{Ker}(\phi) \neq J$. Select a homogeneous element f of largest degree with $f \in \operatorname{Ker}(\phi)$ and $f \notin J$. It follows that f represents a non-trivial element in the socle of S/J. Note that S/J is an artinian monomial complete intersection algebra and its socle is a $\Bbbk$-vector space spanned by $y_1^{a_1} \ldots y_n^{a_n}$. Hence there exists a non-zero element $c \in \Bbbk$ such that $f = cy_1^{a_1} \ldots y_n^{a_n} + g$, where $g \in J$. Thus, $\phi(g) = 0$ and

$$\phi(f) = c(x_1+\cdots+x_{\alpha_1})^{a_1} \ldots (x_{\alpha_{n-1}+1}+\cdots+x_{\alpha_n})^{a_n} = ca_1! \ldots a_n! x_1x_2 \ldots x_m.$$

Since the characteristic of $\Bbbk$ is zero or greater than $m = a_1 + \cdots + a_n$, the element $ca_1! \ldots a_n!$ is invertible in $\Bbbk$. Thus $\phi(f) = ca_1! \ldots a_n! x_1 \ldots x_m \neq 0$ in B. This contradicts $f \in \operatorname{Ker}(\phi)$, completing the proof of the Claim.

By the above Claim, we get the algebra isomorphisms

$$A \simeq S/\operatorname{Ker}(\phi) \simeq \operatorname{Im}(\phi) \subseteq B.$$

It follows that we can identify A with a subalgebra of B. Furthermore, A and B have the same socle degree, namely m. We have the commutative diagrams

$$\begin{tikzcd} A_0 \arrow[r, "\times\ell"] \arrow[d, hook] & A_1 \arrow[r, "\times\ell"] \arrow[d, hook] & \cdots \arrow[r, "\times\ell"] & A_{m-1} \arrow[r, "\times\ell"] \arrow[d, hook] & A_m \arrow[d, hook] \\ B_0 \arrow[r, "\times\ell"] & B_1 \arrow[r, "\times\ell"] & \cdots \arrow[r, "\times\ell"] & B_{m-1} \arrow[r, "\times\ell"] & B_m \end{tikzcd} \tag{3.1}$$

where the vertical maps are the canonical inclusions and $\ell = x_1 + \cdots + x_m$. Since B has the strong Lefschetz property, $\times \ell^{m-2i} : B_i \to B_{m-i}$ is bijective for all $0 \leq i < m/2$. By (3.1), we see that $\times \ell^{m-2i} : A_i \to A_{m-i}$ is injective for all $0 \leq i < m/2$, hence bijective because $\dim_{\Bbbk}(A_i) = \dim_{\Bbbk}(A_{m-i})$. Thus A has the strong Lefschetz property. ∎

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Ho V. N. Phuong
University of Sciences
Hue University
Hue City, Vietnam
E-mail: hvnphuong@husc.edu.vn

Quang Hoa Tran
University of Education
Hue University
Hue City, Vietnam
E-mail: tranquanghoa@hueuni.edu.vn