Intersection of $(1,1)$*-Currents and the Domain of Definition of the Monge-Ampère Operator*

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ABSTRACT. We study the Monge-Ampère operator within the framework of Dinh-Sibony's intersection theory defined via density currents. We show that if u is a plurisubharmonic function belonging to the Błocki-Cegrell class, then the Dinh-Sibony n-fold self-product of $\mathrm{dd}^c u$ exists and coincides with the classically defined Monge-Ampère measure $(\mathrm{dd}^c u)^n$.

1. INTRODUCTION

Let Ω be a domain in $\mathbb{C}^n$ and let u be a plurisubharmonic (p.s.h. for short) function on Ω. A question of central importance in pluripotential theory and its applications is whether one can define the Monge-Ampère measure

$$(\mathrm{dd}^c u)^n = \mathrm{dd}^c u \wedge \cdots \wedge \mathrm{dd}^c u$$

in a meaningful way. Recall that $\mathrm{d}^c := (i/(2\pi))(\bar{\partial} - \partial)$ and $\mathrm{dd}^c = (i/\pi)\,\partial\,\bar{\partial}$.

For bounded p.s.h. functions, the definition of $(\mathrm{dd}^c u)^n$ and the study of its fundamental properties are due to Bedford-Taylor [BT76]. The problem of finding the largest class of p.s.h. functions where the Monge-Ampère operator is suitably defined and continuous under decreasing sequences was studied for a long time, and a complete characterization of this class was finally achieved by Cegrell [Ceg04] and Błocki [Blo06]. We denote this class by $\mathcal{D}(\Omega)$ and call it the *Błocki-Cegrell class*.

The question of defining $(\mathrm{dd}^c u)^n = \mathrm{dd}^c u \wedge \cdots \wedge \mathrm{dd}^c u$ is an instance of the fundamental problem of intersection of currents. Indeed, if we set $T := \mathrm{dd}^c u$, then T is a positive closed $(1,1)$-current on Ω and $(\mathrm{dd}^c u)^n$ is the self-intersection $T^n = T \wedge \cdots \wedge T$. The intersection theory of currents has been quite well developed thanks to the work of many authors. The case of bi-degree $(1,1)$-currents

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is more accessible because of the existence of p.s.h. functions as local potentials. For this reason, this case was soon developed (see [CLN69, BT76, FS95, Dem]). Later on, other notions of intersection were introduced, such as the non-pluripolar product [BT87, BEGZ10] (see also [Vu20-2] for recent developments) and the Andersson-Wulcan product of $(1,1)$-currents with analytic singularities [AW14]. All these generalized notions differ from classical ones by the fact that, in one way or another, one removes the singular set of the currents before intersecting them. As a drawback, there is a mass loss in this procedure.

A general intersection theory for currents of higher bi-degree was developed only later. Most notably, Dinh-Sibony proposed two different notions of intersection, one using what they call superpotentials [DS09] and, more recently, another one based on the notion of density currents, that we consider here. We refer to the original paper [DS18] and also [Vu19] for generalizations and simplified arguments. Both approaches have already found many applications in dynamical systems and foliation theory.

The main goal of the present paper is to study the Monge-Ampère operator from the point of view of the theory of density currents. We now briefly recall this notion. More details are given in Section 2.

Let X be a complex manifold and let $T_1, \dots, T_m$ be positive closed currents on X. Consider the Cartesian product X^m and the positive closed current $\mathbf{T} = T_1 \otimes \cdots \otimes T_m$ on X^m. Let

$$\Delta = \{(x, \dots, x) : x \in X\} \subset X^m$$

be the diagonal and $N\Delta$ be its normal bundle inside X^m. Using a certain type of local coordinates τ in X^m around Δ with values in $N\Delta$, which are called admissible maps, we can consider the current $\tau_* \mathbf{T}$ defined around the zero section of $N\Delta$.

For $\lambda \in \mathbb{C}^*$, let $A_\lambda : N\Delta \to N\Delta$ be the fiberwise multiplication by λ. A *density current* R associated with $(T_1, \dots, T_m)$ is a positive closed current on $N\Delta$ such that there exists a sequence of complex numbers $\{\lambda_k\}_{k \in \mathbb{N}}$ converging to ∞ for which

$$R = \lim_{k \to \infty} (A_{\lambda_k})_* \tau_* \mathbf{T},$$

for every admissible map τ. We then say that the *Dinh-Sibony product* $T_1 \curlywedge \cdots \curlywedge T_m$ of $T_1, \dots, T_m$ exists if there is only one density current R associated with $(T_1, \dots, T_m)$ and $R = \pi^* S$ for some positive closed current on Δ, where $\pi : N\Delta \to \Delta$ is the canonical projection. In that case we define

$$T_1 \curlywedge \cdots \curlywedge T_m := S.$$

Our main result is the following (see Theorem 4.5 below).

Main Theorem. *Let Ω be a domain in $\mathbb{C}^n$ and let $u_1, u_2, \dots, u_m$, $1 \leq m \leq n$, be plurisubharmonic functions in the Błocki-Cegrell class. Then, the Dinh-Sibony* *product of* $\mathrm{dd}^c u_1, \ldots, \mathrm{dd}^c u_m$ *is well defined, and*

$$\mathrm{dd}^c u_1 \curlywedge \cdots \curlywedge \mathrm{dd}^c u_m = \mathrm{dd}^c u_1 \wedge \cdots \wedge \mathrm{dd}^c u_m. \tag{1.1}$$

In particular, for every u *in the Błocki-Cegrell class, the operator*

$$u \mapsto (\mathrm{dd}^c u)^{\curlywedge n} := \mathrm{dd}^c u \curlywedge \cdots \curlywedge \mathrm{dd}^c u$$

is well defined and coincides with the usual Monge-Ampère operator.

The last conclusion of the above theorem says, in other words, that the domain of definition of the Monge-Ampère operator defined via Dinh-Sibony's product contains the Błocki-Cegrell class. The righthand side of (1.1) is obtained in the standard way, that is, by considering sequences of smooth p.s.h. functions decreasing to u_j, $j = 1, \ldots, m$, much as in the case where $m = n$ and all the u_j are equal. The fact that this mixed product is well defined and independent of the chosen sequence is the content of Proposition 4.2 below. Although not explicitly stated in the literature, this fact might be well known among experts. The case $m = n$ can be found in [Ceg04], and the case $m < n$ covered by Proposition 4.2 follows from simple modifications of arguments from [Ceg04] and [Blo06].

Our main theorem is a strengthening of Theorem 1.1 in [KV19] proven by the last two authors, and it follows from a more general result (see Theorem 3.1). This yields an optimal result that covers the most general case where the classical Monge-Ampère operator is well defined and continuous with respect to decreasing sequences.

The proof of Theorem 3.1 in the present work uses different techniques than the ones in [KV19], providing new and clearer arguments. In particular, the integrability assumption on the u_j required in [KV19] cannot be dropped with the techniques used there, so they cannot be applied in our situation. Here, we bypass this difficulty and show that those assumptions are actually unnecessary, yielding the optimal result. In order to achieve that, we obtain almost everywhere vanishing properties for Lelong numbers with respect to singular currents in the considered class (cf. Lemma 3.2). Also, by systematically working with a well-chosen set of test forms (cf. Definition 3.3), we can easily get the compactness of the family of dilated currents (Lemma 3.8 below), which was overlooked in [KV19]. We refer to the end of this introduction and the comments after Theorem 3.1 for an overview of the arguments and the new ingredients of the proof.

It is worth mentioning that the many notions of intersection quoted here are related to one another. As mentioned before, the present paper together with [KV19] show that the intersection via density currents cover all known classical products of $(1,1)$-currents. For higher bi-degree currents it is also known that the density product generalizes the product of currents with continuous super-potentials (see [DNV18]). Concerning generalized notions of products of $(1,1)$-currents, such as the non-pluripolar product and the Andersson-Wulcan product, some comparison results were obtained in [KV19]. The general phenomenon is that, in some sense, these products are dominated by the corresponding density currents. Moreover, the ideas of the present paper were further developed in [Vu20] to prove that the Dinh-Sibony product of $(1,1)$-currents of full mass intersection in Kähler classes on a compact Kähler manifolds exists and is equal to their (relative) non-pluripolar product.

To end this introduction, let us outline the structure and the key points of the proof of our main theorem. Our main result will follow from a more general theorem showing that if a mixed product is well defined in the sense that it is obtained via decreasing sequences of smooth p.s.h. functions, then the corresponding Dinh-Sibony product exists and coincides with the classical one. This is the content of Theorem 3.1 below. Its proof is obtained by induction on the number of currents involved and uses the following key facts: the vanishing of Lelong numbers with respect to the currents obtained in previous steps (Lemma 3.2), the interpretation of Lelong numbers as the mass of dilated currents (Lemma 3.4) and the observation that the dilation procedure in the definition of density currents yields canonical regularizations via decreasing sequences of p.s.h. functions (cf. the proof of Lemma 3.7). A more detailed outline is given below, after the statement of Theorem 3.1. Finally, in Theorem 4.5, we prove our main theorem by verifying that functions in the Błocki-Cegrell class satisfy the assumptions of Theorem 3.1.

2. PRELIMINARIES ON DENSITY CURRENTS

In this section, we recall the definition and basic properties of tangent and density currents. For details, the reader is referred to the original paper [DS18] and to [KV19], [Vu19], [DNV18] for more material.

Let X be a complex manifold of dimension n and V be a smooth complex submanifold of X of dimension ℓ. Let T be a positive closed (p,p)-current on X with $0 \le p \le n$.

Let NV be the normal bundle of V in X and denote by $\pi : NV \to V$ the canonical projection. We identify V with the zero section of NV. Let U be an open subset of X with $U \cap V \neq \emptyset$. An *admissible map* on U is a smooth diffeomorphism τ from U to an open neighbourhood of $V \cap U$ in NV such that τ is the identity map on $V \cap U$ and the restriction of its differential $d\tau$ to $NV|_{V\cap U}$ is the identity. Using a Hermitian metric on X, we can always find an admissible map defined on a small tubular neighbourhood of V (see [DS18, Lemma 4.2]). This map is not holomorphic in general. However, if one only works on a small open set of X, it is easy to obtain holomorphic admissible maps.

For $\lambda \in \mathbb{C}^*$, let $A_\lambda : NV \to NV$ be the multiplication by λ along the fibers of NV. Consider the family of currents $(A_\lambda)_*\tau_*T$ on $NV|_{V\cap U}$ parametrized by $\lambda \in \mathbb{C}^*$. We define a tangent current, following [DS18, KV19, Vu19].

Definition 2.1. A *tangent current* of T along V is a positive closed current R on NV such that there exist a sequence $(\lambda_k)_{k\ge1}$ in $\mathbb{C}^*$ converging to ∞ and a collection of holomorphic admissible maps $\tau_j : U_j \to NV$, $j \in J$, whose domains cover V such that

$$R = \lim_{k\to\infty} (A_{\lambda_k})_*(\tau_j)_* T$$

on $\pi^{-1}(U_j \cap V)$ for every $j \in J$.

Before continuing, let us make some comments on Definition 2.1. In [DS18], the authors considered the situation where X is Kähler and $\operatorname{supp} T \cap V$ is compact. There, a tangent current to T along V is defined as a limit current of the family $(A_\lambda)_*\tau_* T$ as $|\lambda| \to \infty$, where τ is a *global* admissible map, defined on an open tubular neighborhood of V (see [DS18, Definition 4.5]). Then, they proceed to show (cf. [DS18, Proposition 4.4]) that tangent currents are independent of the choice of global admissible maps and can be localized in the following sense: if U is an open subset of X and $S = \lim_{k\to\infty}(A_{\lambda_k})_*\tau_* T$, then for every *local* admissible map $\tau' : U \to NV$ we have $S = \lim_{k\to\infty}(A_{\lambda_k})_*\tau'_* T$ on $\pi^{-1}(U \cap V)$. Therefore, our definition of tangent currents is equivalent to that of [DS18] when $\operatorname{supp} T \cap V$ is compact and X is Kähler. Note also that, in this situation, it is shown in [DS18] that tangent currents always exist.

However, in the cases we consider here, $\operatorname{supp} T \cap V$ is not necessarily compact. Therefore, it is unclear whether we can use the original definition of [DS18]. This is because using only global admissible maps it is hard to ensure that the family $(A_\lambda)_*\tau_* T$ is compact and that the limit currents are independent of τ. That is why we adopt a more flexible definition using local holomorphic admissible maps.

As in the compact setting, tangent currents depend in general on the sequence $(\lambda_k)_{k\geq 1}$. The existence of tangent currents in the local setting is a more delicate matter, and we have to prove it in our particular situation. However, if such currents exist, they are still independent of the choice of admissible maps.

Lemma 2.2. [KV19, Proposition 2.5] *Let $\tau : U \to NV$ be a holomorphic admissible map. Assume there is a sequence $(\lambda_k)_{k\geq 1}$ tending to ∞ such that $(A_{\lambda_k})_*\tau_* T$ converges to some current R on $\pi^{-1}(U \cap V)$. Then, for any other admissible map $\tau' : U' \to NV$, we have*

$$R = \lim_{k\to\infty} (A_{\lambda_k})_*\tau'_* T$$

on $\pi^{-1}(U \cap U' \cap V)$.

A density current is a particular type of tangent current where V is the diagonal inside a product space. More precisely, let $m \geq 1$, and let T_j be positive closed (p_j, p_j)-currents for $1 \leq j \leq m$ on X. We usually assume that

$$p = p_1 + \cdots + p_m \leq n.$$

Let $\mathbf{T} = T_1 \otimes \cdots \otimes T_m$ be their tensor product. Then, $\mathbf{T}$ is a positive closed (p,p)-current on X^m. Let $\Delta = \{(x,\ldots,x) : x \in X\} \subset X^m$ be the diagonal. A *density current* associated with $T_1, \ldots, T_m$ is a tangent current of $\mathbf{T}$ along Δ. A density current is a positive closed (p,p)-current on the normal bundle $N\Delta$ of Δ inside X^m.

Let $\pi : N\Delta \to \Delta$ be the canonical projection. The following definition is given in [DS18].

Definition 2.3. We say that the *Dinh-Sibony product* $T_1 \curlywedge \cdots \curlywedge T_m$ of $T_1, \dots, T_m$ exists if there is a unique density current R associated with $T_1, \dots, T_m$ and $R = \pi^* S$ for some current S on $\Delta = X$. In this case, we define

$$T_1 \curlywedge \cdots \curlywedge T_m := S.$$

3. DINH-SIBONY PRODUCT AND CLASSICAL PRODUCTS

Let Ω be a domain in $\mathbb{C}^n$. For a p.s.h. function u on Ω and a point $x \in \Omega$, we denote by $\nu(u, x)$ the *Lelong number* of u at x. See [Dem] for various definitions and properties of the Lelong number.

The aim of this section is to prove the following general result. Our main theorem will be a consequence of it.

Theorem 3.1. *Let $m \geq 2$ and $p \geq 0$ be such that $m - 1 + p \leq n$. Let $u_1, \dots, u_{m-1}$ be p.s.h. functions on Ω, and let T be a positive closed (p, p)-current on Ω. Assume that for every subset $J = \{j_1, \dots, j_k\} \subset \{1, \dots, m-1\}$, there is a current R_J on Ω so that, for any open set $U \subset \Omega$, for $j \in J$ and any sequence of smooth p.s.h. functions $(u_j^\ell)_{\ell \in \mathbb{N}}$ decreasing to u_j on U as $\ell \to \infty$, one has*

$$\mathrm{dd}^c u_{j_1}^\ell \wedge \cdots \wedge \mathrm{dd}^c u_{j_k}^\ell \wedge T \longrightarrow R_J \quad \text{on } U \text{ as } \ell \to \infty. \tag{3.1}$$

We then define $\mathrm{dd}^c u_{j_1} \wedge \cdots \wedge \mathrm{dd}^c u_{j_k} \wedge T$ as the current R_J. If $J = \emptyset$, we set $R_J := T$.

Then, the Dinh-Sibony product of $\mathrm{dd}^c u_1, \dots, \mathrm{dd}^c u_{m-1}, T$ is well defined, and one has

$$\begin{aligned} \mathrm{dd}^c u_1 \curlywedge \cdots \curlywedge \mathrm{dd}^c u_{m-1} \curlywedge T &= \\ = R_{\{1,\dots,m-1\}} &= \mathrm{dd}^c u_1 \wedge \cdots \wedge \mathrm{dd}^c u_{m-1} \wedge T. \end{aligned} \tag{3.2}$$

Before stating the preparatory results, let us briefly outline the structure of the proof. From the definition of density product we have to show that $R_{1,\lambda} \to \pi^* R_1$ as $\lambda \to \infty$, where $R_{1,\lambda}$ is the dilation of the tensor product of T and $\mathrm{dd}^c u_j$, $j = 1, \dots, m-1$, along the diagonal, and $R_1 := \mathrm{dd}^c u_1 \wedge \cdots \wedge \mathrm{dd}^c u_{m-1} \wedge T$. We will argue by induction on m. After fixing a suitable coordinate system on $(\mathbb{C}^n)^m = (\mathbb{C}^n, y^1) \times \cdots \times (\mathbb{C}^n, y^m)$ it is enough to work with two types of test forms Φ. For each type, the estimates are of a different nature.

- *Forms of type* I: $\Phi = \varphi_1(y^1) \wedge \varphi_2(y^2) \wedge \cdots \wedge \varphi_m(y^m)$, where φ_j are positive (p_j, p_j)-forms on $(\mathbb{C}^n, y^j)$ and at least one among $\varphi_1, \dots, \varphi_{m-1}$ is not of top degree. In this case we have that $\langle R_{1,\lambda}, \Phi \rangle \to 0$ as $\lambda \to \infty$ (see Lemma 3.5). Here, we use Lemma 3.2, saying that the Lelong numbers of u_j are negligible with respect to the currents obtained in previous steps, and a characterization of Lelong numbers in terms of dilated currents (Lemma 3.4).

- *Forms of type* II: $\Phi = \varphi_1(y^1) \wedge \varphi_2(y^2) \wedge \cdots \wedge \varphi_m(y^m)$, where φ_j is a radial (n,n)-form for every $j = 1, \dots, m-1$. In this case we have that $\langle R_{1,\lambda}, \Phi\rangle \to \langle R_1, \pi_* \Phi\rangle$ as $\lambda \to \infty$ (see Lemma 3.7). Here, we use that the sequence of dilations yields canonical regularizations via decreasing sequences of p.s.h. functions.

Working with forms of type I allows us to prove that the limit currents have minimal horizontal dimension and, therefore, are the pullback of some current $R^h_{1,\infty}$ in the diagonal (cf. Lemma 3.11). On the other hand, working with forms of type II lets us recognize the current $R^h_{1,\infty}$ as being R_1.

We now present the auxiliary lemmas we will need. The proof of Theorem 3.1 is given in the end of this section.

Lemma 3.2. *Let the notation and the hypothesis be as in Theorem 3.1. Then, for every* $J \subset \{1, \dots, m-1\}$ *and every* $1 \leq k \leq m-1$ *such that* $k \notin J$*, we have that* $\nu(u_k, \cdot) = 0$ *almost everywhere with respect to the trace measure of* R_J*.*

Proof. We work locally. Let $J \subset \{1, \dots, m-1\}$. Let $(u_j^\ell)_{\ell \in \mathbb{N}}$ be a sequence of smooth p.s.h. functions decreasing to u_j as $\ell \to \infty$ for $j \in J$. Further, let also $k \in \{1, \dots, m-1\} \setminus J$. Let $(u_k^\ell)_{\ell\in\mathbb{N}}$ be a sequence of locally bounded p.s.h. functions decreasing to u_k. We claim that

$$\mathrm{dd}^c u_k^\ell \wedge R_J \to R_{J\cup\{k\}} \quad \text{as } \ell \to \infty. \tag{3.3}$$

Consider first the case where u_k^ℓ is smooth. Let Φ be a test form with compact support and $\varepsilon > 0$ a constant. Using (3.1) and the fact that u_k^ℓ is smooth we can find, for each $\ell \geq 1$ an index s_ℓ satisfying

$$\begin{aligned} &\Big|\Big\langle \mathrm{dd}^c u_k^\ell \wedge R_J - \mathrm{dd}^c u_k^\ell \wedge \bigwedge_{j\in J} \mathrm{dd}^c u_j^{s_\ell} \wedge T, \Phi\Big\rangle\Big| \\ &\qquad = \Big|\Big\langle R_J - \bigwedge_{j\in J} \mathrm{dd}^c u_j^{s_\ell} \wedge T, \mathrm{dd}^c u_k^\ell \wedge \Phi\Big\rangle\Big| \leq \varepsilon. \end{aligned} \tag{3.4}$$

We can choose $(s_\ell)_\ell$ so that it increases to ∞ as $\ell \to \infty$. Hence, $u_j^{s_\ell}$ decreases to u_j for $j \in J$ and by hypothesis, one obtains

$$\mathrm{dd}^c u_k^\ell \wedge \bigwedge_{j\in J} \mathrm{dd}^c u_j^{s_\ell} \wedge T \to R_{J\cup\{k\}} \quad \text{as } s \to \infty.$$

It follows that

$$\Big|\Big\langle \mathrm{dd}^c u_k^\ell \wedge \bigwedge_{j\in J} \mathrm{dd}^c u_j^{s_\ell} \wedge T - R_{J\cup\{k\}}, \Phi\Big\rangle\Big| \leq \varepsilon$$

for ℓ is big enough. Combining this with (3.4) gives

$$|\langle dd^c u_k^\ell \wedge R_J - R_{J\cup\{k\}}, \Phi\rangle| \le 2\varepsilon$$

for ℓ big enough. Therefore, (3.3) follows if u_k^ℓ is smooth.

The case of general u_k^ℓ follows from a regularization argument. Let $u_k^{\ell,\delta}$ be a standard smooth regularization of u_k^ℓ obtaining from convolution against a smoothing kernel, so that $u_k^{\ell,\delta}$ decreases to u_k^ℓ as $\delta \to 0$. For each ℓ, let δ_ℓ be small enough such that

$$|\langle dd^c u_k^\ell \wedge R_J - dd^c u_k^{\ell,\delta_\ell} \wedge R_J, \Phi\rangle| \le \varepsilon. \tag{3.5}$$

We can choose δ_ℓ to be decreasing in ℓ. Hence, u_k^{ℓ,δ_ℓ} are smooth p.s.h. functions decreasing to u_k as $\ell \to \infty$. By the first part of the proof, we see that $dd^c u_k^{\ell,\delta_\ell} \wedge R_J$ converges to $R_{J\cup\{k\}}$. This combined with (3.5) gives

$$|\langle dd^c u_k^\ell \wedge R_J - R_{J\cup\{k\}}, \Phi\rangle| \le 2\varepsilon$$

for ℓ big enough. Hence, (3.3) follows.

Recall that our goal is to prove R_J has no mass on $\{\nu(u_k, \cdot) > 0\}$. Let $w(x) = \|x\|^2$, where x is the standard coordinate system on $\mathbb{C}^n$. Let N be a large constant, and set

$$u_k^\ell := \log(e^{u_k} + 1/\ell e^{Nw}).$$

Then, the u_k^ℓ are locally bounded p.s.h. functions that decrease to u as $\ell \to \infty$. Suppose R_J has positive mass on $V := \{\nu(u_k, \cdot) > 0\}$, that is, $\mathbf{1}_V R_J \ne 0$. Notice that $u_k = -\infty$ on V implies $u_k^\ell = Nw - \log \ell$ on V. It follows that

$$dd^c u_k^\ell \wedge R_J \ge dd^c u_k^\ell \wedge (\mathbf{1}_V R_J) = dd^c(u_k^\ell \mathbf{1}_V R_J) = N dd^c w \wedge (\mathbf{1}_V R_J).$$

Let K be a fixed compact set that is charged by $\mathbf{1}_V R_J$. Then, the mass of $N dd^c w \wedge (\mathbf{1}_V R_J)$ over K equals cN for some constant $c > 0$ independent of ℓ. By the above inequality and (3.3) one gets that the mass of $R_{J\cup\{k\}}$ on K is $\ge cN$. Choosing N large enough gives a contradiction. This finishes the proof. ◻

Let $(x^1, \ldots, x^m)$ be the canonical coordinate system in Ω^m and Δ be the diagonal of Ω^m. Let $y^j := x^j - x^m$ for $1 \le j \le m-1$ and $y^m := x^m$. Then, $(y^1, \ldots, y^{m-1}, y^m)$ forms a new coordinate system on Ω^m and

$$\Delta = \{y^j = 0 : 1 \le j \le m-1\}$$

which is identified with Ω. Using these coordinates, we identify naturally the normal bundle of Δ with the trivial bundle $\pi : (\mathbb{C}^n)^{m-1} \times \Omega \to \Omega$. Observe that the change of coordinates $\varrho : \Omega^m \to (\mathbb{C}^n)^{m-1} \times \Omega$ given by

$$\begin{aligned}\varrho(x^1,\dots,x^m) &= (x^1 - x^m,\dots,x^{m-1} - x^m, x^m)\\ &:= (y^1,\dots,y^m) := (y', y^m) := y\end{aligned}$$

is a holomorphic admissible map. By Lemma 2.2, it will be enough to work only with ϱ.

For $1 \le j \le m-1$, let $T_j := dd^c u_j$, $\tilde{T} := \pi^* T$, and

$$\tilde{u}_j(y', y^m) := \varrho_* u_j(y', y^m) = u_j(y^j + y^m).$$

We can check that $\tilde{u}_j$ is locally integrable with respect to

$$dd^c \tilde{u}_{j+1} \wedge \cdots \wedge dd^c \tilde{u}_{m-1} \wedge \tilde{T}$$

for $j = m-1,\dots,1$; and for every sequence $(u_j^\ell)_{\ell\in\mathbb{N}}$ of smooth p.s.h. functions decreasing to u_j and $\tilde{u}_j^\ell := \varrho_* u_j^\ell$, we have

$$dd^c \tilde{u}_1^\ell \wedge \cdots \wedge dd^c \tilde{u}_{m-1}^\ell \wedge \tilde{T} \to dd^c \tilde{u}_1 \wedge \cdots \wedge dd^c \tilde{u}_{m-1} \wedge \tilde{T} \tag{3.6}$$

as $\ell \to \infty$. (For the meaning of the righthand side, see Definition 3.13 below.) The above assertions follow from reasoning similar to the one in Lemma 2.3 of [KV19]. Consequently, we get

$$\varrho_*(T_1 \otimes \cdots \otimes T_{m-1} \otimes T) = dd^c \tilde{u}_1 \wedge \cdots \wedge dd^c \tilde{u}_{m-1} \wedge \tilde{T}.$$

Now, for $1 \le j \le m-1$, let

$$\begin{aligned}R_{j,\lambda} &:= (A_\lambda)_* \varrho_*(T_j \otimes \cdots \otimes T_{m-1} \otimes T)\\ &= (A_\lambda)_*(dd^c \tilde{u}_j \wedge \cdots \wedge dd^c \tilde{u}_{m-1} \wedge \tilde{T}),\end{aligned}$$

and for $J \subset \{1,\dots,m-1\}$, let

$$R_{J,\lambda} := (A_\lambda)_* \Big(\bigwedge_{j\in J} dd^c \tilde{u}_j \wedge \tilde{T} \Big). \tag{3.7}$$

Define also

$$R_j := R_{\{j,\dots,m-1\}} = dd^c u_j \wedge \cdots \wedge dd^c u_{m-1} \wedge T.$$

We need to show that

$$R_{1,\lambda} \xrightarrow{|\lambda|\to\infty} \pi^* R_1.$$

We will do that by testing $R_{1,\lambda}$ against forms of different types.

For future use, we note that

$$(3.8)\quad R_{1,\lambda} = dd^c u_1(\lambda^{-1}y^1 + y^m) \wedge \cdots \wedge dd^c u_{m-1}(\lambda^{-1}y^{m-1} + y^m) \wedge T(y^m).$$

This is clear when the u_j are smooth and the general case follows by regularizing the u_j and using (3.6).

In the definition below and throughout this paper, $i^n\, dy^j \wedge d\overline{y^j}$ will be a shorthand notation for the standard volume form on $(\mathbb{C}^n, y^j)$, that is,

$$i^n\, dy^j \wedge d\overline{y^j} := \Big(\sum_{k=1}^{n} i\, dy^j_k \wedge d\overline{y^j_k}\Big)^n.$$

Definition 3.3. Let Φ be a differential form on

$$(\mathbb{C}^n, y^1) \times (\mathbb{C}^n, y^2) \times \cdots \times (\mathbb{C}^n, y^m).$$

We say that Φ is a *positive split form* if it can be written as

$$\Phi = \varphi_1(y^1) \wedge \varphi_2(y^2) \wedge \cdots \wedge \varphi_m(y^m),$$

where φ_j are positive (p_j, p_j)-forms on $(\mathbb{C}^n, y^j)$.

An (n,n)-form φ_j on $(\mathbb{C}^n, y^j)$ is *radial* if it is rotation invariant, namely, if it is of the form

$$\varphi_j(y^j) = \chi(\|y^j\|^2) \cdot i^n\, dy^j \wedge d\overline{y^j}$$

for some smooth function χ.

In the sequel we will need the following expression of the Lelong number in terms of tangent currents. We denote by β the standard Kähler form of $\mathbb{C}^n$, and by $B_\rho(0)$ the open ball of radius ρ centered at the origin in $\mathbb{C}^n$.

Lemma 3.4. *Let S be a positive closed (p,p)-current defined near the origin in $\mathbb{C}^n$. Let $A_\lambda(z) = \lambda z$ and set $S_\lambda := (A_\lambda)_* S$. Let λ_k be an increasing sequence tending to ∞ such that S_{λ_k} converges to S^∞. Let*

$$\sigma_{S^\infty} = S^\infty \wedge (1/(n-p)!)\beta^{n-p}$$

be the trace measure of S^∞ and $\nu(S;0)$ be the Lelong number of S at the origin. Then, there is a constant $c_p > 0$ depending only on p such that $\nu(S;0) = \lim_{\lambda_k \to \infty} c_p \sigma_{S_{\lambda_k}}(B_1(0)) = c_p \sigma_{S^\infty}(B_1(0))$ and the limit is decreasing.

Proof. To simplify the notation we may assume the limit of $(A_\lambda)_* S$ as λ tends to infinity exists and is equal to S^∞. Set $c_p := (n-p)!/\pi^{n-p}$. Then, by the definition of Lelong number [Dem] we have

$$\nu(S;0) = \lim_{r \to 0} \frac{1}{\pi^{n-p} r^{2n-2p}} \int_{B_r(0)} S \wedge \beta^{n-p}$$

$$
\begin{aligned}
&= \lim_{|\lambda|\to\infty} \frac{1}{\pi^{n-p}} |\lambda|^{2n-2p} \int_{B_{1/|\lambda|}(0)} S \wedge \beta^{n-p} \\
&= \lim_{|\lambda|\to\infty} \frac{1}{\pi^{n-p}} \int_{B_{1/|\lambda|}(0)} S \wedge (A_\lambda)^*\beta^{n-p} \\
&= \lim_{|\lambda|\to\infty} \frac{1}{\pi^{n-p}} \int_{B_{1/|\lambda|}(0)} (A_\lambda)^*[(A_\lambda)_* S \wedge \beta^{n-p}] \\
&= \lim_{|\lambda|\to\infty} \frac{1}{\pi^{n-p}} \int_{B_1(0)} (A_\lambda)_* S \wedge \beta^{n-p} \\
&\geq \frac{1}{\pi^{n-p}} \int_{B_1(0)} S^\infty \wedge \beta^{n-p} = c_p \sigma_{S^\infty}(B_1(0)).
\end{aligned}
$$

In the second equality we have used that $A_\lambda^*\beta = |\lambda|^2\beta$, and in the last inequality we have used the fact that if a sequence of measures m_λ converges to m and U is open, then $\liminf_\lambda m_\lambda(U) \geq m(U)$. Hence,

$$
\nu(S;0) \geq c_p \sigma_{S^\infty}(B_1(0)). \tag{3.9}
$$

Repeating the above argument on closed balls, and using the fact that if a sequence of measures m_λ converges to m then $\limsup_\lambda m_\lambda(K) \leq m(K)$ for all closed sets K, we get that $\nu(S;0) \leq c_p\sigma_{S^\infty}(\overline{B_1(0)})$. Now, the current S^∞ is invariant by $(A_t)_*$ for every $t \in \mathbb{C}^*$ (see [DS18]), and hence its mass is homogeneous, namely, $\sigma_{S^\infty}(B_\rho(0)) = \rho^{2n-2p}\sigma_{S^\infty}(B_1(0))$ for every $\rho > 0$ and similarly for the closed ball. For $0 < \rho < 1$ this gives

$$
\nu(S;0) \leq c_p\sigma_{S^\infty}(\overline{B_1(0)}) = c_p\rho^{2p-2n}\sigma_{S^\infty}(\overline{B_\rho(0)}) \leq c_p\rho^{2p-2n}\sigma_{S^\infty}(B_1(0)).
$$

Letting $\rho \nearrow 1$ gives $\nu(S;0) \leq c_p\sigma_{S^\infty}(B_1(0))$. Together with (3.9) this gives the desired result.

It is a standard fact that all the above limits are decreasing as $r \to 0$, or equivalently as $\lambda \to \infty$ (see [Dem, III.5]). ◻

Theorem 3.1 will be proved by induction on m. The induction step will make use of the next lemma. Let $u_1, \ldots, u_{m-1}$ and T be as in Theorem 3.1. Then, for $J \subset \{1, \ldots, m-1\}$, let $R_{J,\lambda}$ be the current defined in (3.7) and $R_J = \bigwedge_{j\in J} dd^c u_j \wedge T$, defined as in the statement of Theorem 3.1.

Lemma 3.5. *With the above notation and the hypothesis of Theorem 3.1, assume that $R_{J,\lambda} \to \pi^* R_J$ as $\lambda \to \infty$ for every $J \subset \{1, \ldots, m-1\}$ such that $|J| \leq m-2$.*

Let Φ be a positive split test form with compact support on

$$
(\mathbb{C}^n, y^1) \times \cdots \times (\mathbb{C}^n, y^m).
$$

Assume that Φ is not of bi-degree (n, n) on y^k for some $1 \leq k \leq m-1$. Then, $\langle R_{1,\lambda}, \Phi\rangle \to 0$ as $\lambda \to \infty$.

Proof. By assumption,

$$\Phi = \varphi_1(y^1) \wedge \cdots \wedge \varphi_{m-1}(y^{m-1}) \wedge \varphi_m(y^m),$$

where each φ_j is positive and compactly supported. Observe that we only need to consider forms Φ such that $R_{1,\lambda} \wedge \Phi$ has full degree; otherwise, the last product vanishes and the result is trivial.

Notice that the current $R_{1,\lambda}$ has only terms of degree 0, 1, or 2 on each y^j, $j = 1, \dots, m-1$. Therefore, it suffices to consider the case where φ_j has bi-degree $(n-1, n-1)$ or (n, n) for every $j = 1, \dots, m-1$. Set

$$J = \{j \in \{1, \dots, m-1\} : \varphi_j \text{ has bi-degree } (n, n)\}$$

and

$$K = \{k \in \{1, \dots, m-1\} : \varphi_k \text{ has bi-degree } (n-1, n-1)\}.$$

It follows from the assumption on Φ that K is non-empty and $|J| \leq m-2$. Hence, by hypothesis,

$$\lim_{|\lambda| \to \infty} R_{J,\lambda} = \pi^* R_J.$$

Set $\varphi_J = \bigwedge_{j \in J} \varphi_j$ and $\varphi_K = \bigwedge_{k \in K} \varphi_k$. Since $J \cup K = \{1, \dots, m-1\}$, we have that $\Phi = \varphi_J \wedge \varphi_K \wedge \varphi_m$. It follows from (3.8) that

$$R_{1,\lambda} \wedge \Phi = \bigwedge_{k \in K} (\mathrm{dd}^c u_k(\lambda^{-1} y^k + y^m)) \wedge \varphi_K \wedge R_{J,\lambda} \wedge \varphi_J \wedge \varphi_m.$$

Since $R_{1,\lambda} \wedge \Phi$ is a current of top degree in $(\mathbb{C}^n, y^1) \times \cdots \times (\mathbb{C}^n, y^m)$, it must have bi-degree (n, n) on each y^j (otherwise, $R_{1,\lambda} \wedge \Phi = 0$ and the lemma is trivial). Hence, for $j \in J$ only the derivatives of u_j with respect to y^m will contribute, while for $k \in K$, only the derivatives of u_k with respect to y^k will contribute. This gives

$$R_{1,\lambda} \wedge \Phi = \bigwedge_{k \in K} (\mathrm{dd}^c_{y^k} u_k(\lambda^{-1} y^k + y^m)) \wedge \varphi_K \wedge R_{J,\lambda} \wedge \varphi_J \wedge \varphi_m. \tag{3.10}$$

Here, the symbol $\mathrm{dd}^c_{y^k}$ means that we only consider the (weak) derivatives with respect to the y^k variables. The fact that the above wedge product is well defined is obvious when the u_j are smooth. This is less obvious for non-smooth functions, but it can be justified as in [KV19, Lemma 2.3]. Now, for fixed y^m and $k \in K$ we have that

$$\left| \int_{y^k} \mathrm{dd}^c_{y^k} u_k(\lambda^{-1} y^k + y^m) \wedge \varphi_k(y^k) \right| \leq$$
$$\leq c_k \int_{B_k} \mathrm{dd}^c_{y^k} u_k(\lambda^{-1} y^k + y^m) \wedge \beta^{n-1}(y^k),$$

where $c_k > 0$ is a constant independent of y^m, β is the standard Kähler form on $(\mathbb{C}^n, y^k)$, and B_k is a ball in $(\mathbb{C}^n, y^k)$ containing the support of φ_k. By Lemma 3.4, the integral on the righthand side of the above inequality decreases to a constant independent of y^m times the Lelong number of the $(1,1)$-current $dd^c_{y^k} u_k(y^k + y^m)$ at $y^k = 0$, which is equal to $\nu(u_k, y^m)$. Here, we use that the Lelong number of a positive closed $(1,1)$-current coincides with the Lelong number of any of its local potential (cf. [Dem, III.6.9]). Hence, for every y^m one has

$$\limsup_{|\lambda|\to\infty} \left| \int_{y^k} dd^c_{y^k} u_k(\lambda^{-1} y^k + y^m) \wedge \varphi_k(y^k) \right| \lesssim \nu(u_k, y^m).$$

Combining this with (3.10), the hypothesis that $R_{J,\lambda} \to \pi^* R_J$ as $\lambda \to \infty$, and Lemma 3.6 below, one obtains

$$\limsup_{|\lambda|\to\infty} |\langle R_{1,\lambda} \wedge \Phi\rangle| \lesssim \int_{y^m} \Big(\prod_{k\in K} \nu(u_k, y^m)\Big) R_J \wedge \varphi_m.$$

The last integral in the above inequality vanishes because $\nu(u_k, \cdot) = 0$ almost everywhere with respect to R_J, by Lemma 3.2. Therefore,

$$\limsup_{|\lambda|\to\infty} |\langle R_{1,\lambda} \wedge \Phi\rangle| = 0,$$

concluding the proof of the lemma. □

We have used the following well-known result.

Lemma 3.6. *Let X be a locally compact Hausdorff space. Let m_λ be a sequence of Radon measures on X whose supports are contained in a fixed compact subset of X. Assume that $m_\lambda \to m$ as $\lambda \to \infty$. Then, for any sequence $(f_\lambda)_\lambda$ of continuous functions decreasing pointwise to a function f, we have that*

$$\limsup_{\lambda\to\infty} \int_X f_\lambda \, dm_\lambda \le \int_X f \, dm.$$

Lemma 3.7. *Under the assumptions of Theorem 3.1, let*

$$\Phi = \varphi_1(y^1) \wedge \cdots \wedge \varphi_{m-1}(y^{m-1}) \wedge \varphi_m(y^m)$$

be a positive split test form with compact support on $(\mathbb{C}^n, y^1) \times \cdots \times (\mathbb{C}^n, y^m)$. Assume that φ_j is a radial (n,n)-form for every $j = 1, \ldots, m-1$. Then,

$$\langle R_{1,\lambda}, \Phi\rangle \to \langle R_1, \pi_* \Phi\rangle \quad \text{as } \lambda \to \infty.$$

Proof. After multiplying Φ by a positive constant, we can assume that

$$\int_{(\mathbb{C}^n, y^j)} \varphi_j = 1 \quad \text{for every } j = 1, \ldots, m-1.$$

Notice that $\pi_*\Phi = \varphi_m$ and φ_m has bidegree $(n-m-p+1, n-m-p+1)$.

For $j = 1, \ldots, m-1$, define

$$u_j^\lambda(y^m) := \int_{(\mathbb{C}^n, y^j)} u_j(\lambda^{-1}y^j + y^m)\varphi_j(y^j).$$

Observe that u_j^λ is a convolution against a (radially symmetric) smoothing kernel on a disc of radius $|\lambda|^{-1}$ centered at y^m. Hence, u_j^λ is a smooth p.s.h. function on $(\mathbb{C}^n, y^m)$ decreasing pointwise to $u_j(y^m)$ as $\lambda \to \infty$ (see Chapter I.4.18 in [Dem]). By (3.1) we get that

$$\mathrm{dd}^c u_1^\lambda \wedge \cdots \wedge \mathrm{dd}^c u_{m-1}^\lambda \wedge T \xrightarrow{\lambda\to\infty} R_1. \tag{3.11}$$

Recall from (3.8) that

$$R_{1,\lambda} = \mathrm{dd}^c u_1(\lambda^{-1}y^1 + y^m) \wedge \cdots \wedge \mathrm{dd}^c u_{m-1}(\lambda^{-1}y^{m-1} + y^m) \wedge T(y^m).$$

Using the fact that the bidegree of each φ_j, $j = 1, \ldots, m-1$, is maximal, one has

$$\mathrm{dd}^c u_j(\lambda^{-1}y^j + y^m) \wedge \varphi_j = \mathrm{dd}^c_{y^m} u_j(\lambda^{-1}y^j + y^m) \wedge \varphi_j \quad j = 1, \ldots, m-1.$$

Hence,

$$\begin{aligned} R_{1,\lambda} \wedge \Phi = \mathrm{dd}^c_{y^m} u_1(\lambda^{-1}y^1 + y^m) \wedge \varphi_1(y^1) \wedge \cdots \\ \cdots \wedge \mathrm{dd}^c_{y^m} u_{m-1}(\lambda^{-1}y^{m-1} + y^m) \\ \wedge \varphi_{m-1}(y^{m-1}) \wedge T(y^m) \wedge \varphi_m(y^m). \end{aligned}$$

Taking the integral of both sides of the above equality and using Fubini's theorem, one obtains

$$\begin{aligned} &\langle R_{1,\lambda}, \Phi\rangle \\ &= \int_{(\mathbb{C}^n, y^m)} \bigg(\Big(\int_{(\mathbb{C}^n, y^1)} \mathrm{dd}^c_{y^m} u_1(\lambda^{-1}y^1 + y^m) \wedge \varphi_1(y^1) \Big) \wedge \cdots \\ &\qquad \cdots \wedge \Big(\int_{(\mathbb{C}^n, y^{m-1})} \mathrm{dd}^c_{y^m} u_{m-1}(\lambda^{-1}y^{m-1} + y^m) \wedge \varphi_{m-1}(y^{m-1}) \Big) \bigg) \\ &\qquad \wedge T(y^m) \wedge \varphi_m(y^m) \\ &= \int_{(\mathbb{C}^n, y^m)} \mathrm{dd}^c u_1^\lambda(y^m) \wedge \cdots \wedge \mathrm{dd}^c u_{m-1}^\lambda(y^m) \wedge T(y^m) \wedge \varphi_m(y^m) \\ &= \langle \mathrm{dd}^c u_1^\lambda \wedge \cdots \wedge \mathrm{dd}^c u_{m-1}^\lambda \wedge T, \varphi_m\rangle. \end{aligned}$$

By (3.11), the last quantity tends to $\langle R_1, \varphi_m\rangle = \langle R_1, \pi_*\Phi\rangle$ as $\lambda \to \infty$. This finishes the proof. □

The following result is an important consequence of the previous lemmas.

Lemma 3.8. *Under the assumptions of Lemma 3.5, the mass of* $R_{1,\lambda}$ *on compact sets is uniformly bounded.*

Proof. Let $\omega := \sum_{j=1}^{m}\sum_{k=1}^{n} i\,\mathrm{d}y_k^j \wedge \mathrm{d}\overline{y_k^j}$ be the standard Kähler form on $(\mathbb{C}^n)^m = (\mathbb{C}^n, y^1) \times \cdots \times (\mathbb{C}^n, y^m)$, and set $\Theta := \omega^{nm-m+1-p}$. In order to prove the desired assertion, using the fact that $R_{1,\lambda}$ is positive, it is enough to check that the mass of the trace measure $R_{1,\lambda} \wedge \Theta$ is uniformly bounded on compact subsets of Ω^m.

Notice that the form Θ is a linear combination of positive split forms. Therefore, in order to obtain the above bound, it will be enough to prove that $\langle R_{1,\lambda}, \Phi\rangle$ is uniformly bounded for any fixed positive split test form

$$\Phi = \varphi_1(y^1) \wedge \cdots \wedge \varphi_{m-1}(y^{m-1}) \wedge \varphi_m(y^m)$$

with compact support.

If φ_j is not of top degree for some $j = 1, \ldots, m-1$, then, by Lemma 3.5, we have that $|\langle R_{1,\lambda}, \Phi\rangle| \to 0$ as $\lambda \to \infty$. In particular, $|\langle R_{1,\lambda}, \Phi\rangle|$ is uniformly bounded. Hence, we can assume that φ_j has bidegree (n,n) for every $j = 1, \ldots, m-1$. In this case, since φ_j is always bounded by some radial positive test form, we can assume furthermore that φ_j is radial for every j. From this observation and Lemma 3.7, we have $\langle R_{1,\lambda}, \Phi\rangle \to \langle R_1, \pi_*\Phi\rangle$ as $\lambda \to \infty$. In particular, $|\langle R_{1,\lambda}, \Phi\rangle|$ is uniformly bounded. This finishes the proof of the lemma. □

We now recall from [DS18, Section 3] the notion of horizontal dimension of currents on vector bundles. Actually, the authors consider projective fibrations, that is, the projectivization $\mathbb{P}(E)$ of a given holomorphic vector bundle E. Here, we phrase the definitions and results for vector bundles instead. The proofs can be easily adapted from the ones in [DS18].

Let V be a Kähler manifold of dimension ℓ with Kähler form ω_V and let $\pi : E \to V$ be a holomorphic vector bundle over V.

Definition 3.9. Let S be a non-zero positive closed current on E. The *horizontal dimension* (h-dimension for short) of S is the largest integer j such that $S \wedge \pi^*\omega_V^j \neq 0$.

We will need the following characterization of currents of minimal h-dimension.

Lemma 3.10. *Let* S *be a positive closed* (p,p)*-current on* E *with* $p \leq \ell$*. Assume that the* h*-dimension of* S *is smaller than or equal to* $\ell - p$*. Then, the* h*-dimension of* S *is equal to* $\ell - p$ *and there is a positive closed* (p,p)*-current* S^h *on* V *such that* $S = \pi^*(S^h)$*.*

Proof. (See [DS18, Lemma 3.4].) □

Now, let $V = \Omega \subset (\mathbb{C}^n, y^m)$, $\omega_V = \sum_{k=1}^n \mathrm{d}y_k^m \wedge \mathrm{d}\bar{y}_k^m := \beta(y^m)$ be the standard Kähler form on Ω and E be the trivial bundle $\pi : (\mathbb{C}^n)^{m-1} \times \Omega \to \Omega$, $\pi(y', y^m) = y^m$.

Recall from Lemma 3.8 that $(R_{1,\lambda})_\lambda$ is a relatively compact family of positive closed $(m-1+p, m-1+p)$-currents on E.

Lemma 3.11. *In the assumptions of Lemma 3.5, let $R_{1,\infty}$ be a limit point of the family $R_{1,\lambda}$ as $\lambda \to \infty$. Then, the h-dimension of $R_{1,\infty}$ is minimal, equal to $n-m+1-p$. In particular, there is a positive closed $(m-1+p, m-1+p)$-current $R^h_{1,\infty}$ on Ω such that $R_{1,\infty} = \pi^* R^h_{1,\infty}$.*

Proof. Let λ_k be a sequence that tends to ∞ such that $R_{1,\lambda_k} \to R_{1,\infty}$. By Lemma 3.10, we only need to show that $R_{1,\infty} \wedge \pi^*\beta^{n-m-p+2}(y^m) = 0$. To do this, it is enough to verify that

$$\langle R_{1,\infty} \wedge \pi^*\beta^{n-m-p+2}(y^m), \Phi\rangle = 0$$

for every positive split test form Φ.

Let $\Phi = \varphi_1(y^1) \wedge \cdots \wedge \varphi_{m-1}(y^{m-1}) \wedge \varphi_m(y^m)$ be such a form. As in the beginning of the proof of Lemma 3.5, we may assume that each φ_j, $j = 1, \dots, m-1$, has bidegree (n,n) or $(n-1,n-1)$. Since the total bidegree of Φ is (p', p'), where

$$p' = nm - (n-m-p+2) - (m-1+p) = nm - n - 1,$$

at least one of φ_j, $j = 1, \dots, m-1$, has bidegree $(n-1, n-1)$. In this case, by Lemma 3.5, one has $\langle R_{1,\lambda} \wedge \pi^*\beta^{n-m-p+2}(y^m), \Phi\rangle \to 0$ as $\lambda \to \infty$. This finishes the proof. ◻

We are now in position to prove Theorem 3.1.

End of proof of Theorem 3.1. Recall our notation

$$\begin{aligned} R_{j,\lambda} &:= (A_\lambda)_*(\mathrm{dd}^c\tilde{u}_j \wedge \cdots \wedge \mathrm{dd}^c\tilde{u}_{m-1} \wedge \tilde{T}) && (1 \le j \le m-1),\\ R_{J,\lambda} &:= (A_\lambda)_*\Big(\bigwedge_{j\in J} \mathrm{dd}^c\tilde{u}_j \wedge \tilde{T}\Big) && (J \subset \{1, \dots, m-1\}),\\ R_j &:= R_{\{j,\dots,m-1\}} = \mathrm{dd}^c u_j \wedge \cdots \wedge \mathrm{dd}^c u_{m-1} \wedge T && (1 \le j \le m-1). \end{aligned}$$

Recall also that proving (3.2) is equivalent to proving that

$$R_{1,\lambda} \xrightarrow{|\lambda|\to\infty} \pi^* R_1.$$

We will proceed by induction on m. When $m = 1$ the result is obvious. Now let $m \ge 2$ and assume $R_{J,\lambda} \to \pi^* R_J$ as $\lambda \to \infty$ for every $J \subset \{1, \dots, m-1\}$ such that $|J| \le m-2$. When $m = 2$ this assumption is vacuous. Then, the hypothesis of Lemma 3.5 is satisfied. By Lemma 3.8, the family $(R_{1,\lambda})_\lambda$ is relatively compact. Let $R_{1,\infty} = \lim_{\lambda_k\to\infty} R_{1,\lambda_k}$ be one of its limit points. By Lemma 3.11, there is a positive closed $(m-1+p, m-1+p)$-current $R^h_{1,\infty}$ on Ω such that $R_{1,\infty} = \pi^* R^h_{1,\infty}$. We need to show that $R^h_{1,\infty} = R_1$.

Let φ_m be a test form on $(\mathbb{C}^n, y^m)$. Take $\varphi_1, \dots, \varphi_{m-1}$ positive radial (n,n)-forms with compact support such that $\int_{(\mathbb{C}^n, y^j)} \varphi_j = 1$ for every $j = 1, \dots, m-1$. Then, $\Phi := \varphi_1 \wedge \cdots \wedge \varphi_{m-1} \wedge \varphi_m$ is such that $\pi_*\Phi = \varphi_m$. Using Lemma 3.7, we get

$$\begin{aligned}\langle R^h_{1,\infty}, \varphi_m\rangle &= \langle R^h_{1,\infty}, \pi_*\Phi\rangle = \langle \pi^* R^h_{1,\infty}, \Phi\rangle = \langle R_{1,\infty}, \Phi\rangle \\ &= \lim_{\lambda_k\to\infty} \langle R_{1,\lambda_k}, \Phi\rangle = \langle R_1, \pi_*\Phi\rangle = \langle R_1, \varphi_m\rangle.\end{aligned}$$

As φ_m is arbitrary, we get $R^h_{1,\infty} = R_1$. This concludes the proof of the theorem. □

Remark 3.12. In the statement of Theorem 3.1, if we consider the case where $m-1+p > n$, then the arguments in the above proof still work, and we obtain that the associated density current vanishes.

Let R be a positive closed current and v be a p.s.h. function. If v is locally integrable with respect to (the trace measure) of R, we define, following Bedford-Taylor,

$$dd^c v \wedge R := dd^c(vR).$$

For a collection $v_1, \dots, v_s$ of p.s.h. functions, we can apply the above definition recursively, as long as the integrability conditions are satisfied.

Definition 3.13. We say that the intersection of $dd^c v_1, \dots, dd^c v_s, R$ is *classically well defined* if, for every non-empty subset $J = \{j_1, \dots, j_k\}$ of $\{1, \dots, s\}$, we have that v_{j_k} is locally integrable with respect to the trace measure of R, and inductively, we have that v_{j_r} is locally integrable with respect to the trace measure of $dd^c v_{j_{r+1}} \wedge \cdots \wedge dd^c v_{j_k} \wedge R$ for $r = k-1, \dots, 1$, and the product $dd^c v_{j_1} \wedge \cdots \wedge dd^c v_{j_k} \wedge R$ is continuous under decreasing sequences of p.s.h. functions.

The last definition is slightly more restrictive than the one given in [KV19]. We have the following comparison result between the Dinh-Sibony product and the above notion of wedge products. This result is a direct consequence of Theorem 3.1.

Corollary 3.14. *Let $m \geq 2$ and $p \geq 0$ be such that $m-1+p \leq n$. Let $u_1, \dots, u_{m-1}$ be p.s.h. functions on Ω, and let T be a positive closed (p,p)-current on Ω. Assume that $dd^c u_1 \wedge \cdots \wedge dd^c u_{m-1} \wedge T$ is classically well defined. Then, the Dinh-Sibony product of $dd^c u_1, \dots, dd^c u_{m-1}, T$ is well defined and*

$$dd^c u_1 \curlywedge \cdots \curlywedge dd^c u_{m-1} \curlywedge T = dd^c u_1 \wedge \cdots \wedge dd^c u_{m-1} \wedge T.$$

We note that [KV19, Theorem 1.1] asserts a similar conclusion, but there's an imprecision in the proof of the result as stated there.

4. PRODUCTS IN THE BŁOCKI-CEGRELL CLASS AND THE DOMAIN OF DEFINITION OF THE MONGE-AMPÈRE OPERATOR

In this section we apply Theorem 3.1 to the study of the domain of definition of the Monge-Ampère operator via Dinh-Sibony's intersection product. Let Ω be a domain in $\mathbb{C}^n$. Denote by $\mathrm{PSH}(\Omega)$ the set of p.s.h. functions on Ω.

Definition 4.1. The *Błocki-Cegrell class* on Ω is the subset $\mathcal{D}(\Omega)$ of $\mathrm{PSH}(\Omega)$ consisting of functions u with the following property: there exists a measure μ in Ω such that for every open set $U \subset \Omega$ and every sequence $(u_\ell)_\ell$ of smooth p.s.h. functions on U decreasing to u pointwise as $\ell \to \infty$, we have that $(dd^c u_\ell)^n$ converges to μ as $\ell \to \infty$.

For $u \in \mathcal{D}(\Omega)$, we define $(dd^c u)^n := \mu$, where μ is the above measure.

The class $\mathcal{D}(\Omega)$ is the largest subset of $\mathrm{PSH}(\Omega)$ where we can define a Monge-Ampère operator that coincides with the usual one for smooth p.s.h. functions and which is continuous under decreasing sequences (see [Ceg04, Blo06]).

We first need the following result ensuring the existence of the mixed products in the Błocki-Cegrell class.

Proposition 4.2. *Let Ω be a domain in $\mathbb{C}^n$, and let $u_1, u_2, \dots, u_m$, $1 \le m \le n$, be p.s.h. functions in $\mathcal{D}(\Omega)$. Then, there exists a positive closed (m,m)-current S_m such that for every open set $U \subset \Omega$ and every sequence $(u_j^\ell)_\ell$ of smooth p.s.h. functions on U decreasing to u_j pointwise as $\ell \to \infty$, we have that*

$$dd^c u_1^\ell \wedge \cdots \wedge dd^c u_m^\ell \to S_m \quad \text{on } U \text{ as } \ell \to \infty.$$

For $u_1, u_2, \dots, u_m \in \mathcal{D}(\Omega)$, we define their wedge product by

$$dd^c u_1 \wedge \cdots \wedge dd^c u_m := S_m,$$

where S_m is the current that appears in the above proposition. In particular, for $u \in \mathcal{D}(\Omega)$, one sees that $(dd^c u)^n$ is the Monge-Ampère measure given in Definition 4.1.

As mentioned in the Introduction, Proposition 4.2 is known when $m = n$, and the case $m < n$ might also be known to experts. We give a proof here for completeness, following closely the proof of [Blo06, Theorem 1.1]. A simplifying step in [Blo06] is the fact that it suffices to work with test functions that are p.s.h. on a ball and vanish on its boundary. In the case $m < n$, this step is replaced by the following lemma.

Lemma 4.3. *Let $\mathbb{B}_1 \Subset \mathbb{B}_2 \Subset \Omega$ be balls. Let $\mathcal{A}$ be the vector space generated by forms of the type $h\, dd^c v_1 \wedge \cdots \wedge dd^c v_{n-m}$, where $h, v_1, \dots, v_{n-m}$ are p.s.h. functions on $\mathbb{B}_2$ which are continuous up to $\partial\mathbb{B}_2$ and vanish on $\partial\mathbb{B}_2$. Then, every smooth $(n-m, n-m)$-form ψ compactly supported in $\mathbb{B}_1$ is in $\mathcal{A}$.*

Proof. It is a standard fact that every smooth $(n-m, n-m)$-form ψ compactly supported in $\mathbb{B}_1$ can be written as a linear combination of forms of type $\eta := h\, i\gamma_1 \wedge \bar{\gamma}_1 \wedge \cdots \wedge i\gamma_{n-m} \wedge \bar{\gamma}_{n-m}$, where h is a smooth function with compact support in $\mathbb{B}_1$ and $\gamma_1, \ldots, \gamma_{n-m}$ are $(1,0)$-forms with constant coefficients (see [Dem, III.1.4]). Hence, it is enough to prove the desired assertion for η as above.

Write $\gamma_\ell = \sum_{j=1}^n a_{j\ell}\, dz_j$, for $1 \le \ell \le n-m$, where $a_{j\ell} \in \mathbb{C}$. Observe $i\gamma_\ell \wedge \bar{\gamma}_\ell = dd^c v_\ell$, where $v_\ell(z) := \pi |\sum_{j=1}^n a_{j\ell} z_j|^2$, where $(z_1, \ldots, z_n)$ are the standard coordinates on $\mathbb{C}^n$. Let $\tilde{v}_\ell$ be the envelope constructed from v_ℓ as in Lemma 4.4 for $1 \le \ell \le n-m$. We have that $\tilde{v}_\ell = v_\ell$ on $\mathbb{B}_1$, $\tilde{v}_\ell \in \mathrm{PSH}(\mathbb{B}_2) \cap C^0(\bar{\mathbb{B}}_2)$, and $\tilde{v}_\ell = 0$ on $\partial\mathbb{B}_2$. This combined with the fact that h is compactly supported in $\mathbb{B}_1$ gives $\eta = h\, dd^c \tilde{v}_1 \wedge \cdots \wedge dd^c \tilde{v}_{n-m}$. On the other hand, since $\mathbb{B}_2$ is a ball, we can express $h = h_1 - h_2$ where h_1, h_2 are smooth p.s.h. functions such that $h_1 = h_2 = 0$ on $\partial\mathbb{B}_2$. We deduce that $\eta \in \mathcal{A}$. This finishes the proof. □

For the proof of Proposition 4.2, we need the following result about Monge-Ampère measures of envelopes. The first part is classical (see [BT76, Wal69]), while the second part is contained in the proof of Theorem 1.1 in [Blo06].

Lemma 4.4. *Let* $\mathbb{B}_1 \Subset \mathbb{B}_2 \Subset \Omega$ *be balls compactly contained in* Ω*. For a negative continuous function* $v \in \mathrm{PSH}(\Omega)$*, set*

$$\tilde{v} := \sup\{w \in \mathrm{PSH}(\mathbb{B}_2) : w < v \text{ on } \mathbb{B}_1 \text{ and } w < 0 \text{ on } \mathbb{B}_2\}.$$

Then, $\tilde{v}$ *is a p.s.h. function on* $\mathbb{B}_2$ *which is continuous on* $\bar{\mathbb{B}}_2$ *and satisfies the following:*

(1) $\tilde{v} = 0$ *on* $\partial\mathbb{B}_2$*;*
(2) $\tilde{v} = v$ *on* $\bar{\mathbb{B}}_1$*;*
(3) $(dd^c \tilde{v})^n = 0$ *on* $\mathbb{B}_2 \setminus \bar{\mathbb{B}}_1$.

Moreover, if $u \in \mathcal{D}(\Omega)$*, then for any sequence* $(u_\ell)_{\ell \ge 1}$ *of smooth p.s.h. functions on* Ω *decreasing to* u*, we have*

$$\sup_{\ell \ge 1} \int_{\mathbb{B}_2} (dd^c \tilde{u}_\ell)^n < +\infty.$$

Proof of Proposition 4.2. By using Lemma 4.3, the proof is parallel to that of Theorem 1.1 in [Blo06]. We include the main differences in the argument for completeness. Since the problem is local, in order to get the desired assertion, it suffices to prove that there exists a current S_m on Ω such that for every ball $\mathbb{B}_1 \Subset \Omega$ and every sequence $(u_j^\ell)_{\ell \ge 1}$ of smooth p.s.h. functions on Ω decreasing to u_j for $1 \le j \le m$, we have $dd^c u_1^\ell \wedge \cdots \wedge dd^c u_m^\ell \to S_m$ on $\mathbb{B}_1$ as $\ell \to \infty$.

Let $\mathbb{B}_2 \Subset \Omega$ be a ball containing $\bar{\mathbb{B}}_1$. Let $h, v_1, \ldots, v_{n-m} \in \mathrm{PSH}(\mathbb{B}_2) \cap C^0(\bar{\mathbb{B}}_2)$ be functions vanishing on $\partial\mathbb{B}_2$. Put $\eta := h\, dd^c v_1 \wedge \cdots \wedge dd^c v_{n-m}$. Let $\tilde{u}_j$ be the envelope constructed from u_j as in Lemma 4.4, for $1 \le j \le m$. We have that

$$\tilde{u}_j = u_j \quad \text{on } \mathbb{B}_1 \tag{4.1}$$

$\tilde{u}_j$ is continuous up to $\partial\mathbb{B}_2$ and is equal to 0 on $\partial\mathbb{B}_2$ for $1 \le j \le m$. Let

$$S_m^\ell := \mathrm{dd}^c u_1^\ell \wedge \cdots \wedge \mathrm{dd}^c u_m^\ell,$$
$$\tilde{S}_m^\ell := \mathrm{dd}^c \tilde{u}_1^\ell \wedge \cdots \wedge \mathrm{dd}^c \tilde{u}_m^\ell.$$

We will prove that $\langle \tilde{S}_m^\ell, \eta\rangle$ is convergent. By [Ceg04, Corollary 5.6], we have

$$\begin{aligned}\int_{\mathbb{B}_2} &\mathrm{dd}^c \tilde{u}_1^\ell \wedge \cdots \wedge \mathrm{dd}^c \tilde{u}_m^\ell \wedge \mathrm{dd}^c v_1 \wedge \cdots \wedge \mathrm{dd}^c v_{n-m} \\ &\le \left(\int_{\mathbb{B}_2} (\mathrm{dd}^c \tilde{u}_1^\ell)^n\right)^{1/n} \cdots \left(\int_{\mathbb{B}_2} (\mathrm{dd}^c \tilde{u}_m^\ell)^n\right)^{1/n} \\ &\quad \times \left(\int_{\mathbb{B}_2} (\mathrm{dd}^c v_1)^n\right)^{1/n} \cdots \left(\int_{\mathbb{B}_2} (\mathrm{dd}^c v_{n-m})^n\right)^{1/n}.\end{aligned}$$

This combined with Lemma 4.4 yields that $\langle \tilde{S}_m^\ell, \eta\rangle$ is uniformly bounded as $\ell \to \infty$. With this last property, we can follow the exact same arguments from the proof of [Blo06, Theorem 1.1]. This gives that $\lim_{\ell\to\infty}\langle \tilde{S}_m^\ell, \eta\rangle$ exists and is independent of the choice of the sequences $(u_j^\ell)_{\ell\ge 1}$. Using this and Lemma 4.3, for every smooth form φ compactly supported in $\mathbb{B}_1$, we obtain that $\langle \tilde{S}_m^\ell, \varphi\rangle$ converges to a number independent of the choice of $(u_j^\ell)_{\ell\ge 1}$ as $\ell \to \infty$. On the other hand, by (4.1), we get

$$\langle S_m^\ell, \varphi\rangle = \langle \tilde{S}_m^\ell, \varphi\rangle.$$

Consequently, the limit $\lim_{\ell\to\infty}\langle S_m^\ell, \varphi\rangle$ exists and is independent of the choice of $(u_j^\ell)_{\ell\ge 1}$ as $\ell \to \infty$. Hence, the current S_m defined by setting $\langle S_m, \varphi\rangle := \lim_{\ell\to\infty}\langle \tilde{S}_m^\ell, \varphi\rangle$ satisfies the desired property. This concludes the proof of Proposition 4.2. ◻

Theorem 4.5. *Let Ω be a domain in $\mathbb{C}^n$, and let $u_1, u_2, \ldots, u_m$, $1 \le m \le n$, be functions in $\mathcal{D}(\Omega)$. Then, the Dinh-Sibony product of $\mathrm{dd}^c u_1, \ldots, \mathrm{dd}^c u_m$ is well defined, and*

$$\mathrm{dd}^c u_1 \curlywedge \cdots \curlywedge \mathrm{dd}^c u_m = \mathrm{dd}^c u_1 \wedge \cdots \wedge \mathrm{dd}^c u_m.$$

In particular, for $u \in \mathcal{D}(\Omega)$, note that the Dinh-Sibony Monge-Ampère operator $u \mapsto (\mathrm{dd}^c u)^{\curlywedge n}$ is well defined and coincides with the usual one.

Proof. We will apply Theorem 3.1 to $u_1, u_2, \ldots, u_{m-1}$ and $T = dd^c u_m$. Let $J \subset \{1, \ldots, m-1\}$. Then, the current $R_J = \bigwedge_{j\in J} dd^c u_j \wedge dd^c u_m$ is well defined, by Proposition 4.2. Now, we check the hypothesis of Theorem 3.1 for R_J. Let $(u_j^\ell)_\ell$ be a sequence of smooth p.s.h. functions decreasing to u_j for $j \in J$. We need to show that $\bigwedge_{j\in J} dd^c u_j^\ell \wedge dd^c u_m$ converges to R_J as $\ell \to \infty$. Let $(u_m^\ell)_\ell$ be a sequence of smooth functions decreasing to u_m. Let Φ be a smooth test form with compact support and $\varepsilon > 0$ a constant. For every ℓ, since $dd^c u_m^\ell \to dd^c u_m$, there exists $s_\ell \in \mathbb{N}$ such that

$$\Big| \Big\langle \bigwedge_{j\in J} dd^c u_j^\ell \wedge dd^c u_m^{s_\ell} - \bigwedge_{j\in J} dd^c u_j^\ell \wedge dd^c u_m, \Phi \Big\rangle \Big| \le \varepsilon. \tag{4.2}$$

We can choose s_ℓ so that s_ℓ is decreasing in ℓ. Hence, $u_m^{s_\ell}$ decreases to u_m. By Proposition 4.2, we get $\bigwedge_{j\in J} dd^c u_j^\ell \wedge dd^c u_m^{s_\ell} \to R_J$ as $\ell \to \infty$. This combined with (4.2) gives

$$\Big| \Big\langle \bigwedge_{j\in J} dd^c u_j^\ell \wedge dd^c u_m^{s_\ell} - R_J, \Phi \Big\rangle \Big| \le 2\varepsilon$$

for ℓ big enough. Hence, $\bigwedge_{j\in J} dd^c u_j^\ell \wedge dd^c u_m$ converges to R_J as $\ell \to \infty$. In other words, we have checked the hypothesis of Theorem 3.1 for R_J. The desired assertion follows. The proof is thus finished. □

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