Journal Pre-proof

Differential Theory of Zero-Dimensional Schemes

Martin Kreuzer, Tran N.K. Linh, Le N. Long

PII: S0022-4049(24)00212-3
DOI: https://doi.org/10.1016/j.jpaa.2024.107815
Reference: JPAA 107815

To appear in: *Journal of Pure and Applied Algebra*

Received date: 4 April 2023
Revised date: 10 September 2024

Please cite this article as: M. Kreuzer et al., Differential Theory of Zero-Dimensional Schemes, *J. Pure Appl. Algebra* (2024), 107815, doi: https://doi.org/10.1016/j.jpaa.2024.107815.

Differential Theory of Zero-Dimensional Schemes

Martin Kreuzer[a,*], Tran N. K. Linh[b], Le N. Long[b,a]

[a] *Faculty of Informatics and Mathematics, University of Passau, Passau, D-94030, Germany*
[b] *Department of Mathematics, University of Education - Hue University, 34 Le Loi, Hue, Vietnam*

Abstract

To study a 0-dimensional scheme $\mathbb{X}$ in $\mathbb{P}^n$ over a perfect field K, we use the module of Kähler differentials $\Omega^1_{R/K}$ of its homogeneous coordinate ring R and its exterior powers, the higher modules of Kähler differentials $\Omega^m_{R/K}$. One of our main results is a characterization of weakly curvilinear schemes $\mathbb{X}$ by the Hilbert polynomials of the modules $\Omega^m_{R/K}$ which allows us to check this property algorithmically without computing the primary decomposition of the vanishing ideal of $\mathbb{X}$. Further main achievements are precise formulas for the Hilbert functions and Hilbert polynomials of the modules $\Omega^m_{R/K}$ for a fat point scheme $\mathbb{X}$ which extend and settle previous partial results and conjectures. Underlying these results is a novel method: we first embed the homogeneous coordinate ring R into its truncated integral closure $\widetilde{R}$. Then we use the corresponding map from the module of Kähler differentials $\Omega^1_{R/K}$ to $\Omega^1_{\widetilde{R}/K}$ to find a formula for the Hilbert polynomial $\mathrm{HP}(\Omega^1_{R/K})$ and a sharp bound for the regularity index $\mathrm{ri}(\Omega^1_{R/K})$. Next we extend this to formulas for the Hilbert polynomials $\mathrm{HP}(\Omega^m_{R/K})$ and bounds for the regularity indices of the higher modules of Kähler differentials. As a further application, we characterize uniformity conditions on $\mathbb{X}$ using the Hilbert functions of the Kähler differential modules of $\mathbb{X}$ and its subschemes.

Keywords: Kähler differential module, zero-dimensional scheme, regularity index, Hilbert function, curvilinear scheme, fat point scheme
2010 MSC: 13N05, 13D40, 14N05

1. Introduction

The study of 0-dimensional subschemes $\mathbb{X}$ of a projective space $\mathbb{P}^n$, in particular of finite sets of points, has a long and rich history. Traditional tools used in this area are the homogeneous vanishing ideal $I_{\mathbb{X}}$ of $\mathbb{X}$ in $P = K[x_0, \dots, x_n]$,

*Corresponding author
Email addresses: martin.kreuzer@uni-passau.de (Martin Kreuzer), tnkhanhlinh@hueuni.edu.vn (Tran N. K. Linh), lengoclong@dhsphue.edu.vn (Le N. Long)

the homogeneous coordinate ring $R = P/I_{\mathbb{X}}$, its Hilbert function $\mathrm{HF}_{\mathbb{X}}(i) = \dim_K(R_i)$, and the canonical module ω_R of R. In several previous works, the authors introduced and started using the Kähler differential module $\Omega^1_{R/K}$ of R and even the entire Kähler differential algebra $\Omega^\bullet_{R/K} = \Lambda_R(\Omega^1_{R/K})$ in order to advance this topic (see [4], [7], and [8]).

In the present paper we systematize this approach, extend it by several new techniques, solve and unify previous partial results and conjectures, and derive a number of useful applications to some computational tasks. Let us describe the individual contributions in more detail.

In Section 2 we recall some basic objects related to a 0-dimensional scheme $\mathbb{X}$ in the projective space $\mathbb{P}^n$ over a perfect field K, in particular its homogeneous vanishing ideal $I_{\mathbb{X}} \subseteq P = K[X_0,\dots,X_n]$, its homogeneous coordinate ring $R = P/I_{\mathbb{X}}$, and its Hilbert function $\mathrm{HF}_{\mathbb{X}}(i) = \dim_K(R_i)$ for $i \in \mathbb{Z}$. We always assume that the support of $\mathbb{X}$ is contained in the affine space $D_+(X_0)$ and denote its affine coordinate ring by $S = K[X_1,\dots,X_n]/I^{\mathrm{deh}}_{\mathbb{X}}$. After identifying the homogeneous ring of quotients of R with $S[x_0,x_0^{-1}]$, we introduce a new tool, namely the embedding of R into its *truncated integral closure* $\widetilde{R} \cong S[x_0]$.

These techniques come to fruition in Section 3, where we introduce and study the Kähler differential module $\Omega^1_{R/K}$ and its Hilbert function $\mathrm{HF}_{\Omega^1_{R/K}}$. The main new results are the description of $\mathrm{HP}(\Omega^1_{R/K})$ and a sharp bound $\mathrm{ri}(\Omega^1_{R/K}) \le 2\,r_{\mathbb{X}} + 1$ for the (Hilbert) regularity index of $\Omega^1_{R/K}$. Here $r_{\mathbb{X}}$ is the regularity index of $\mathbb{X}$, i.e., the first degree from where on the Hilbert function agrees with the value of the Hilbert polynomial (see Prop. 3.4). After finding the Hilbert function and the Hilbert polynomial of $\tilde{R}$ (see Prop. 3.7), we construct the canonical map $\Phi : \Omega^1_{R/K} \to \Omega^1_{\widetilde{R}/K}$ explicitly, show that $\mathrm{Ker}(\Phi)$ is exactly the torsion submodule of $\Omega^1_{R/K}$, and use it to prove our first main result, namely the formula

$$\mathrm{HP}(\Omega^1_{R/K}) \;=\; \deg(\mathbb{X}) + \dim_K(\Omega^1_{S/K})$$

for the Hilbert polynomial of $\Omega^1_{R/K}$. Notice that, in the current setting, this Hilbert polynomial agrees with the multiplicity $\mathrm{mult}(\Omega^1_{R/K})$ of the finitely generated graded R-module $\Omega^1_{R/K}$.

Further tools are provided in Section 4 in order to study the torsion submodule $T\Omega^1_{R/K}$ and other important submodules of $\Omega^1_{R/K}$. The Euler derivation $\delta : R \to R$ given by $\delta(f) = i\,f$ for $f \in R_i$ gives rise to the *Euler form* $\varepsilon : \Omega^1_{R/K} \to R$ which satisfies $\varepsilon(dx_i) = x_i$ for $i = 0,\dots,n$. The Koszul complex over ε is called the *Euler-Koszul complex* of R, and the image of $\varepsilon^{(2)} : \Lambda^2\Omega^1_{R/K} \to \Omega^1_{R/K}$ is called the *Koszul submodule* $U_{R/K} = \langle x_i dx_j - x_j dx_i \mid 0 \le i < j \le n\rangle$ of $\Omega^1_{R/K}$. Both $T\Omega^1_{R/K}$ and $U_{R/K}$ are contained in $\mathrm{Ker}(\varepsilon)$ (see Props. 4.2 and 4.4). The Koszul submodule is equal to $\mathrm{Ker}(\varepsilon)$ if $\mathrm{char}(K)$ is zero or $\ge 2r_{\mathbb{X}} + 1$, and in the other cases the (small) difference between the two submodules is laid out in detail in Prop. 4.6.

Next up, we apply the preceding results in Section 5 to the higher Kähler differential modules $\Omega^m_{R/K}$, where $m \ge 1$. After recalling the presentation and computation of $\Omega^m_{R/K}$ (see Prop. 5.2), we use the embedding $R \hookrightarrow \tilde{R}$ to prove the formula

$$\mathrm{HP}(\Omega^m_{R/K}) \;=\; \dim_K(\Omega^m_{S/K}) + \dim_K(\Omega^{m-1}_{S/K})$$

for the Hilbert polynomial of $\Omega^m_{R/K}$ and the sharp bound $\mathrm{ri}(\Omega^m_{R/K}) \le r_{\mathbb{X}} + m$ for its regularity index (see Prop. 5.4).

In final three sections we apply and extend these algebraic results to characterize and compute geometric properties of the scheme $\mathbb{X}$. In Section 6 we look at (weakly) curvilinear schemes which are defined by the property that the maximal ideals of their local rings are unigenerated. Notice that this generalizes slightly the condition that these local rings are of the form $K[z]/\langle p^k\rangle$ with an irreducible polynomial p. Special cases are, of course, smooth schemes for which it is known that they can be characterized by $\Omega^1_{S/K} = 0$, or, equivalently, by $\mathrm{HP}(\Omega^1_{R/K}) = \deg(\mathbb{X})$ (see Prop. 6.1). Note that this allows us to check smoothness of $\mathbb{X}$ without computing the primary decomposition of $I_{\mathbb{X}}$. In a similar vein, we characterize non-smooth weakly curvilinear schemes by $\Omega^1_{S/K} \ne 0$ and $\Omega^m_{S/K} = 0$ for $m \ge 2$. Equivalently, the scheme $\mathbb{X}$ is weakly curvilinear, but not smooth, if and only if

$$\mathrm{HP}(\Omega^1_{R/K}) > \deg(\mathbb{X}), \quad \mathrm{HP}(\Omega^2_{R/K}) = \mathrm{HP}(\Omega^1_{R/K}) - \deg(\mathbb{X}), \quad \text{and} \quad \mathrm{HP}(\Omega^m_{R/K}) = 0 \text{ for } m \ge 3$$

(see Prop. 6.6). Thus we can check algorithmically whether $\mathbb{X}$ is weakly curvilinear without computing a primary decomposition. Using the dimensions of the residue class fields of the local rings of $\mathbb{X}$, we can also write down explicit formulas for the Hilbert polynomials of $\Omega^1_{R/K}$ and $\Omega^2_{R/K}$ in the weakly curvilinear case (see Cor. 6.8).

The next application is contained in Section 7 where we consider fat point schemes $\mathbb{X}$. They are defined by vanishing ideals of the form $I_{\mathbb{X}} = I^{m_1}_{p_1} \cap \cdots \cap I^{m_t}_{p_t}$ where the points p_i in the support of $\mathbb{X}$ are assumed to be K-rational and $m_i \ge 1$. The case of a reduced scheme, i.e., the case $m_1 = \cdots = m_t = 1$, is easily characterized by $\mathrm{HP}(\Omega^1_{R/K}) = t$ and $\mathrm{HP}(\Omega^m_{R/K}) = 0$ for $m \ge 2$ (see Remark 7.1). The general case reduces to the study of Kähler differential modules of rings of the form $S = A/\mathfrak{q}^k$, where $A = K[X_1, \dots, X_n]$ and $\mathfrak{q} = \langle X_1, \dots, X_n\rangle$. Here we obtain explicit formulas for the values of the Hilbert function of $\Omega^m_{S/K}$ which only depend upon the value of $\delta = \dim_K(d\mathfrak{q}^k \wedge \Omega^{m-1}_{A/K})$ (see Prop. 7.2).

If $\mathrm{char}(K) = 0$ or $\mathrm{char}(K) > k$, we succeed in determining this value δ using a number of subtle and detailed arguments (see Props. 7.4, 7.5, and 7.6). Thus we are able to derive the explicit formula

$$\dim_K(\Omega^m_{S/K}) \;=\; \binom{n}{m}\binom{n+k-2}{n} + \binom{m+k-2}{m}\binom{n+k-2}{n-m-1}$$

for $S = K[X_1, \dots, X_n]/\langle X_1, \dots, X_n\rangle^k$ and $\mathrm{char}(K) = 0$ or $\mathrm{char}(K) > k$ (see Thm. 7.8). Under a more stringent assumption about the characteristic of K, we also provide another proof based on the exactness of the Euler-Koszul complex (see Prop. 7.9 and Remark 7.10).