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The Weak Lefschetz Property of Artinian Algebras Associated to Paths and Cycles

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Dedicated to Professor Ngo Viet Trung on the occasion of his 70th birthday

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Abstract
Given a base field $\Bbbk$ of characteristic zero, for each graph G, we associate the artinian algebra $A(G)$ defined by the edge ideal of G and the squares of the variables. We study the weak Lefschetz property of $A(G)$. We classify some classes of graphs with relatively few edges, including paths and cycles, such that its associated artinian ring has the weak Lefschetz property.

Keywords Artinian algebras · Edge ideals · Independence polynomials · Weak Lefschetz property

Mathematics Subject Classification (2020) 13E10 · 13F20 · 13F55 · 05C31

1 Introduction

A graded artinian algebra $A = [A]_0 \oplus [A]_1 \oplus \cdots \oplus [A]_D$ over a field $\Bbbk$ has the weak Lefschetz property (WLP for short) if there exists a linear form $\ell \in [A]_1$ such that each multiplication maps $\cdot\ell : [A]_i \to [A]_{i+1}$ have maximal rank for all i, while A has the strong Lefschetz property (SLP for short) if there exists a linear form ℓ such that each multiplication map $\cdot\ell^j : [A]_i \longrightarrow [A]_{i+j}$ has maximal rank for all i and all j. The study of Lefschetz properties of graded algebras has connections to several areas of mathematics. Many authors have studied the problem from many different points of view, applying tools from representation theory, algebraic topology, differential geometry, commutative algebra, among others (see, for instance, [6, 16, 22, 24–28, 30–33, 37]). Even the characteristic of $\Bbbk$ plays an interesting role in the study of the Lefschetz properties; see, for example, [7–9, 20, 21, 26].

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The case of artinian $\Bbbk$-algebras defined by monomial ideals, while being rather accessible, is far from simple and the literature concerning their Lefschetz properties is quite extensive; see, for instance, [2, 3, 11, 12, 22, 23, 26, 29] and the references therein. In this work, we focus on a special class of artinian algebras defined by quadratic monomials, given as follows. Let $G = (V, E)$ be a simple graph on the vertex set $V = \{1, 2, \ldots, n\}$ and $R = \Bbbk[x_1, \ldots, x_n]$ be the standard graded polynomial ring over $\Bbbk$. The *edge ideal* of G is given by

$$I(G) = (x_i x_j \mid \{i, j\} \in E) \subset R.$$

Then, we say that

$$A(G) = \frac{R}{(x_1^2, \ldots, x_n^2) + I(G)}$$

is the *artinian algebra associated to* G. We are interested in the following question.

Question 1.1 For which graphs G does $A(G)$ have the WLP or the SLP? If $A(G)$ does not have the WLP or the SLP, in which degrees do the multiplication maps fail to have maximal rank?

The algebra $A(G)$ has been studied in [38] where the second author classifies the WLP/SLP for some special classes of graphs including the complete graphs, the star graphs, the Barbell graphs, and the wheel graphs. Note that artinian algebras defined by quadratic monomial relations were considered in previous work by Michałek–Miró-Roig [23], and Migliore–Nagel–Schenck [29]. These rings can be regarded as special cases of a more general construction due to Dao and Nair [11] where they associate to each simplicial complex Δ on n vertices the ring

$$A(\Delta) = \frac{R}{\left(x_1^2, \ldots, x_n^2\right) + I_\Delta},$$

in which I_Δ is the Stanley-Reisner ideal of Δ. A recent work due to Cooper et al. [10] also investigates the WLP of $A(G)$ where the focus was on whiskered graphs.

Our main goal in this note is to classify some important classes of graphs G where $A(G)$ has the weak Lefschetz property, such as paths, cycles, and certain tadpole graphs. More precisely, denote by P_n, C_n, Pan_n the paths, cycles, pan graphs (namely a cycle together with a pendant attached to one vertex), respectively. Our main results are the following.

Theorem 1.2 (Theorems 4.2, 4.4 and 4.7) *Assume that* $\mathrm{char}(\Bbbk) = 0$.

(1) *For an integer* $n \geq 1$*, the ring* $A(P_n)$ *has the WLP if and only if* $n \in \{1, 2, \ldots, 7, 9, 10, 13\}$.
(2) *For an integer* $n \geq 3$*, the ring* $A(C_n)$ *has the WLP if and only if* $n \in \{3, 4, \ldots, 11, 13, 14, 17\}$.
(3) *For an integer* $n \geq 3$*, the ring* $A(\mathrm{Pan}_n)$ *has the WLP if and only if* $n \in \{3, 4, \ldots, 10, 12, 13, 16\}$.

The proof combines Macaulay2 [13] computations with inductive arguments based on the unimodality of the independence polynomials of the relevant graphs. We hope that our main results will inspire further research on Question 1.1.

Our paper is structured as follows. In the next section we recall relevant terminologies and results on artinian algebras, Lefschetz properties, and graph theory. In Section 3, we investigate the unimodality and the mode of the independence polynomials of familiar graphs, such as paths, cycles and pan graphs. These results are useful to study the WLP of artinian algebras associated to these graphs. In Section 4, we prove our main theorems (see Theorems 4.2, 4.4 and 4.7) on the WLP of the artinian algebras associated to paths, cycles and the pan graphs.

2 Preliminaries

In this section we recall some standard terminologies and notations from commutative algebra and combinatorial commutative algebra, as well as some results needed later on. For a general introduction to artinian rings and the weak and strong Lefschetz properties we refer the readers to [17] and [28].

2.1 The Weak Lefschetz Property

In this paper we consider artinian algebras defined by monomial ideals, and in this case it suffices to choose the Lefschetz element to be the sum of the variables.

Proposition 2.1 [26, 35] *Let $I \subset R = \mathbb{k}[x_1, \dots, x_n]$ be an artinian monomial ideal. Then $A = R/I$ has the WLP if and only if $\ell = x_1 + x_2 + \cdots + x_n$ is a Lefschetz element for A.*

A necessary condition for the WLP and SLP of an artinian algebra A is the unimodality of the Hilbert series of A.

Definition 2.2 Let $A = \oplus_{j \geq 0}[A]_j$ be a standard graded $\mathbb{k}$-algebra. The *Hilbert series* of A is the power series $\sum \dim_{\mathbb{k}}[A]_i t^i$ and is denoted by $HS(A, t)$. The *Hilbert function* of A is the function $h_A : \mathbb{N} \longrightarrow \mathbb{N}$ defined by $h_A(j) = \dim_{\mathbb{k}}[A]_j$.

If A is an artinian graded algebra, then $[A]_i = 0$ for $i \gg 0$. Denote

$$D = \max\{i \mid [A]_i \neq 0\},$$

the *socle degree* of A. In this case, the Hilbert series of A is a polynomial

$$HS(A, t) = 1 + h_1 t + \cdots + h_D t^D,$$

where $h_i = \dim_{\mathbb{k}}[A]_i > 0$. By definition, the degree of the Hilbert series for an artinian graded algebra A is equal to its socle degree D. Since A is artinian and non-zero, this number also agrees with the *Castelnuovo-Mumford regularity* of A, i.e.,

$$\operatorname{reg}(A) = D = \deg(HS(A, t)).$$

The algebra A is called *level* if its socle is concentrated in one degree.

Definition 2.3 A polynomial $\sum_{k=0}^n a_k t^k \in \mathbb{R}[t]$ with non-negative coefficients is called *unimodal* if there is some m, such that

$$a_0 \leq a_1 \leq \cdots \leq a_{m-1} \leq a_m \geq a_{m+1} \geq \cdots \geq a_n.$$

Set $a_{-1} = 0$. The *mode* of the unimodal polynomial $\sum_{k=0}^n a_k t^k$ is defined to be the unique integer i between 0 and n such that

$$a_{i-1} < a_i \geq a_{i+1} \geq \cdots \geq a_n.$$

Proposition 2.4 [17, Proposition 3.2] *If A has the WLP or SLP then the Hilbert series of A is unimodal.*

Finally, to study the failure of the WLP of tensor products of $\mathbb{k}$-algebras, the following simple lemma turns out to be quite useful.

Lemma 2.5 [5, Lemma 7.8] *Let $A = A' \otimes_{\mathbb{k}} A''$ be a tensor product of two graded artinian $\mathbb{k}$-algebras A' and A''. Let $\ell' \in A'$ and $\ell'' \in A''$ be linear elements, and set $\ell = \ell' + \ell'' = \ell' \otimes 1 + 1 \otimes \ell'' \in A$. Then*

(a) *If the multiplication maps* $\cdot\ell' : [A']_i \longrightarrow [A']_{i+1}$ *and* $\cdot\ell'' : [A'']_j \longrightarrow [A'']_{j+1}$ *are both not surjective, then neither is the map*

$$\cdot\ell : [A]_{i+j+1} \longrightarrow [A]_{i+j+2}.$$

(b) *If the multiplication maps* $\cdot\ell' : [A']_i \longrightarrow [A']_{i+1}$ *and* $\cdot\ell'' : [A'']_j \longrightarrow [A'']_{j+1}$ *are both not injective, then neither is the map*

$$\cdot\ell : [A]_{i+j} \longrightarrow [A]_{i+j+1}.$$

2.2 Graph Theory

From now on, by a graph we mean a simple graph $G = (V, E)$ with the vertex set $V = V(G)$ and the edge set $E = E(G)$. We start by recalling some basic definitions.

Definition 2.6 The *disjoint union* of the graphs G_1 and G_2 is a graph $G = G_1 \cup G_2$ having as vertex set the disjoint union of $V(G_1)$ and $V(G_2)$, and as edge set the disjoint union of $E(G_1)$ and $E(G_2)$. In particular, $\cup_n G$ denotes the disjoint union of $n > 1$ copies of the graph G.

Definition 2.7 Let $G = (V, E)$ be a graph.

(i) A subset X of V is called an *independent set* of G if for any $i, j \in X$, $\{i, j\} \notin E$, i.e., the vertices in X are pairwise non-adjacent. If an independent set X has k elements, then we say that X is an *independent set of size* k or a k-independent set of G.
(ii) An independent set X is called *maximal* if for every vertices $v \in V \setminus X$, $X \cup \{v\}$ is not an independent set of G.
(iii) The *independence number* of a graph G is the largest cardinality of an independent set of G. We denote this value by $\alpha(G)$.
(iv) A graph G is said to be *well-covered* if every maximal independent set of G has the same size $\alpha(G)$.

Definition 2.8 The *independence polynomial* of a graph G is a polynomial in one variable t whose coefficient of t^k is given by the number of independent sets of size k of G. We denote this polynomial by $I(G; t)$, i.e.,

$$I(G; t) = \sum_{k=0}^{\alpha(G)} s_k(G)t^k,$$

where $s_k(G)$ is the number of independent sets of size k in G. Note that $s_0(G) = 1$ since $\emptyset$ is an independent set of any graph G.

The independence polynomial of a graph was defined by Gutman and Harary in [14] as a generalization of the matching polynomial of a graph. For a vertex $v \in V$, define $N(v) = \{w \mid w \in V \text{ and } \{v, w\} \in E\}$ and $N[v] = N(v) \cup \{v\}$. The following equalities are very useful for the calculation of the independence polynomial for various families of graphs (see, for instance, [14, 18]).

Proposition 2.9 *Let* G_1, G_2, G *be the graphs. Assume that* $G = (V, E)$, $w \in V$ *and* $e = \{u, v\} \in E$. *Then the following equalities hold:*

(i) $I(G; t) = I(G \setminus w; t) + t \cdot I(G \setminus N[w]; t)$;
(ii) $I(G; t) = I(G \setminus e; t) - t^2 \cdot I(G \setminus (N(u) \cup N(v)); t)$;
(iii) $I(G_1 \cup G_2; t) = I(G_1; t)I(G_2; t)$.

2.3 Artinian Algebras Associated to Graphs

Let $G = (V, E)$ be a graph, with the set of vertices $V = \{1, 2, \dots, n\}$. Let $R = \Bbbk[x_1, \dots, x_n]$ be a standard graded polynomial ring over $\Bbbk$. The edge ideal of G is the ideal

$$I(G) = (x_i x_j \mid \{i, j\} \in E) \subset R.$$

Then, we say that

$$A(G) = \frac{R}{(x_1^2, \dots, x_n^2) + I(G)}$$

is the *artinian algebra associated to* G. The algebra $A(G)$ contains significant combinatorial information about G, as witnessed by

Proposition 2.10 *The Hilbert series of $A(G)$ is equal to the independence polynomial of G, i.e.,*

$$HS(A(G); t) = I(G; t) = \sum_{k=0}^{\alpha(G)} s_k(G) t^k.$$

As a consequence, the Castelnuovo–Mumford regularity of $A(G)$ is $\operatorname{reg}(A(G)) = \alpha(G)$ *and $A(G)$ is level if and only G is well-covered.*

Therefore, the WLP/SLP of $A(G)$ has strong consequences on the unimodality of the independence polynomial of G. Indeed, if $I(G; t)$ is not unimodal, then $A(G)$ fails the WLP by Proposition 2.4. Thus, to study the WLP/ SLP of $A(G)$, it is enough to consider the graphs whose independence polynomials are unimodal. Concerning the unimodality of the independence polynomial of graphs, we have the following famous conjecture.

Conjecture 2.11 [1] If G is a tree or forest, then the independence polynomial of G is unimodal.

To our best knowledge, until now, the largest class of graphs for which the independence polynomial is known to be unimodal is the class of claw-free graphs [15]. Recall that a graph is said to be *claw-free* if it does not admit the complete bipartite graph $K_{1,3}$ as an induced subgraph. Conjecture 2.11 remains widely open. The following example due to Bhattacharyya and Kahn [4] shows that one cannot expect the statement of Conjecture 2.11 to be true for bipartite graphs.

Example 2.12 Given positive integers m and $n > m$, let $G = (V, E)$ with $V = V_1 \cup V_2 \cup V_3$, where V_1, V_2, V_3 are disjoint; $|V_1| = n - m$ and $|V_2| = |V_3| = m$; E consists of the edges of the complete bipartite graph with the bipartition $V_1 \cup V_2$ and a perfect matching between V_2 and V_3. Then G is a bipartite graph and for every $i \geq 0$, $s_i(G) = (2^i - 1)\binom{m}{i} + \binom{n}{i}$. Therefore, for $m \geq 95$ and $n = \lfloor m \log_2(3) \rfloor$, $I(G; t)$ is not unimodal. As a consequence, $A(G)$ fails the WLP.

It is known that the Lefschetz properties depend strongly on the characteristic of the field.

Example 2.13 An empty graph is simply a graph with no edges. We denote the empty graph on n vertices by E_n. Then

$$A(E_n) = R/(x_1^2, \dots, x_n^2) \quad \text{and } I(E_n; t) = (1 + t)^n.$$

A well-known result of Stanley on complete intersections says that $A(E_n)$ has the SLP if $\operatorname{char}(\Bbbk) = 0$ or $\operatorname{char}(\Bbbk) > n$ [36, 37]. This result does not hold if $\operatorname{char}(\Bbbk) \leq n$, even for the

WLP. Indeed, in the case where $\mathrm{char}(\Bbbk) = 2$, it was known that $A(E_n)$ has the WLP if and only if $n = 3$ [7, 20]. In [20] Kustin and Vraciu showed that if $n \geq 5$ and $\mathrm{char}(\Bbbk) = p$ is odd, then $A(E_n)$ has the WLP if and only if $p \geq \lfloor \frac{n+3}{2} \rfloor$.

The complete graph on n vertices, denoted by K_n, is the graph where each vertex is adjacent to every other. It follows that

$$A(K_n) = R/(x_1, \ldots, x_n)^2 \quad \text{and } I(K_n; t) = 1 + nt.$$

It is easy to see that $A(K_n)$ has the SLP for any field $\Bbbk$. Concerning disjoint unions of complete graphs, we have the following.

Proposition 2.14 [29, Theorem 4.8] *Let* $\mathrm{char}(\Bbbk) \neq 2$ *and* $A(G)$ *be the artinian algebra associated to* $G = \cup_{i=1}^{r} K_{n_i}$. *Assume* $n_1 \geq n_2 \geq \cdots \geq n_r \geq 1$. *Then* $A(G)$ *has the WLP if and only if one of the following holds:*

(1) $n_2 = \cdots = n_r = 1$, *i.e.,* G *is the disjoint union of a complete graph* K_{n_1} *and an empty graph on* $r - 1$ *vertices.*
(2) $n_3 = \cdots = n_r = 1$ *and* r *is odd.*

In particular for every $n \geq 2$, *if* G *is the disjoint union of* n *complete graphs, none of which is a singleton, then* $A(G)$ *does not have the WLP.*

3 Independence Polynomial of Some Graphs

In this section, we provide some results on the independence polynomial of some familiar graphs, namely paths, cycles, and pan graphs. These results will be useful to prove our main theorems in the next section.

3.1 Paths

Let P_n be the path on n vertices ($n \geq 1$) (Fig. 1).

Proposition 3.1 *The independence polynomial of* P_n *is*

$$I(P_n; t) = \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-i}{i} t^i.$$

Moreover, for every $n \geq 1$, $I(P_n; t)$ *is unimodal, with the mode*

$$\lambda_n = \left\lceil \frac{5n + 2 - \sqrt{5n^2 + 20n + 24}}{10} \right\rceil.$$

Proof Hopkins and Staton [19] showed that

$$I(P_n; t) = F_{n+1}(t),$$

Fig. 1 The path P_6

where $F_n(t),\ n \geq 0$, are the so-called Fibonacci polynomials, which are defined recursively by

$$F_0(t) = 1;\ F_1(t) = 1;\ F_n(t) = F_{n-1}(t) + tF_{n-2}(t).$$

Based on this recurrence, one can deduce that

$$I(P_n; t) = \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-i}{i} t^i.$$

The unimodality of the independence polynomial of P_n is implied from the fact that the independence polynomial of a claw-free graph is unimodal [15]. Now we determine the mode of $I(P_n; t)$. Let i be an integer such that $0 \leq i \leq \lfloor \frac{n+1}{2} \rfloor$ and

$$\binom{n+1-i}{i} \geq \binom{n-i}{i+1}.$$

This is clearly true if $i \geq n/2$. If $i < n/2$, we have

$$\begin{aligned}
&\binom{n+1-i}{i} \geq \binom{n-i}{i+1} \\
\Leftrightarrow &\frac{n+1-i}{(n-2i)(n-2i+1)} \geq \frac{1}{i+1} \\
\Leftrightarrow &5i^2 - (5n+2)i + n^2 - 1 \leq 0 \\
\Leftrightarrow &\frac{5n+2-\sqrt{5n^2+20n+24}}{10} \leq i \leq \frac{5n+2+\sqrt{5n^2+20n+24}}{10}.
\end{aligned}$$

As the inequality on the right holds for any $i \leq \lfloor \frac{n+1}{2} \rfloor$, we have

$$\binom{n+1-i}{i} \geq \binom{n-i}{i+1} \Leftrightarrow i \geq \frac{5n+2-\sqrt{5n^2+20n+24}}{10}.$$

This means that the mode of $I(P_n; t)$ is equal to $\lambda_n = \lceil \frac{5n+2-\sqrt{5n^2+20n+24}}{10} \rceil$. □

We summarize the important properties of the mode of $I(P_n; t)$.

Lemma 3.2 *For any* $n \geq 1$, *one has the following.*

(i) $\lambda_{n+1} \geq \lambda_n$;
(ii) $\lambda_{n+3} - 1 \leq \lambda_n \leq \lambda_{n+4} - 1$;
(iii) $\lambda_{n+11} \geq \lambda_n + 3$.

Proof Set $\alpha_n = \frac{5n+2-\sqrt{5n^2+20n+24}}{10}$. A straightforward computation shows that

$$\alpha_{n+1} \geq \alpha_n;\ \ \alpha_{n+3} - 1 \leq \alpha_n \leq \alpha_{n+4} - 1 \text{ and } \alpha_{n+11} \geq \alpha_n + 3.$$

The lemma follows from basic properties of the ceiling function. □

Table 1 provides information about the initial values of the mode of the independence polynomial $I(G; t)$ for the classes of graphs considered in this paper, by using `Macaulay2` [34]. A dash indicates an undefined value.

Table 1 Graphs and modes of their independence polynomials

G	mode of $I(G;t)$ \ n	1	2	3	4	5	6	7	8	9	10	11	12	13
P_n	λ_n	0	1	1	1	2	2	2	2	3	3	3	4	4
C_n	ρ_n	–	–	1	1	1	2	2	2	3	3	3	3	4
CE_n	χ_n	–	–	–	1	1	2	2	2	2	3	3	3	4
Pan_n	ζ_n	–	–	1	1	2	2	2	3	3	3	3	4	4

3.2 Cycles

Let C_n be the cycle on n vertices ($n \geq 3$) (Fig. 2).

Proposition 3.3 *The independence polynomial of C_n is*

$$\begin{aligned} I(C_n;t) &= I(P_{n-1};t) + tI(P_{n-3};t) \\ &= 1 + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{i}\binom{n-i-1}{i-1} t^i. \end{aligned}$$

Moreover, $I(C_n;t)$ is unimodal, with the mode $\rho_n = \lceil \frac{5n-4-\sqrt{5n^2-4}}{10} \rceil$ for all $n \geq 3$.

Proof Hopkins and Staton [19] showed that

$$I(C_n;t) = 1 + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{i}\binom{n-i-1}{i-1} t^i.$$

The unimodality of the independence polynomial of C_n is implied from the fact that the independence polynomial of a claw-free graph is unimodal [15]. Arguing as in the proof of Proposition 3.1, solving for $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1$ from

$$\frac{n}{i}\binom{n-i-1}{i-1} \geq \frac{n}{i+1}\binom{n-i-2}{i}$$

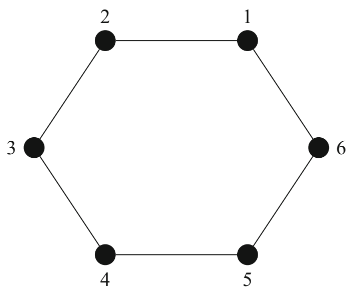

Fig. 2 The cycle C_6

we get

$$(i+1)(n-i-1) \geq (n-2i-1)(n-2i).$$

Equivalently $5i^2 - i(5n-4) + n^2 - 2n + 1 \leq 0$. Thus

$$\frac{5n-4-\sqrt{5n^2-4}}{10} \leq i \leq \frac{5n-4+\sqrt{5n^2-4}}{10}.$$

This implies that the mode of $I(C_n;t)$ is equal to $\rho_n = \left\lceil \frac{5n-4-\sqrt{5n^2-4}}{10} \right\rceil$, as desired. □

Lemma 3.4 *For all $n \geq 5$, there are inequalities $\lambda_{n-1} \leq \rho_n \leq \lambda_{n-4} + 1 \leq \lambda_n$.*

Proof By Lemma 3.2, $\lambda_{n-4} + 1 \leq \lambda_n$, hence it suffices to show that

$$\lambda_{n-1} \leq \rho_n \leq \lambda_{n-4} + 1.$$

For the inequality on the left, we have to show that

$$\begin{aligned}
&\frac{5(n-1)+2-\sqrt{5(n-1)^2+20(n-1)+24}}{10} \leq \frac{5n-4-\sqrt{5n^2-4}}{10}\\
\Leftrightarrow\, & 5n-3-\sqrt{5n^2+10n+9} \leq 5n-4-\sqrt{5n^2-4}\\
\Leftrightarrow\, & \sqrt{5n^2-4}+1 \leq \sqrt{5n^2+10n+9}\\
\Leftrightarrow\, & 5n^2-3+2\sqrt{5n^2-4} \leq 5n^2+10n+9\\
\Leftrightarrow\, & \sqrt{5n^2-4} \leq 5n+6 \Leftrightarrow (5n+6)^2-(5n^2-4) \geq 0\\
\Leftrightarrow\, & 20n^2+60n+40 \geq 0,
\end{aligned}$$

which is clear.

For the inequality on the right, we have to show that

$$\begin{aligned}
&\frac{5n-4-\sqrt{5n^2-4}}{10} \leq \frac{5(n-4)+2-\sqrt{5(n-4)^2+20(n-4)+24}}{10}+1\\
\Leftrightarrow\, & 5n-4-\sqrt{5n^2-4} \leq 5n-8-\sqrt{5n^2-20n+24}\\
\Leftrightarrow\, & \sqrt{5n^2-20n+24}+4 \leq \sqrt{5n^2-4}\\
\Leftrightarrow\, & 5n^2-20n+24+16+8\sqrt{5n^2-20n+24} \leq 5n^2-4 \quad \text{(by squaring)}\\
\Leftrightarrow\, & 8\sqrt{5n^2-20n+24} \leq 20n-44\\
\Leftrightarrow\, & 2\sqrt{5n^2-20n+24} \leq 5n-11\\
\Leftrightarrow\, & 4(5n^2-20n+24) \leq (5n-11)^2\\
\Leftrightarrow\, & 5n^2-30n+25 \geq 0 \Leftrightarrow 5(n-1)(n-5) \geq 0
\end{aligned}$$

which is true for all $n \geq 5$. The proof is complete. □

3.3 Pans

The n-pan graph is the graph obtained by joining a cycle graph C_n to a singleton graph K_1 with a bridge. We denote this graph by Pan_n.

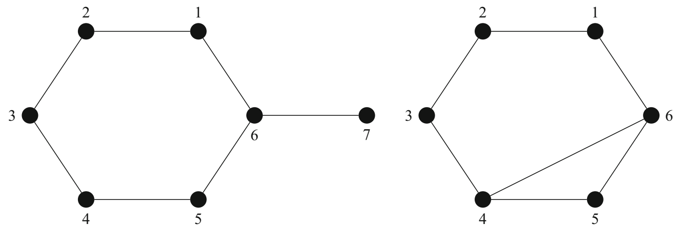

Fig. 3 Pan_6 and CE_6

Our goal is to show the independence polynomial of Pan_n is unimodal. For this, we consider a family of graphs formed by adding an edge $\{n-2, n\}$ to the cycles C_n $(n \geq 4)$. We denote this graph by CE_n (Fig. 3).

Note that CE_n is a claw-free graph, and hence its independence polynomial is unimodal [15].

Lemma 3.5 *The independence polynomial of* CE_n *is*

$$\begin{aligned} I(\mathrm{CE}_n; t) &= \sum_{i=0}^{\alpha(\mathrm{CE}_n)} s_i(\mathrm{CE}_n) t^i \\ &= I(P_{n-1}; t) + t I(P_{n-4}; t) \\ &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \left[\binom{n-i}{i} + \binom{n-i-2}{i-1} \right] t^i. \end{aligned}$$

Let χ_n *be the mode of* $I(\mathrm{CE}_n; t)$ *and* λ_n *be the mode of* $I(P_n; t)$ *as in Proposition 3.1. For any* $n \geq 5$, *one has* $\lambda_{n-1} \leq \chi_n \leq \lambda_{n-4} + 1$.

Proof The first assertion follows from applying Proposition 2.9 (i) for the vertex numbered n.

Let $1 \leq i \leq \lambda_{n-1}$. We need to show that

$$s_{i-1}(\mathrm{CE}_n) < s_i(\mathrm{CE}_n),$$

namely,

$$\binom{n-i+1}{i-1} + \binom{n-i-1}{i-2} < \binom{n-i}{i} + \binom{n-i-2}{i-1}.$$

This is clear for $i = 1$, so we assume that $i \geq 2$.

Since $i \leq \lambda_{n-1}$, $\binom{n-i+1}{i-1} \leq \binom{n-i}{i}$. It suffices to show that

$$\begin{aligned} &\binom{n-i-1}{i-2} < \binom{n-i-2}{i-1} \\ \Leftrightarrow &(n-i-1)(i-1) < (n-2i)(n-2i+1) \\ \Leftrightarrow &5i^2 - (5n+2)i + n^2 + 2n - 1 > 0 \\ \Leftrightarrow &i < \frac{5n+2-\sqrt{5n^2-20n+24}}{10} \text{ or } i > \frac{5n+2+\sqrt{5n^2-20n+24}}{10}. \end{aligned}$$

As $i \le \lambda_{n-1}$, it is enough to show that

$$\begin{aligned}
&\frac{5(n-1)+2-\sqrt{5(n-1)^2+20(n-1)+24}}{10} < \frac{5n+2-\sqrt{5n^2-20n+24}}{10} - 1\\
\Leftrightarrow &5n-3-\sqrt{5n^2+10n+9} < 5n-8-\sqrt{5n^2-20n+24}\\
\Leftrightarrow &5+\sqrt{5n^2-20n+24} < \sqrt{5n^2+10n+9}\\
\Leftrightarrow &\sqrt{5n^2-20n+24} < 3n-4 \quad \text{(after squaring and simplifying)}\\
\Leftrightarrow &n^2-n-2>0\\
\Leftrightarrow &(n+1)(n-2)>0,
\end{aligned}$$

which is clear for any $n \ge 4$. It follows that $\lambda_{n-1} \le \chi_n$. It remains to show that if $\lfloor \frac{n}{2} \rfloor \ge i \ge \lambda_{n-4}+1$ (note that $\lfloor \frac{n}{2} \rfloor$ is the independence number of CE_n), then

$$s_i(\mathrm{CE}_n) \ge s_{i+1}(\mathrm{CE}_n) \iff \binom{n-i}{i} + \binom{n-i-2}{i-1} \ge \binom{n-i-1}{i+1} + \binom{n-i-3}{i}.$$

By Lemma 3.2, $i \ge \lambda_{n-4}+1 \ge \lambda_{n-1}$, so $\binom{n-i}{i} \ge \binom{n-i-1}{i+1}$ thanks to Proposition 3.1. We have to show that

$$\begin{aligned}
\binom{n-i-2}{i-1} \ge \binom{n-i-3}{i}&\\
&\Leftrightarrow i(n-i-2) \ge (n-2i-2)(n-2i-1)\\
&\Leftrightarrow 5i^2-(5n-8)i+n^2-3n+2 \le 0\\
&\Leftrightarrow \frac{5n-8-\sqrt{5n^2-20n+24}}{10} \le i \le \frac{5n-8+\sqrt{5n^2-20n+24}}{10}.
\end{aligned}$$

Since $\lfloor \frac{n}{2} \rfloor \ge i \ge \lambda_{n-4}+1$, and by simple computations,

$$\left\lfloor \frac{n}{2} \right\rfloor \le \frac{5n-8+\sqrt{5n^2-20n+24}}{10} \quad \text{for all } n \ge 5,$$

the inequality on the right of the last chain is always true. Thus it is enough to prove the inequality on the left, which would be true if

$$\begin{aligned}
&\frac{5n-8-\sqrt{5n^2-20n+24}}{10} \le \frac{5(n-4)+2-\sqrt{5(n-4)^2+20(n-4)+24}}{10}+1\\
\Leftrightarrow &\frac{5n-8-\sqrt{5n^2-20n+24}}{10} \le \frac{5n-18+\sqrt{5n^2-20n+24}}{10}+1,
\end{aligned}$$

which is clear. Thus $\chi_n \le \lambda_{n-4}+1$. The proof is complete. □

Note that Pan_n is not a claw-free graph. Hence, we need to show the unimodality of its independence polynomial. We have the following.

Lemma 3.6 *The independence polynomial $I(\mathrm{Pan}_n; t)$ of the n-pan graph is unimodal. Let ζ_n be the mode of $I(\mathrm{Pan}_n; t)$. Then for all $n \ge 5$, there are inequalities*

$$\chi_{n+1} \le \zeta_n \le \rho_n + 1 \le \lambda_n + 1 \le \chi_{n+1} + 1, \tag{3.1}$$

where $\lambda_n, \rho_n, \chi_n$ are as in Propositions 3.1, 3.3 and Lemma 3.5.

Proof By using Proposition 2.9 (i) for the vertex of K_1, Propositions 3.1 and 3.3, we have

$$\begin{aligned}
I(\mathrm{Pan}_n; t) &= \sum_{i=0}^{\alpha(\mathrm{Pan}_n)} s_i(\mathrm{Pan}_n)t^i \\
&= I(C_n; t) + tI(P_{n-1}; t) \\
&= I(P_{n-1}; t) + t\big(I(P_{n-3}; t) + I(P_{n-1}; t)\big) \\
&= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor + 1} \left[\binom{n-i}{i} + \binom{n-i-1}{i-1} + \binom{n-i+1}{i-1}\right] t^i.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
s_i(\mathrm{Pan}_n) &= \binom{n-i}{i} + \binom{n-i-1}{i-1} + \binom{n-i+1}{i-1} \\
&= \binom{n-i+1}{i} + \binom{n-i-1}{i-1} + \binom{n-i}{i-2} \\
&\quad \left(\text{using} \binom{n}{p} = \binom{n-1}{p} + \binom{n-1}{p-1}\right) \\
&= s_i(\mathrm{CE}_{n+1}) + \binom{n-i}{i-2} \quad \text{(by Lemma 3.5).} \qquad (3.2)
\end{aligned}$$

We first have the following assertion.
CLAIM 1: $s_{i-1}(\mathrm{Pan}_n) < s_i(\mathrm{Pan}_n)$ for any $i \leq \chi_{n+1}$.

Proof of Claim 1: For any $1 \leq i \leq \chi_{n+1}$, $s_{i-1}(\mathrm{CE}_{n+1}) \leq s_i(\mathrm{CE}_{n+1})$. Therefore by (3.2), it suffices to show that

$$\begin{aligned}
&\binom{n-i+1}{i-3} < \binom{n-i}{i-2} \\
\Longleftrightarrow\, &(n-i+1)(i-2) < (n-2i+3)(n-2i+4) \\
\Longleftrightarrow\, &5i^2 - (5n+17)i + n^2 + 9n + 14 > 0 \\
\Longleftrightarrow\, &i < \frac{5n+17-\sqrt{5n^2-10n+9}}{10} \text{ or } i > \frac{5n+17+\sqrt{5n^2-10n+9}}{10}.
\end{aligned}$$

By Lemma 3.5,

$$\begin{aligned}
i \leq \chi_{n+1} &\leq \lambda_{n-3} + 1 \\
&< \frac{5(n-3)+2-\sqrt{5(n-3)^2+20(n-3)+24}}{10} + 2 \\
&= \frac{5n+7-\sqrt{5n^2-10n+9}}{10}.
\end{aligned}$$

Consequently, it suffices to show that

$$\frac{5n+7-\sqrt{5n^2-10n+9}}{10} \leq \frac{5n+17-\sqrt{5n^2-10n+9}}{10},$$

which is clear. Thus, we finish the proof of Claim 1.

Now, by again Proposition 2.9, we have

$$\begin{aligned} I(\mathrm{Pan}_n;t) &= \sum_{i=0}^{\alpha(\mathrm{Pan}_n)} s_i(\mathrm{Pan}_n)t^i \\ &= I(C_n;t) + tI(P_{n-1};t), \end{aligned}$$

we get $s_i(\mathrm{Pan}_n) = s_i(C_n) + s_{i-1}(P_{n-1})$. Next we have the following.
CLAIM 2: $s_i(\mathrm{Pan}_n) \geq s_{i+1}(\mathrm{Pan}_n)$ for any $i \geq \rho_n + 1$.

Proof of Claim 2: Since $i \geq \rho_n + 1$ and $n \geq 5$, $i - 1 \geq \rho_n \geq \lambda_{n-1}$ by Lemma 3.4. It follows that $s_i(C_n) \geq s_{i+1}(C_n)$ and $s_{i-1}(P_{n-1}) \geq s_i(P_{n-1})$. Thus $s_i(\mathrm{Pan}_n) \geq s_{i+1}(\mathrm{Pan}_n)$, as desired.

By Lemmas 3.4 and 3.5, $\rho_n \leq \lambda_n \leq \chi_{n+1}$, which yields the last two inequalities in (3.1). Moreover, it follows from Claims 1 and 2 that $s_{i-1}(\mathrm{Pan}_n) < s_i(\mathrm{Pan}_n)$ for any $i \leq \chi_{n+1}$ and $s_i(\mathrm{Pan}_n) \geq s_{i+1}(\mathrm{Pan}_n)$ for any $i \geq \chi_{n+1} + 1$. Thus, the independence polynomial $I(\mathrm{Pan}_n;t)$ of the n-pan graph is unimodal. Moreover, $\chi_{n+1} \leq \zeta_n$ by Claim 1 and $\zeta_n \leq \rho_n + 1$ by Claim 2. This concludes the proof. □

4 WLP for Algebras Associated to Paths and Cycles

In this section, we study the WLP for artinian algebras associated to paths and cycles. From now on, we always assume $\mathrm{char}(\Bbbk) = 0$ and denote by ℓ the sum of variables in the polynomial ring we are working with.

4.1 Paths

The artinian algebra associated to P_n is

$$A(P_n) = R/K,$$

where $K = (x_1^2, \ldots, x_n^2) + (x_1x_2, x_2x_3, \ldots, x_{n-1}x_n) \subset R = \Bbbk[x_1, \ldots, x_n]$. The following lemma is useful to an inductive argument on the WLP of $A(P_n)$.

Lemma 4.1 *For every integer i, there is a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & [A(P_{n-2})]_{i-1} & \longrightarrow & [A(P_n)]_i & \longrightarrow & [A(P_{n-1})]_i & \longrightarrow & 0 \\ & & \downarrow{\cdot\ell} & & \downarrow{\cdot\ell} & & \downarrow{\cdot\ell} & & \\ 0 & \longrightarrow & [A(P_{n-2})]_i & \longrightarrow & [A(P_n)]_{i+1} & \longrightarrow & [A(P_{n-1})]_{i+1} & \longrightarrow & 0. \end{array}$$

Proof Assume $A(P_n) = R/K$ and set $I = K + (x_n)$ and $J = (K : x_n)$. Then $A(P_{n-1}) \cong R/I$ and $A(P_{n-2}) \cong R/J$ and we have the following exact sequence

$$0 \longrightarrow R/J(-1) \xrightarrow{\cdot x_n} R/K \longrightarrow R/I \longrightarrow 0.$$

The desired conclusion follows. □

We now prove our first main result.

Theorem 4.2 *The ring $A(P_n)$ has the WLP if and only if $n \in \{1, 2, \ldots, 7, 9, 10, 13\}$.*

Proof Using `Macaulay2` [34] to compute the Hilbert series of the rings $A(P_n)$ and $A(P_n)/\ell A(P_n)$ with $1 \leq n \leq 17$, we see that $A(P_n)$ has the WLP for each $n \in \{1, 2, \ldots, 7, 9, 10, 13\}$. Furthermore, for each $n \in \{8, 11, 14, 15, 17\}$, $A(P_n)$ only fails the surjectivity in the multiplication map by ℓ from degree λ_n to degree $\lambda_n + 1$. However, for $n \in \{12, 16\}$, $A(P_n)$ only fails the injectivity in the multiplication map by ℓ from degree $\lambda_n - 1$ to degree λ_n.

It remains to show the following.

CLAIM: The multiplication map $\cdot\ell : [A(P_n)]_{\lambda_n} \longrightarrow [A(P_n)]_{\lambda_n+1}$ is not surjective for all $n \geq 17$.

We will prove the above claim by induction on n, having just established the case $n = 17$. For $n \geq 18$, we consider the multiplication map

$$\cdot\ell : [A(P_n)]_{\lambda_n} \longrightarrow [A(P_n)]_{\lambda_n+1}.$$

By Lemma 3.2, one has $\lambda_{n-1} \leq \lambda_n \leq \lambda_{n-1} + 1$, hence we consider the following two cases.

Case 1: $\lambda_n = \lambda_{n-1}$. In the diagram of Lemma 4.1, where $i = \lambda_{n-1} = \lambda_n$, the right vertical map is not surjective by the induction hypothesis, so neither is the middle vertical map.

Case 2: $\lambda_n = \lambda_{n-1} + 1$. By Lemma 3.2, one has $\lambda_{n-1} = \lambda_{n-2} = \lambda_{n-3}$. In this case, we must have $n \geq 20$ since $\lambda_{16} = \lambda_{17} = \lambda_{18} = \lambda_{19} = 5$.

Assume $A(P_n) = R/K$ and set $I = K + (x_{n-2})$ and $J = K : x_{n-2}$. Then we have the following exact sequence

$$0 \longrightarrow R/J(-1) \xrightarrow{\cdot x_{n-2}} R/K \longrightarrow R/I \longrightarrow 0,$$

where $R/J \cong A(P_{n-4}) \otimes_{\Bbbk} \Bbbk[x_n]/(x_n^2)$ and $R/I \cong A(P_{n-3}) \otimes_{\Bbbk} A(P_2)$, with

$$A(P_2) = \Bbbk[x_{n-1}, x_n]/(x_{n-1}, x_n)^2.$$

This exact sequence gives rise to the following commutative diagram, with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & [R/J]_{\lambda_n-1} & \longrightarrow & [A(P_n)]_{\lambda_n} & \longrightarrow & [R/I]_{\lambda_n} & \longrightarrow & 0 \\ & & \downarrow{\cdot\ell} & & \downarrow{\cdot\ell} & & \downarrow{\cdot\ell} & & \\ 0 & \longrightarrow & [R/J]_{\lambda_n} & \longrightarrow & [A(P_n)]_{\lambda_n+1} & \longrightarrow & [R/I]_{\lambda_n+1} & \longrightarrow & 0. \end{array}$$

To prove that the middle vertical map is not surjective, it suffices to show that the right vertical map

$$\cdot\ell : [R/I]_{\lambda_n} \longrightarrow [R/I]_{\lambda_n+1}$$

is not surjective. By the inductive hypothesis, $A(P_{n-3})$ fails the surjectivity from degree $\lambda_n - 1$ to degree λ_n, as $\lambda_{n-3} = \lambda_n - 1$. Clearly, the Hilbert function of $A(P_2)$ is $(1, 2)$, and hence $A(P_2)$ fails the surjectivity from degree 0 to degree 1. Then by Lemma 2.5 (a), $R/I \cong A(P_{n-3}) \otimes_{\Bbbk} A(P_2)$ fails the surjectivity from degree λ_n to degree $\lambda_n + 1$, as desired. □

The above theorem shows that $A(P_n)$ fails the WLP since surjectivity fails for any $n \geq 17$. The next result also prove that $A(P_n)$ fails the injectivity for some cases.

Proposition 4.3 *Recall the mode λ_n of the independence polynomial of $I(P_n; t)$. If $n \geq 12$ is an integer such that $\lambda_n = \lambda_{n-1} + 1$, then $A(P_n)$ fails the injectivity from degree $\lambda_n - 1$ to degree λ_n.*

Proof We prove the above proposition by induction on $n \geq 12$. A computation with `Macaulay2` [34] shows that the proposition holds for $n \in \{12, 16, 20\}$. This covers all cases from 12 to 20 due to Lemma 3.2. Now consider an $n \geq 21$ such that $\lambda_n = \lambda_{n-1} + 1$. Set

$$\begin{aligned} n_1 &= \max\{j \mid j < n \text{ and } \lambda_j = \lambda_{j-1} + 1\}, \\ n_2 &= \max\{j \mid j < n_1 \text{ and } \lambda_j = \lambda_{j-1} + 1\}, \\ m &= \max\{j \mid j < n_2 \text{ and } \lambda_j = \lambda_{j-1} + 1\}. \end{aligned}$$

Then, by Lemma 3.2, $9 \leq n - m \leq 11$. We have the following exact sequence

$$0 \to A(P_m) \otimes_{\Bbbk} A(P_{n-m-3})(-1) \xrightarrow{\cdot x_{m+2}} A(P_n) \to A(P_{m+1}) \otimes_{\Bbbk} A(P_{n-m-2}) \to 0.$$

By using this exact sequence, it suffices to show that

$$\cdot\ell : [A(P_m) \otimes_{\Bbbk} A(P_{n-m-3})]_{\lambda_n - 2} \longrightarrow [A(P_m) \otimes_{\Bbbk} A(P_{n-m-3})]_{\lambda_n - 1}$$

is not injective. By the inductive hypothesis, $A(P_m)$ fails the injectivity from degree $\lambda_m - 1$ to λ_m. Observe that $\lambda_m = \lambda_n - 3$ and $6 \leq n - m - 3 \leq 8$. Hence by Table 1, $\lambda_{n-m-3} = 2$ and consequently, $A(P_{n-m-3})$ fails the injectivity from degree 2 to degree 3. By Lemma 2.5 (b), $A(P_m) \otimes_{\Bbbk} A(P_{n-m-3})$ fails the injectivity from degree $\lambda_m + 1 = \lambda_n - 2$ to $\lambda_n - 1$, as desired. □

4.2 Cycles

The artinian algebra associated to the cycle on n vertices is

$$(C_n) = R/K,$$

where $K = (x_1^2, \ldots, x_n^2) + (x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1) \subset R = \Bbbk[x_1, \ldots, x_n]$. Our second main result is the following.

Theorem 4.4 *The algebra $A(C_n)$ has the WLP if and only if $n \in \{3, 4, \ldots, 11, 13, 14, 17\}$.*

Proof Recall that ρ_n is the mode of the independence polynomial of C_n. Using `Macaulay2` [34] to compute the Hilbert series of $A(C_n)$ and $A(C_n)/\ell A(C_n)$ with $3 \leq n \leq 20$, we can check that:

- $A(C_n)$ has the WLP for each $3 \leq n \leq 17$ and $n \notin \{12, 15, 16\}$;
- for $n \in \{12, 15, 18, 19\}$, then $A(C_n)$ fails the surjectivity from degree ρ_n to degree $\rho_n + 1$;
- for $n \in \{16, 20\}$, then $A(C_n)$ fails the injectivity from degree $\rho_n - 1$ to degree ρ_n.

Now assume that $n \geq 21$. By Lemmas 3.2 and 3.4, $\lambda_{n-1} \leq \rho_n \leq \lambda_{n-4} + 1 \leq \lambda_{n-1} + 1$. Consider the following two cases.

Case 1: $\rho_n = \lambda_{n-1}$. In this case, we will show that $A(C_n)$ fails the WLP due to the failure of the surjectivity from degree ρ_n to degree $\rho_n + 1$. Indeed, write $A(C_n) = R/K$, and let $I = K + (x_n)$ and $J = K : x_n$. Then $A(P_{n-1}) \cong R/I$ and $A(P_{n-3}) \cong R/J$ and we have the following exact sequence

$$0 \longrightarrow R/J(-1) \xrightarrow{\cdot x_n} R/K \longrightarrow R/I \longrightarrow 0.$$

This yields a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & [A(P_{n-3})]_{\rho_n -1} & \longrightarrow & [A(C_n)]_{\rho_n} & \longrightarrow & [A(P_{n-1})]_{\rho_n} & \longrightarrow & 0 \\ & & \downarrow \cdot\ell & & \downarrow \cdot\ell & & \downarrow \cdot\ell & & \\ 0 & \longrightarrow & [A(P_{n-3})]_{\rho_n} & \longrightarrow & [A(C_n)]_{\rho_n+1} & \longrightarrow & [A(P_{n-1})]_{\rho_n+1} & \longrightarrow & 0. \end{array}$$

The proof of Theorem 4.2 shows that the multiplication map

$$\cdot\ell : [A(P_{n-1})]_{\rho_n} \longrightarrow [A(P_{n-1})]_{\rho_n+1}$$

is not surjective for any $n \geq 18$. Hence the middle vertical map

$$\cdot\ell : [A(C_n)]_{\rho_n} \longrightarrow [A(C_n)]_{\rho_n+1}$$

is not surjective, as desired.

Case 2: $\rho_n = \lambda_{n-1} + 1$. In this case, Lemmas 3.2 and 3.4 yield $\lambda_{n-1} = \lambda_{n-4}$.

Denote $y_1 = x_{n-1}$, $y_2 = x_{n-2}$. We have the following diagram

$$\begin{array}{ccccc} [A(C_n)]_{\rho_n} & \xrightarrow{/(x_n)} & [A(P_{n-1})]_{\lambda_{n-1}+1} & \xrightarrow{/(x_{n-3})} & \left[A(P_{n-4}) \otimes_{\Bbbk} \frac{\Bbbk[y_1,y_2]}{(y_1,y_2)^2}\right]_{\lambda_{n-4}+1} \\ \downarrow \cdot\ell & & & & \downarrow \cdot\ell \\ [A(C_n)]_{\rho_n+1} & \longrightarrow & [A(P_{n-1})]_{\lambda_{n-1}+2} & \longrightarrow & \left[A(P_{n-4}) \otimes_{\Bbbk} \frac{\Bbbk[y_1,y_2]}{(y_1,y_2)^2}\right]_{\lambda_{n-4}+2}. \end{array}$$

By the proof of Theorem 4.2 and the fact that $n - 4 \geq 17$, the map

$$A(P_{n-4}) \xrightarrow{\cdot\ell} A(P_{n-4})$$

fails the surjectivity at degree λ_{n-4}. Since the map

$$\Bbbk[y_1, y_2]/(y_1, y_2)^2 \xrightarrow{\cdot(y_1+y_2)} \Bbbk[y_1, y_2]/(y_1, y_2)^2$$

fails the surjectivity at degree 0, Lemma 2.5 (a) yields that the right vertical map of the diagram fails the surjectivity at degree $\lambda_{n-4} + 1$.

By the surjectivity of the horizontal maps in the diagram, we conclude that left vertical map in the diagram fails the surjectivity at degree $\lambda_{n-4} + 1 = \rho_n$. Hence $A(C_n)$ does not have the WLP. This concludes the proof. □

Next, we consider a special case of the tadpole graphs. The *tadpole* graph $T_{3,n}$ is obtained by joining a cycle C_3 to a path P_n with a bridge (Fig. 4).

Clearly, $T_{3,n}$ is a claw-free graph. Therefore, the independence polynomial of $T_{3,n}$ is unimodal [15]. By Proposition 2.9 for either of the vertices on the left of the three cycle, we

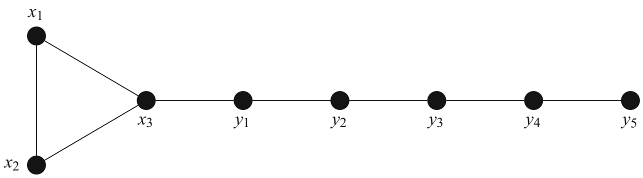

Fig. 4 Tadpole $T_{3,5}$

have

$$I(T_{3,n};t) = I(P_{n+2};t) + tI(P_n;t) = I(C_{n+3};t).$$

By Proposition 3.3, it follows that the mode of $I(T_{3,n};t)$ is equal to that of $I(C_{n+3};t)$, which is

$$\rho_{n+3} = \left\lceil \frac{5(n+3) - 4 - \sqrt{5(n+3)^2 - 4}}{10} \right\rceil.$$

Corollary 4.5 *The algebra $A(T_{3,n})$ has the WLP if and only if $n \in \{1, 3, 4, 7\}$.*

Proof The artinian algebra associated to $T_{3,n}$ is

$$A(T_{3,n}) = \frac{\Bbbk[x_1, x_2, x_3, y_1, \ldots, y_n]}{(x_1^2, x_2^2, x_3^2, y_1^2, \ldots, y_n^2) + (x_1x_2, x_2x_3, x_3x_1, x_3y_1, y_1y_2, \ldots, y_{n-1}y_n)}.$$

Using `Macaulay2` [34] to compute the Hilbert series of $A(T_{3,n})$ and $A(T_{3,n})/\ell A(T_{3,n})$ with $1 \le n \le 17$, we can check that:

- $A(T_{3,n})$ has the WLP for each $n \in \{1, 3, 4, 7\}$;
- for $n \in \{2, 5, 8, 9, 11, 12, 14, 15, 16, 17\}$, then $A(T_{3,n})$ fails the surjectivity from degree ρ_{n+3} to degree $\rho_{n+3} + 1$;
- for $n \in \{2, 6, 10, 13, 14, 17\}$, then $A(T_{3,n})$ fails the injectivity from degree $\rho_{n+3} - 1$ to degree ρ_{n+3}.

Now assume that $n \ge 18$. By Lemmas 3.2 and 3.4, $\lambda_{n+2} \le \rho_{n+3} \le \lambda_{n+2} + 1$. We consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & [A(P_n)]_{\rho_{n+3}-1} & \xrightarrow{\cdot x_1} & [A(T_{3,n})]_{\rho_{n+3}} & \longrightarrow & [A(P_{n+2})]_{\rho_{n+3}} & \longrightarrow & 0 \\ & & \downarrow{\cdot \ell} & & \downarrow{\cdot \ell} & & \downarrow{\cdot \ell} & & \\ 0 & \longrightarrow & [A(P_n)]_{\rho_{n+3}} & \xrightarrow{\cdot x_1} & [A(T_{3,n})]_{\rho_{n+3}+1} & \longrightarrow & [A(P_{n+2})]_{\rho_{n+3}+1} & \longrightarrow & 0. \end{array}$$

The proof proceeds along the same lines as that of Theorem 4.4, replacing $A(C_{n+3})$ by $A(T_{3,n})$ and noting that $n + 3 \ge 21$. □

4.3 Pans

To study the WLP of rings associated to pans, we first examine the WLP of rings associated to CE_n (Fig. 5). The latter is by definition

$$A(\mathrm{CE}_n) = \frac{\Bbbk[x_1, \ldots, x_n]}{(x_1^2, \ldots, x_n^2) + (x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1) + (x_{n-2}x_n)}.$$

Theorem 4.6 *For an integer $n \ge 4$, the algebra $A(\mathrm{CE}_n)$ has the WLP if and only if $n \in \{4, 5, \ldots, 8, 10, 11, 14\}$.*

Proof By using `Macaulay2` [34] to compute the Hilbert series of $A(\mathrm{CE}_n)/\ell A(\mathrm{CE}_n)$ and $A(\mathrm{CE}_n)$ with $4 \le n \le 20$, we can check that:

- $A(\mathrm{CE}_n)$ has the WLP for each $4 \le n \le 14$ and $n \notin \{9, 12, 13\}$;
- for $n \in \{9, 12, 15, 16, 18, 19\}$, $A(\mathrm{CE}_n)$ fails the surjectivity from degree χ_n to degree $\chi_n + 1$;

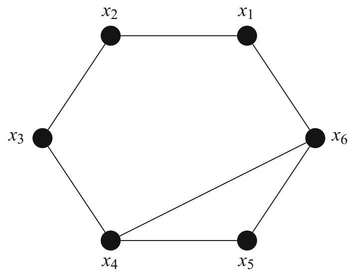

Fig. 5 The graph CE_6

- for $n \in \{13, 17, 20\}$, $A(\mathrm{CE}_n)$ fails the injectivity from degree $\chi_n - 1$ to degree χ_n.

Now assume that $n \geq 21$. We will prove that $A(\mathrm{CE}_n)$ fails the surjectivity from degree χ_n to degree $\chi_n + 1$. The proof is similar to that of Theorem 4.4. Recall that by Lemmas 3.2 and 3.5,

$$\lambda_{n-1} \leq \chi_n \leq \lambda_{n-4} + 1 \leq \lambda_{n-1} + 1. \tag{4.1}$$

We consider the following two cases.

Case 1: $\chi_n = \lambda_{n-1}$. Consider the exact sequence

$$0 \longrightarrow A(P_{n-4})(-1) \xrightarrow{\cdot x_n} A(\mathrm{CE}_n) \longrightarrow A(P_{n-1}) \longrightarrow 0.$$

The proof of Theorem 4.2 shows that the multiplication map

$$\cdot\ell : [A(P_{n-1})]_{\chi_n} \longrightarrow [A(P_{n-1})]_{\chi_n+1}$$

is not surjective for any $n \geq 18$. Hence, the map

$$\cdot\ell : [A(\mathrm{CE}_n)]_{\chi_n} \longrightarrow [A(\mathrm{CE}_n)]_{\chi_n+1}$$

is also not surjective, as desired.

Case 2: $\chi_n = \lambda_{n-1} + 1$. In this case, the chain (4.1) yields $\lambda_{n-1} = \lambda_{n-4}$. As in the proof of Theorem 4.4, denoting $y_1 = x_{n-1}$, $y_2 = x_{n-2}$, we have the following diagram

$$\begin{array}{ccccc} [A(\mathrm{CE}_n)]_{\chi_n} & \xrightarrow{/(x_n)} & [A(P_{n-1})]_{\lambda_{n-1}+1} & \xrightarrow{/(x_{n-3})} & \left[A(P_{n-4}) \otimes_{\Bbbk} \frac{\Bbbk[y_1,y_2]}{(y_1,y_2)^2}\right]_{\lambda_{n-4}+1} \\ \downarrow{\cdot\ell} & & & & \downarrow{\cdot\ell} \\ [A(\mathrm{CE}_n)]_{\chi_n+1} & \longrightarrow & [A(P_{n-1})]_{\lambda_{n-1}+2} & \longrightarrow & \left[A(P_{n-4}) \otimes_{\Bbbk} \frac{\Bbbk[y_1,y_2]}{(y_1,y_2)^2}\right]_{\lambda_{n-4}+2} \end{array}.$$

Since the right vertical map of the diagram fails the surjectivity at degree $\lambda_{n-4} + 1$, we conclude that the left vertical map in the diagram fails the surjectivity at degree $\lambda_{n-4}+1 = \chi_n$, as desired. □

Finally, we show the last main result.

Theorem 4.7 *The algebra* $A(\mathrm{Pan}_n)$ *associated to the pan graph* Pan_n *has the WLP if and only if* $n \in \{3, 4, \ldots, 10, 12, 13, 16\}$.

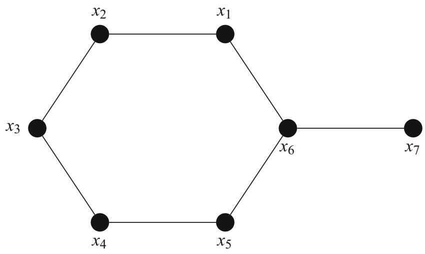

Fig. 6 Pan_6

Proof The artinian algebra associated to Pan_n (see Fig. 6) is

$$A(\mathrm{Pan}_n) = \frac{\Bbbk[x_1, \ldots, x_{n+1}]}{(x_1^2, \ldots, x_{n+1}^2) + (x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_{n+1}) + (x_1x_n)}.$$

By using `Macaulay2` [34] to compute the Hilbert series of $A(\mathrm{Pan}_n)$ and $\dfrac{A(\mathrm{Pan}_n)}{\ell A(\mathrm{Pan}_n)}$ with $3 \le n \le 20$, we can check that:

- $A(\mathrm{Pan}_n)$ has the WLP for each $3 \le n \le 16$ and $n \notin \{11, 14, 15\}$;
- for $n \in \{11, 14, 17, 18, 20\}$, $A(\mathrm{Pan}_n)$ fails the surjectivity from degree ζ_n to degree ζ_n+1;
- for $n \in \{15, 19\}$, $A(\mathrm{Pan}_n)$ fails the injectivity from degree $\zeta_n - 1$ to degree ζ_n.

Now assume that $n \ge 21$. Recall that by Lemma 3.6,

$$\chi_{n+1} \le \zeta_n \le \rho_n + 1 \le \lambda_n + 1 \le \chi_{n+1} + 1. \tag{4.2}$$

We consider the following two cases.
Case 1: $\zeta_n = \chi_{n+1}$. In this case, $A(\mathrm{Pan}_n)$ fails the surjectivity from degree ζ_n to degree ζ_n+1 by using the exact sequence

$$0 \longrightarrow A(P_{n-3})(-2) \xrightarrow{\cdot x_1x_{n+1}} A(\mathrm{Pan}_n) \longrightarrow A(\mathrm{CE}_{n+1}) \longrightarrow 0.$$

Indeed, the proof of Theorem 4.6 shows that the multiplication map

$$\cdot\ell : [A(\mathrm{CE}_{n+1})]_{\zeta_n} \longrightarrow [A(\mathrm{CE}_{n+1})]_{\zeta_n+1}$$

is not surjective for any $n \ge 21$. Hence the map

$$\cdot\ell : [A(\mathrm{Pan}_n)]_{\zeta_n} \longrightarrow [A(\mathrm{Pan}_n)]_{\zeta_n+1}$$

is also not surjective, as desired.
Case 2: $\zeta_n = \chi_{n+1} + 1$. In this case, the chain (4.2) yields $\lambda_n = \rho_n = \zeta_n - 1$. Since $\lambda_n - \lambda_{n-3} \le 1$ by Lemma 3.2, we have the following two subcases.
Subcase 2.1: $\lambda_n = \lambda_{n-3}$. As in the proof of Theorem 4.4, denote $y_1 = x_n$, $y_2 = x_{n+1}$, we have the following diagram

$$\begin{array}{ccccc}
[A(\mathrm{Pan}_n)]_{\zeta_n} & \xrightarrow{/(x_{n-1})} & [A(P_n)]_{\lambda_n+1} & \xrightarrow{/(x_1)} & \left[A(P_{n-3}) \otimes_\Bbbk \frac{\Bbbk[y_1,y_2]}{(y_1,y_2)^2}\right]_{\lambda_{n-3}+1} \\
\downarrow{\cdot\ell} & & & & \downarrow{\cdot\ell} \\
[A(\mathrm{Pan}_n)]_{\zeta_n+1} & \longrightarrow & [A(P_n)]_{\lambda_n+2} & \longrightarrow & \left[A(P_{n-3}) \otimes_\Bbbk \frac{\Bbbk[y_1,y_2]}{(y_1,y_2)^2}\right]_{\lambda_{n-3}+2}
\end{array}.$$

Since the right vertical map of the above diagram fails the surjectivity at degree $\lambda_{n-3}+1 = \zeta_n$, we conclude that the left vertical map fails the surjectivity at the same degree.
Subcase 2.2: $\lambda_n = \lambda_{n-3} + 1$. Set

$$m = \max\{j \mid j \leq n \text{ and } \lambda_j = \lambda_{j-1} + 1\}.$$

Then by Lemma 3.2, $n-2 \leq m \leq n$ and $\lambda_m = \lambda_n$. First, we consider the case where $m \neq n$. Set

$$y = \begin{cases} x_{n-2} & \text{if } m = n-2, \\ x_{n+1} & \text{if } m = n-1. \end{cases}$$

Then we have the following diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & [A(P_m)]_{\zeta_n-2} & \xrightarrow{\cdot y} & [A(\text{Pan}_n)]_{\zeta_n-1} \\ & & \downarrow{\cdot\ell} & & \downarrow{\cdot\ell} \\ 0 & \longrightarrow & [A(P_m)]_{\zeta_n-1} & \xrightarrow[\cdot y]{} & [A(\text{Pan}_n)]_{\zeta_n}. \end{array}$$

Since $\zeta_n - 2 = \lambda_n - 1 = \lambda_m - 1$, we have the first vertical map of the diagram fails the injectivity at degree $\lambda_m - 1$ by Proposition 4.3. It follows that the second vertical map of the diagram fails the injectivity at degree $\zeta_n - 1$.

To complete the proof of the theorem, we consider the case where $m = n$. In this case, one has $\rho_n = \lambda_n = \lambda_{n-1} + 1$. By Lemmas 3.2 and 3.4, $\lambda_{n-1} = \lambda_{n-4} = \lambda_{n-5} + 1$. Hence $\lambda_{n-4} = \zeta_n - 2$. Now we consider the following diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & [A(P_{n-4})]_{\zeta_n-3} & \xrightarrow{\cdot x_{n-4}x_{n-2}} & [A(\text{Pan}_n)]_{\zeta_n-1} \\ & & \downarrow{\cdot\ell} & & \downarrow{\cdot\ell} \\ 0 & \longrightarrow & [A(P_{n-4})]_{\zeta_n-2} & \xrightarrow[\cdot x_{n-4}x_{n-2}]{} & [A(\text{Pan}_n)]_{\zeta_n}. \end{array}$$

By Proposition 4.3, the first vertical map of the diagram fails the injectivity at degree $\lambda_{n-4} - 1 = \zeta_n - 3$. It follows that the second vertical map of the diagram fails the injectivity at degree $\zeta_n - 1$. Thus we complete the proof. □

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