



# Asymptotic regularity of invariant chains of edge ideals

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## Abstract

We study chains of nonzero edge ideals that are invariant under the action of the monoid  $\text{Inc}$  of increasing functions on the positive integers. We prove that the sequence of Castelnuovo–Mumford regularity of ideals in such a chain is eventually constant with limit either 2 or 3, and we determine explicitly when the constancy behavior sets in. This provides further evidence to a conjecture on the asymptotic linearity of the regularity of  $\text{Inc}$ -invariant chains of homogeneous ideals. The proofs reveal unexpected combinatorial properties of  $\text{Inc}$ -invariant chains of edge ideals.

**Keywords** Invariant ideal · Castelnuovo–Mumford regularity · Edge ideal · Induced matching · Asymptotic behavior

**Mathematics Subject Classification** 13A50 · 13C15 · 13D02 · 13F20 · 16P70 · 16W22

## 1 Introduction

Let  $\mathbb{k}$  be a field,  $R_n = \mathbb{k}[x_1, \dots, x_n]$  a polynomial ring in  $n$  variables, and  $R = \bigcup_{n \geq 1} R_n = \mathbb{k}[x_1, x_2, \dots]$  be the polynomial ring in infinitely many variables. Denote by  $\mathbb{N}$  the set of positive integers, and  $\text{Inc}$  the monoid of strictly increasing functions

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$\pi : \mathbb{N} \rightarrow \mathbb{N}$ , with the operation being composition of maps. Then,  $\text{Inc}$  acts on  $R$  by endomorphisms via the rule

$$\pi(x_i) = x_{\pi(i)}, \quad \text{for each } i \in \mathbb{N}.$$

A family of ideals  $\mathcal{I} = (I_n)_{n \geq 1}$ , where  $I_n \subseteq R_n$ , is called an *Inc-invariant chain* if for all  $1 \leq m < n$  and for all  $\pi \in \text{Inc}$  such that  $\pi(m) \leq n$ , we have  $\pi(I_m) \subseteq I_n$ . A related notion is that of *Inc-invariant ideals*: An ideal  $I \subseteq R$  is called *Inc-invariant* if  $\pi(I) \subseteq I$  for any  $\pi \in \text{Inc}$ . Recently, there was considerable interest in *Inc-invariant chains* and *Inc-invariant ideals*, due to several reasons. First, *Inc-invariant chains* and *Inc-invariant ideals* appear in various places. For instance, let  $\text{Sym}(n)$  be the symmetric group on  $n$  objects and  $\text{Sym} = \bigcup_{n \geq 1} \text{Sym}(n)$  the direct limit of these groups. Then, any *Sym-invariant ideal*  $I \subseteq R$  is also *Inc-invariant* [18]. Let  $\mathcal{I} = (I_n)_{n \geq 1}$  be any *Sym-invariant chain*, i.e. each  $I_n$  is *Sym*( $n$ )-invariant and  $\pi(I_m) \subseteq I_n$  whenever  $1 \leq m < n$ ,  $\pi \in \text{Sym}(n)$ . Then, any such *Sym-invariant chain* is also *Inc-invariant*. Special instances of *Sym-invariant ideals* are abound in representation theory [3, 22] and algebraic statistics [1, 10]. Second, *Inc-invariant chains* and *Inc-invariant ideals* enjoy remarkable finiteness properties due to their highly symmetrical nature. By work of Cohen [4], Aschenbrenner–Hillar [1], Hillar–Sullivant [10], Nagel–Römer [18], it is known that  $R$  is *Inc-Noetherian*, in particular any *Inc-invariant ideal*  $I \subseteq R$  is generated by finitely many *Inc-orbits*. Moreover, any *Inc-invariant chain* of ideals stabilizes, namely for integers  $1 \leq m \leq n$ , let

$$\text{Inc}_{m,n} = \{\pi \in \text{Inc} \mid \pi(m) \leq n\},$$

then for any *Inc-invariant chain*  $\mathcal{I} = (I_n)_{n \geq 1}$ , there exists some finite  $r \geq 1$  such that  $I_n = \text{Inc}_{r,n}(I_r)R_n$  for all  $n \geq r$ . We call the smallest such  $r$  the *stability index* of  $\mathcal{I}$ , denoted  $\text{ind}(\mathcal{I})$ .

The present work can be seen as the continuation of an active program aiming at studying asymptotic properties of *Inc-* and *Sym-invariant chains* of ideals; see, for example [6, 9, 11, 13, 17, 18] for various aspects of this program. In [13, Conjecture 4.12], Le, Nagel, the second author, and Römer proposed

**Conjecture 1.1** *Let  $\mathcal{I} = (I_n)_{n \geq 1}$  be an *Inc-invariant chain* of homogeneous ideals. Then for all  $n \gg 0$ ,  $\text{reg } I_n = Cn + D$  for certain constants  $C$  and  $D$  depending only on  $\mathcal{I}$ . Here  $\text{reg}(-)$  denotes the Castelnuovo–Mumford regularity.*

We say that  $\mathcal{I} = (I_n)_{n \geq 1}$  is a *saturated chain* if  $I_n = I_{n+1} \cap R_n$  for all  $n \geq 1$ . In [13], the conjecture was partially solved in some cases, notably when (the quotient ring defined by)  $I_n$  is Artinian for all  $n$ , or when  $\mathcal{I}$  is a *saturated chain*, and  $I_n$  is a squarefree monomial ideal for all  $n$ . The main result of [14] implies that the limit  $\lim_{n \rightarrow \infty} (\text{reg } I_n/n)$  always exists for an arbitrary invariant chain of monomial ideals, supporting the validity of Conjecture 1.1. The conjecture was verified by Murai [16] and Raicu [19] (using different approaches) if  $\mathcal{I} = (I_n)_{n \geq 1}$  is a *Sym-invariant chain* of monomial ideals. On the other hand, Conjecture 1.1 remains open when the chain  $\mathcal{I}$  is not saturated, even if each  $I_n$  is a squarefree monomial ideal.

As usual, an ideal  $I_n \subseteq R_n$  is called an *edge ideal* if it is generated by squarefree quadratic monomials. Related to the theory of invariant chains of edge ideals, algebraic and combinatorial aspects of FI-graphs were studied in [20, 21]. The goal of this paper is to prove the following results.

**Theorem 1.2** (Theorem 7.1) *Let  $\mathcal{I} = (I_n)_{n \geq 1}$  be an Inc-invariant (not necessarily saturated) chain of eventually nonzero edge ideals. Let  $r = \text{ind}(\mathcal{I}) \geq 1$ . Then for all  $n \geq N = N(r)$ , there is an equality  $\text{reg } I_n = \text{reg } I_{n+1}$ , and moreover  $\text{reg } I_n \in \{2, 3\}$ .*

It follows from [13, Corollary 4.8] that if  $\mathcal{I} = (I_n)_{n \geq 1}$  is an Inc-invariant chain of eventually nonzero squarefree monomial ideals, then the sequence  $(\text{reg } I_n)_{n \geq 1}$  is bounded by a constant. On the other hand, there was no information about this constant. Theorem 1.2 provides the surprising information that for an Inc-invariant chain of nonzero edge ideals, not only does the limit  $\lim_{n \rightarrow \infty} \text{reg } I_n$  exist, but it can only be either 2 or 3. We are able to supply an explicit upper bound for the position when the sequence  $(\text{reg } I_n)_{n \geq 1}$  becomes constant, namely the number  $N(r)$  in Theorem 1.2 is at most  $2(r^2 + 5r)$  (see Theorem 7.1 and Remark 7.2). In addition, we provide an effective characterization when  $\text{reg } I_n = 2$  for all large enough  $n$ .

**Theorem 1.3** (Theorem 7.1) *Keep using notations of Theorem 1.2. Let  $G_n$  be the graph on the vertex set  $[n] = \{1, 2, \dots, n\}$  corresponding to  $I_n$  for each  $n$ , and let  $E(G_r) = \{\{i_1, j_1\}, \dots, \{i_s, j_s\}\}$  where we assume that  $1 \leq i_t < j_t \leq r$ ,  $i_1 \leq i_2 \leq \dots \leq i_s$  and if  $i_t = i_{t+1}$  then  $j_t < j_{t+1}$ . Denote  $q = \max\{1 \leq t \leq s : i_t = i_1\}$ . Denote by  $\text{indmatch}(G_n)$  the induced matching number of the graph  $G_n$ . Then, the following statements are equivalent:*

- (1)  $\text{reg } I_n = 2$  for all  $n \gg 0$ ;
- (2) Either  $j_q = \max\{j_1, \dots, j_s\}$ , or it holds that  $\min\{j_t - i_t : t = 1, \dots, s\} = 1$  and  $\text{indmatch}(G_{3r}) = 1$ .

The proofs of Theorems 1.2 and 1.3 are combinatorial and reveal unexpected properties of the graphs  $G_n$ . A fairly simple but crucial observation is Lemma 3.3, which translates questions about the chain  $(I_n)_{n \geq 1}$  and the graphs  $G_n$  to questions about integral points inside certain right isosceles triangles on the coordinate plane. Most of the results in this paper were found thanks to the intuitive approach provided by this observation. Our main tools in proving Theorems 1.2 and 1.3 are two key Lemmas 3.5 and 3.6, which, thanks to Lemma 3.3, are statements about induced  $2K_2$  (i.e. two disjoint edges) subgraphs of  $G_n$  when  $n$  is large. In particular, we show in Theorem 3.1 that for all  $n \geq 3r$ ,  $G_n$  has induced matching number  $\text{indmatch}(G_n) \in \{1, 2\}$  and the equality  $\text{indmatch}(G_n) = \text{indmatch}(G_{n+1})$  holds.

There was a lot of work on the algebraic and combinatorial properties of edge ideals; see, for example, [15] and the references therein. The proofs of our main results also use some known results on this topic, e.g., the celebrated Fröberg theorem characterizing edge ideals with a 2-linear resolution [7]. Nevertheless, our work is more than just routine applications of the existing literature. For example, besides Lemmas 3.5 and 3.6 on induced  $2K_2$  subgraphs of  $G_n$ , a novelty of our approach is given by Construction 5.7, where we construct long induced *anticycles* (i.e., complementary graphs of cycles) of  $G_n$  for a large family of chains. It is also worth mentioning that to

prove the inequality  $\text{reg } I_n \leq 3$  for  $n \gg 0$  in Theorem 1.2, we show that  $G_n$  possesses certain “cochordal filtration”, see the proof of Theorem 6.1. Related to this last point, we note that the well-known inequality  $\text{reg } I_n \geq 1 + \text{indmatch}(G_n)$  is strict in general for an Inc-invariant chain of edge ideals  $(I_n)_{n \geq 1}$ . To summarize, our results suggest that there is an interesting interplay between the algebraic properties of  $I_n$  and the combinatorics of  $G_n$ . We hope that our main results and techniques may shed further light on Conjecture 1.1 and on the combinatorics of Inc-invariant chains of monomial ideals in general. Based on experiments with Macaulay2 [8], we are tempted to propose the following

**Conjecture 1.4** *Let  $\mathcal{I} = (I_n)_{n \geq 1}$  be an Inc-invariant chain of eventually nonzero squarefree monomial ideals with  $r = \text{ind}(\mathcal{I})$ , and let  $d = d(I_r)$  be the maximal generating degree of  $I_r$ . Then,  $\text{reg } I_n \leq 2d - 1$  for all  $n \gg 0$ .*

**Organization.** We discuss some basic results and constructions in Sect. 2. Within this part, Sect. 2.4 illustrates our main results in the special case of saturated chains. Sections 3, 4, and 5 form the technical core of this work. In Sect. 3, we prove that for large  $n$ , the graph  $G_n$  corresponding to the ideal  $I_n$  in any Inc-invariant chain of edge ideals has no induced  $3K_2$  subgraph, and its induced matching number is a constant (either 1 or 2). In Sect. 4, we show that while the complementary graph  $G_n^c$  may contain induced cycles (Example 4.3), the length of such cycles has to be large (Proposition 4.1). In Sect. 5, we construct long induced anticycles of the graph  $G_n$  under some extra assumptions, see Proposition 5.2 and particularly Construction 5.7. This section is technically the most challenging part of the manuscript. On the other hand, in essence Sect. 5 does not depend on the remaining parts, and Proposition 5.2 is the only result in this section that matters later on. For a quick first reading, the reader may simply take Proposition 5.2 for granted and proceed directly to Sect. 6. After the preparations in Sects. 3–5, we derive the more transparent and interesting results on Inc-invariant chains of edge ideals in the rest of this paper. In Sect. 6, we prove that under the hypotheses of Theorem 1.2,  $\text{reg } I_n \leq 3$  for all large enough  $n$ . We then deduce our main results, Theorems 1.2 and 1.3, in Sect. 7.

## 2 Preliminaries

### 2.1 Equivariant chains of ideals

Let  $R_n = \mathbb{k}[x_1, \dots, x_n]$  be a standard graded Noetherian polynomial ring where  $n \geq 1$ , and let  $R_0 = \mathbb{k}$ . Denote by  $G(J)$  the set of minimal monomial generators of a monomial ideal  $J$  of  $R_n$ .

**Definition 2.1** (Maximal index in the support) For a monomial  $m \in R_n$ , denote by  $\text{Msupp}(m)$  the maximal index  $1 \leq i \leq n$  such that  $x_i$  divides  $m$ . If  $m = 1$ , we set  $\text{Msupp}(1) = 0$ . For a monomial ideal  $J \subseteq R_n$ , define  $\text{Msupp}(J) = \max\{\text{Msupp}(m) : m \in G(J)\}$ .

The following slight generalization of saturated chains was introduced in [14].

**Definition 2.2** (Quasi-saturated chains) Let  $\mathcal{I} = (I_n)_{n \geq 1}$  be an Inc-invariant chain of ideals with  $r = \text{ind}(\mathcal{I})$ . Set  $p_n = \text{Msupp}(I_n)$ . We say that  $\mathcal{I}$  is *quasi-saturated* if  $I_{n+1} \cap R_{p_n} = I_n \cap R_{p_n}$  for all  $n \geq r$ .

Any saturated chain is also a quasi-saturated chain. We will invoke the following simple lemma several times. Both assertions are immediate from the fact that for  $n \geq r = \text{ind}(\mathcal{I})$ , it holds that  $\langle \text{Inc}_{n,n+1}(I_n) \rangle_{R_{n+1}} = I_{n+1}$ .

**Lemma 2.3** Let  $\mathcal{I} = (I_n)_{n \geq 1}$  be an Inc-invariant chain of nonzero proper monomial ideals with  $r = \text{ind}(\mathcal{I})$ . Denote  $p = \text{Msupp } I_r$ , so  $1 \leq p \leq r$ . Then we have  $\text{Msupp}(I_{n+1}) \leq \text{Msupp}(I_n) + 1$ , and in particular  $\text{Msupp } I_n \leq n - r + p$  for all  $n \geq r$ .

The following is a consequence of [14, Lemma 4.7 and its proof] by setting the integer  $e = 0$  in its setup.

**Proposition 2.4** Let  $\mathcal{I} = (I_n)_{n \geq 1}$  be an Inc-invariant chain of eventually nonzero proper monomial ideals with  $r = \text{ind}(\mathcal{I})$ . Let  $p = \text{Msupp}(I_r)$ . Consider collection  $\mathcal{J} = (J_n)_{n \geq 1}$  of ideals given by

$$J_n = \begin{cases} \langle I_n \cap R_{n-r+p-1} \rangle_{R_n}, & \text{if } n \geq r + 1, \\ 0, & \text{if } n \leq r. \end{cases}$$

The following statements hold.

- (i)  $\mathcal{J}$  is an Inc-invariant chain of eventually nonzero monomial ideals and  $I_n \subseteq J_{n+1}$  for all  $n \geq r$ .
- (ii) There is an equality  $\text{ind } \mathcal{J} = r + 1$ .
- (iii) The chain  $\mathcal{I}$  is quasi-saturated if and only if  $J_{r+1} = \langle I_r \rangle_{R_{r+1}}$ .
- (iv) There is an equality  $\text{reg } J_n = \text{reg}(I_n, x_{n-r+p})$  for all  $n \geq r + 1$ .

The following invariant is useful for inductive arguments.

**Definition 2.5** Let  $J \subseteq R_n$  be a nonzero proper monomial ideal such that  $\text{Msupp}(J) = p \geq 1$ . Let  $\delta(J)$  denote the maximal degree of an element of  $G(J)$ . Then, the  $q$ -invariant of  $J$  is defined to be the number

$$q(J) = \sum_{j=0}^{\delta(J)} \dim_{\mathbb{k}} \left( \frac{R_p}{J \cap R_p} \right)_j.$$

Let  $\mathcal{I} = (I_n)_{n \geq 1}$  be an Inc-invariant chain with  $r = \text{ind}(\mathcal{I})$ . We call  $q(\mathcal{I}) := q(I_r)$  the  $q$ -invariant of the chain  $\mathcal{I}$ .

If  $J = R_n$ , we use the convention that  $q(J) = 0$ . Hence,  $q(J) = 0$  if and only if  $J = R_n$ .

**Example 2.6** Let  $J = (x_1x_3, x_2x_4) \subseteq R_5 = \mathbb{k}[x_1, \dots, x_5]$ . Then  $\delta(J) = 2$ ,  $\text{Msupp}(J) = 4$ , so

$$q(J) = \sum_{j=0}^2 \dim_{\mathbb{k}} \left( \frac{R_4}{J \cap R_4} \right)_j = 1 + 4 + 8 = 13.$$

The next lemma is immediate.

**Lemma 2.7** *The following statements hold.*

- (1) *For any monomial ideal  $I \subseteq R_n$ , there is an equality  $q(I) = q(\langle I \rangle_{R_{n+1}})$ .*
- (2) *Let  $I \subseteq J$  be nonzero monomial ideals of  $R_n$  such that  $\delta(I) \geq \delta(J)$  and  $\text{Msupp}(I) \geq \text{Msupp}(J)$ . Then there is an inequality*

$$q(I) \geq q(J).$$

*The equality holds if and only if  $I = J$ .*

## 2.2 Castelnuovo–Mumford regularity

Let  $S = \mathbb{k}[x_1, \dots, x_n]$  be a standard graded polynomial ring where  $n \geq 1$ . Let  $M$  be a finitely generated graded  $S$ -module. Then the (*Castelnuovo–Mumford*) regularity of  $M$  is

$$\text{reg } M = \sup\{j - i : \text{Tor}_i^S(\mathbb{k}, M)_j \neq 0\}.$$

The following useful lemma on regularity of monomial ideals is from [5, Lemma 2.10] and [2, Corollary 3.3 and Theorem 4.7].

**Lemma 2.8** *Let  $J$  be a monomial ideal of  $S$  and  $x$  any variable of  $S$ . Then there is a chain*

$$\max\{\text{reg}(J : x), \text{reg}(J, x)\} \leq \text{reg } J \in \{\text{reg}(J : x) + 1, \text{reg}(J, x)\}.$$

*Here  $(J, x)$  denotes the ideal  $J + (x)$ .*

## 2.3 Edge ideals and graphs

Let  $G$  be a simple graph on the vertex set  $[n] = \{1, \dots, n\}$ , and let  $E(G)$  be its edge set. The *edge ideal* of  $G$  is the following ideal of  $R_n$ :

$$I(G) = (x_i x_j \mid \{i, j\} \in E(G)).$$

An *induced subgraph*  $H$  of  $G$  is a graph with vertex set  $V(H) \subseteq V(G)$  and edge set  $E(H) \subseteq E(G)$ , such that two vertices  $i, j \in V(H)$  are adjacent in  $H$  if and only if they are adjacent in  $G$ .

The complementary graph of  $G$  is denoted by  $G^c$ . The *cycle*  $C_m$  is the graph on  $m \geq 3$  vertices  $v_1, \dots, v_m$  with the edge set

$$E(C_m) = \{\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_m, v_1\}\}.$$

We call  $m$  the *length* of the cycle  $C_m$ . An *anticycle* is the complementary graph  $C_m^c$  of the cycle  $C_m$  for some  $m \geq 4$ . By abuse of terminology, we call  $m$  the *length* of

the anticycle  $C_m^c$  and say that an anticycle has *consecutive vertices*  $v_1, v_2, \dots, v_m$  if its complementary graph is the cycle with edges  $v_1v_2, \dots, v_{m-1}v_m, v_mv_1$ .

We say that  $G$  is *chordal* if it has no induced cycle of length at least 4. Dually, we say  $G$  is *cochordal* if  $G^c$  is chordal.

For each  $g \geq 1$ , the graph  $gK_2$  is a union of  $g$  disjoint edges. The *induced matching number* of  $G$  is

$$\text{indmatch}(G) = \max\{g \mid G \text{ has an induced } gK_2 \text{ subgraph}\}.$$

Clearly  $C_4^c$  is isomorphic to  $2K_2$ , therefore if  $G$  is a cochordal graph with at least one edge, then as  $G$  does not contain any induced  $C_4^c$ ,  $\text{indmatch}(G) = 1$ .

The first part of the following lemma is contained in [12, Lemma 2.2] and [23, Lemma 7]. The second part is contained in [7, Theorem 1].

**Lemma 2.9** *Let  $G$  be a graph with at least one edge. The following statements hold.*

- (1) *There is an inequality  $\text{reg } I(G) \geq 1 + \text{indmatch}(G)$ .*
- (2) (Fröberg’s theorem) *We have  $\text{reg } I(G) = 2$  if and only if  $G$  is a cochordal graph.*

For a subset  $U$  of the vertex set  $V(G)$  of  $G$ , the *deletion* of  $G$  to  $U$ , denoted  $G \setminus U$ , has the vertex set  $V(G \setminus U) := V(G) \setminus U$  and the edge set

$$E(G \setminus U) := \{\{x, y\} \in E(G) \mid x, y \in V(G) \setminus U\}.$$

For a vertex  $v \in G$ , its *open neighborhood*  $N_G(v)$  is the set of vertices  $u \neq v$  that are adjacent to  $v$ , and its *closed neighborhood* is

$$N_G[v] := N_G(v) \cup \{v\}.$$

Let  $G$  be a graph on  $[n]$ ,  $v \in V(G)$  a vertex. Let  $S = \mathbb{k}[x_1, \dots, x_n]$ ,  $J$  a monomial ideal of  $S$ , and  $y$  a linear form of  $S$ . We have a standard exact sequence

$$0 \rightarrow \left(\frac{S}{J : y}\right)(-1) \rightarrow \frac{S}{J} \rightarrow \frac{S}{(J, y)} \rightarrow 0.$$

Using this sequence for  $J = I(G)$  and  $y = x_v$ , we may prove the following statement.

**Lemma 2.10** ([5, Lemma 3.1]) *There is an inequality*

$$\text{reg } I(G) \leq \max\{\text{reg } I(G \setminus N_G[v]) + 1, \text{reg } I(G \setminus v)\}.$$

### 2.4 Regularity of saturated chains

The following result together with its simple proof serve as a motivation for Theorems 1.2 and 1.3.

**Lemma 2.11** *Let  $\mathcal{I} = (I_n)_{n \geq 1}$  be a quasi-saturated chain of eventually nonzero edge ideals with  $\text{ind}(\mathcal{I}) = r$ . Then, we have an equality  $\text{reg } I_n = 2$  for all  $n \geq r$ , i.e.,  $I_n$  has a 2-linear resolution for all such  $n$ .*

**Proof** By Fröberg’s theorem (Lemma 2.9(2)), we have to show that the graph  $G_n$  corresponding to  $I_n$  is cochordal for all  $n \geq r$ .

Assume the contrary, that  $i_1, \dots, i_m$  is an induced cycle of length  $m \geq 4$  in  $G_n^c$  for some  $n \geq r$ . By reindexing, we may assume that  $i_1 = \min\{i_j : 1 \leq j \leq m\}$ . Note that  $i_1, \dots, i_m$  are pairwise distinct.

We claim that  $i_2 < \min\{i_3, \dots, i_{m-1}\}$ . If this is not the case, then  $i_2 > i_j$  for some  $3 \leq j \leq m - 1$ . Since  $\{i_1, i_j\} \notin G_n^c$ ,  $x_{i_1}x_{i_j} \in I_n$ , and the Inc-invariance of  $\mathcal{I}$  implies that  $x_{i_1}x_{i_2} \in I_{n+i_2-i_j}$ . Let  $p = \text{Msupp}(I_n)$ . The quasi-saturatedness of  $\mathcal{I}$  implies that  $I_s \cap R_p \subseteq I_n$  for all  $s \geq n$ . Now  $x_{i_1}x_{i_2} \in I_{n+i_2-i_j} \cap R_p \subseteq I_n$ , hence  $\{i_1, i_2\} \in G_n$ , a contradiction. Similarly  $i_m < \min\{i_3, \dots, i_{m-1}\}$ .

Hence,  $\min\{i_2, i_3, \dots, i_m\} = \min\{i_2, i_m\}$ . We assume that  $\min\{i_2, i_3, \dots, i_m\} = i_2$ , the remaining case being similar. Now  $\{i_2, i_m\} \notin G_n^c$ , so  $x_{i_2}x_{i_m} \in I_n$ . From above  $i_m < i_3$ , so arguing as in the last paragraph, we deduce  $x_{i_2}x_{i_3} \in I_n$ . This means  $\{i_2, i_3\} \in G_n$ , a contradiction. Hence  $G_n$  is cochordal for all  $n \geq r$ . The proof is completed.  $\square$

**Example 2.12** The behavior of the regularity for non-quasi-saturated chains of edge ideals is much more delicate. For instance, let  $(I_n)_{n \geq 1}$  be the Inc-invariant chain with  $I_n = (0)$  for  $n \leq 9$ ,  $I_{10} = (x_1x_{10}, x_2x_4, x_3x_5, x_7x_9)$  and stability index  $r = 10$ . Computations with Macaulay2 [8] give the following data.

$n$	10	11	12	13	14	15	16	17	18	19
$\text{reg } I_n$	5	4	3	4	4	3	3	3	3	2

In particular, the sequence  $(\text{reg } I_n)_{n \geq 1}$  might be fairly irregular at the beginning. By Proposition 7.4, we have  $\text{reg } I_n = 2$  for all  $n \geq 3r = 30$ .

### 3 Induced matchings

The goal of this section is to prove

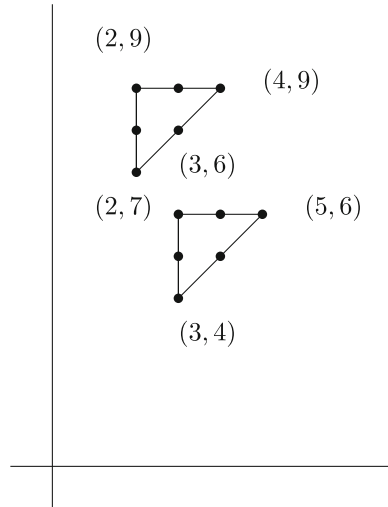
**Theorem 3.1** *Let  $\mathcal{I} = (I_n)_{n \geq 1}$  be an Inc-invariant chain of eventually nonzero edge ideals with the stability index  $\text{ind } \mathcal{I} = r \geq 1$ . Let  $G_n$  be the graph corresponding to  $I_n$ . Then for all  $n \geq 3r$ , the graph  $G_n$  does not contain any induced  $3K_2$  subgraph. In other words, we have  $\text{indmatch}(G_n) \leq 2$  for all  $n \geq 3r$ .*

Furthermore,  $\text{indmatch}(G_n) = \text{indmatch}(G_{n+1})$  for all  $n \geq 3r$ .

The main work is done by two key Lemmas 3.5 and 3.6. In the sequel, we use the following notations.



**Fig. 1** An invariant chain with  $I_7 = (x_2x_7, x_3x_4)$



**Notation 3.2** For a point  $(i, j) \in \mathbb{R}^2$  and a positive real number  $n$ , denote by  $\Delta((i, j), n)$  the right isosceles triangle with the vertices  $(i, j), (i, j+n), (i+n, j+n)$ .

For example, Fig. 1 depicts the triangles  $\Delta((2, 7), 2)$  and  $\Delta((3, 4), 2)$ . Questions about Inc-invariant chains of edge ideals can be translated into questions about integral points in the triangles  $\Delta((i, j), n)$  via the following simple but crucial observation. Below, for  $x, y \in \mathbb{R}$ , denote by  $(x, y)^\leq$  the point  $(\min(x, y), \max(x, y)) \in \mathbb{R}^2$ .

**Lemma 3.3** Let  $1 \leq i < j \leq r$  and  $n \geq r$  be positive integers. Then for integers  $u < v$ , the following are equivalent:

- (1)  $x_u x_v \in \text{Inc}_{r,n}(x_i x_j)$ ;
- (2)  $(u, v) \in \Delta((i, j), n - r)$ .

In particular, let  $\mathcal{I} = (I_n)_{n \geq 1}$  be an Inc-invariant chain of eventually nonzero edge ideals with  $r = \text{ind}(\mathcal{I})$ , and denote  $E(G_r) = \{\{i_1, j_1\}, \dots, \{i_s, j_s\}\}$  where  $1 \leq i_p < j_p \leq r$  for  $p = 1, \dots, s$ . Then for each  $n \geq 0$ ,

$$E(G_{n+r}) = \bigcup_{p=1}^s \{ \{u, v\} : u, v \in \mathbb{Z}, (u, v)^\leq \in \Delta((i_p, j_p), n) \}.$$

**Proof** The second assertion follows from the fact that  $I_{n+r} = \langle \text{Inc}_{r,n+r}(I_r) \rangle_{R_{n+r}}$  and the first assertion. Let us prove the latter.

(1)  $\Rightarrow$  (2): Since  $x_u x_v \in \text{Inc}_{r,n}(x_i x_j)$ , for some  $\pi \in \text{Inc}_{r,n}$ , we have  $u = \pi(i), v = \pi(j)$ . Since  $\pi$  is increasing,  $\pi(r) \leq n$ , and  $1 \leq i < j \leq r$ , we have

$$0 \leq \pi(i) - i \leq \pi(j) - j \leq \pi(r) - r \leq n - r.$$

This implies  $(u, v) \in \Delta((i, j), n - r)$ .

(2)  $\Rightarrow$  (1): Assume that  $(u, v) \in \Delta((i, j), n - r)$ . Then there is a chain

$$0 \leq u - i \leq v - j \leq n - r.$$

Choose  $\pi : \mathbb{N} \rightarrow \mathbb{N}$  such that

$$\pi(t) = \begin{cases} t + u - i, & \text{if } t \leq j - 1, \\ t + v - j, & \text{if } t \geq j. \end{cases}$$

Then  $\pi \in \text{Inc}_{r,n}$ , as  $\pi(r) = r + v - j \leq n$ . It is also clear that  $\pi(i) = u, \pi(j) = v$ , hence  $x_u x_v = \pi(x_i x_j) \in \text{Inc}_{r,n}(x_i x_j)$ , as desired.  $\square$

**Example 3.4** Let  $\mathcal{I}$  be an Inc-invariant chain with  $I_7 = (x_2x_7, x_3x_4)$  and  $I_n = \text{Inc}_{7,n}(I_7)$  for  $n \geq 7$ . Then using Lemma 3.3, we can compute

$$\begin{aligned} I_9 &= \langle \text{Inc}_{7,9}(x_2x_7) \rangle_{R_9} + \langle \text{Inc}_{7,9}(x_3x_4) \rangle_{R_9} \\ &= (x_2x_7, x_2x_8, x_2x_9, x_3x_8, x_3x_9, x_4x_9) + (x_3x_4, x_3x_5, x_3x_6, x_4x_5, x_4x_6, x_5x_6). \end{aligned}$$

See Fig. 1. More generally, for each  $n > 7$ , the generators of the ideal  $I_n$  correspond to the lattices points in the union of the two triangles  $\Delta((2, 7), n - 7)$  and  $\Delta((3, 4), n - 7)$ .

Let  $A_1 = (i_1, j_1), A_2 = (i_2, j_2), A_3 = (i_3, j_3) \in \mathbb{R}_{\geq 1}^2$  be three given points such that  $i_p < j_p$ . For each  $n$ , let  $\Delta_{p,n} = \Delta((i_p, j_p), n)$ .

The first key lemma in this section shows that if  $A_1 = (i_1, j_1), A_2 = (i_2, j_2) \in \mathbb{R}_{\geq 1}^2$  are such that  $i_p < j_p, i_1 \leq i_2, n$  is sufficiently large, and  $(u_p, v_p) \in \Delta_{p,n}, p = 1, 2$  are such that the rectangle with vertices  $(u_i, v_j), 1 \leq i, j \leq 2$  has exactly two vertices in  $\bigcup_{p=1}^2 \Delta_{p,n}$  (namely  $(u_1, v_1)$  and  $(u_2, v_2)$ ), then  $u_1$  has to be small, while  $v_2$  has to be large. In particular,  $(u_1, v_1)$  has to lie to the left of  $\Delta_{2,n}$ , while  $(u_2, v_2)$  has to lie above  $\Delta_{1,n}$  (see Fig. 2).

**Lemma 3.5** Assume that  $A_1 = (i_1, j_1), A_2 = (i_2, j_2) \in \mathbb{R}_{\geq 1}^2$  are (not necessarily distinct) points such that  $i_p < j_p, p = 1, 2$  and  $i_1 \leq i_2$ . Let  $n \geq \max\{2(j_1 - i_1 + i_2 - j_2), 1\}$  be an integer. Let  $(u_p, v_p) \in \Delta_{p,n}, p = 1, 2$  be two points such that  $(u_1, v_2), (u_2, v_1) \notin \bigcup_{p=1}^2 \Delta_{p,n}$ . Then the following inequalities hold:

- (1)  $u_1 < v_1, u_2 < v_2$ ;
- (2)  $u_1 < i_2$  and  $v_2 > j_1 + n$ ;
- (3)  $i_1 < i_2$  and  $j_1 < j_2$ , thus  $A_1$  and  $A_2$  are distinct.

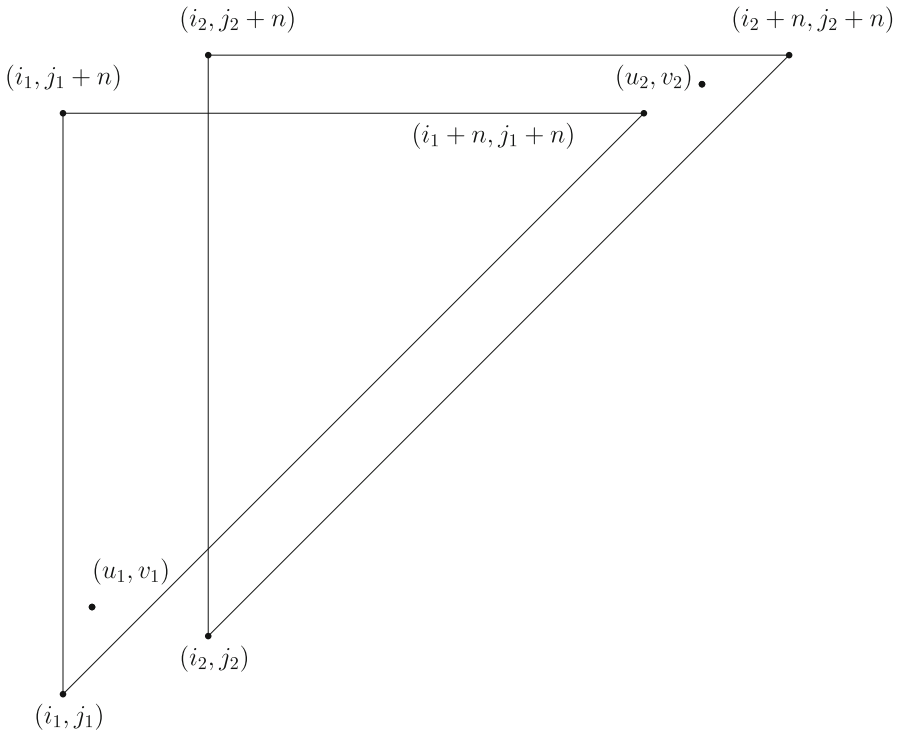
**Proof** Denote  $\Delta_p = \Delta_{p,n}$  for short. Note that

$$\Delta_p = \{(u, v) \in \mathbb{R}^2 : 0 \leq u - i_p \leq v - j_p \leq n\}.$$

For (1): Since  $v_p \geq u_p + j_p - i_p$  and  $j_p > i_p$ , we get  $v_p > u_p$ , namely (1) is true.

For (2): By the hypothesis  $(u_2, v_1) \notin \Delta_1$ , so the following chain is invalid

$$0 \leq u_2 - i_1 \leq v_1 - j_1 \leq n.$$



**Fig. 2** Two points  $(u_1, v_1), (u_2, v_2)$  satisfying the hypotheses of Lemma 3.5

But  $u_2 \geq i_2$ , so  $u_2 - i_1 \geq i_2 - i_1 \geq 0$ , namely the first inequality of the last chain is valid. The last inequality is also valid since  $(u_1, v_1) \in \Delta_1$ . Hence the middle one is false, namely

$$u_2 - i_1 > v_1 - j_1 \geq u_1 - i_1.$$

The last inequality follows from  $(u_1, v_1) \in \Delta_1$ . From the last chain, we get  $u_1 < u_2$ .

By the hypothesis  $(u_1, v_2) \notin \Delta_2$ , so the following chain is invalid

$$0 \leq u_1 - i_2 \leq v_2 - j_2 \leq n.$$

The middle and the last inequalities are true since from  $(u_2, v_2) \in \Delta_2$ , and  $u_1 < u_2$ , we deduce  $u_1 - i_2 < u_2 - i_2 \leq v_2 - j_2 \leq n$ . Hence the first inequality of the last display is false, namely  $u_1 < i_2$ .

Next we have to show that  $v_2 > n + j_1$ . Assume the contrary that  $v_2 \leq n + j_1$ . By the hypothesis,  $(u_1, v_2) \notin \Delta_1$ , hence the following chain is invalid

$$0 \leq u_1 - i_1 \leq v_2 - j_1 \leq n. \tag{3.1}$$

The first inequality holds because  $(u_1, v_1) \in \Delta_1$ . By assumption, the last inequality holds as well. Therefore, the middle inequality does not hold, namely  $v_2 < u_1 - i_1 + j_1 \leq v_1$ , using  $(u_1, v_1) \in \Delta_1$ . Hence as  $u_1 < i_2$ ,

$$v_2 < \min\{v_1, i_2 - i_1 + j_1\}. \quad (3.2)$$

By the hypothesis  $(u_2, v_2) \notin \Delta_2$ , so the following chain is invalid

$$0 \leq u_2 - i_2 \leq v_1 - j_2 \leq n.$$

Thanks to (3.2) and  $(u_2, v_2) \in \Delta_2$ ,  $0 \leq u_2 - i_2 \leq v_2 - j_2 < v_1 - j_2$ . Hence, the first two inequalities in the last display are valid, and the last one is not. In other words,

$$v_1 > n + j_2. \quad (3.3)$$

By the hypothesis,  $(u_2, v_1) \notin \Delta_1$ , so the following chain is invalid

$$0 \leq u_2 - i_1 \leq v_1 - j_1 \leq n.$$

The first and the last inequalities are true, since  $u_2 \geq i_2 \geq i_1$  and  $(u_1, v_1) \in \Delta_1$ . Hence, the middle one is false, namely  $u_2 > v_1 + i_1 - j_1$ . Together with (3.3), this yields

$$u_2 > n + j_2 + i_1 - j_1. \quad (3.4)$$

Now  $(u_2, v_2) \in \Delta_2$ , so  $u_2 \leq v_2 - j_2 + i_2$ . Combining with (3.4) and (3.2), we get

$$n + j_2 + i_1 - j_1 < u_2 \leq v_2 - j_2 + i_2 < 2i_2 - j_2 - i_1 + j_1.$$

This yields

$$n < 2(j_1 - i_1 + i_2 - j_2),$$

contradicting the hypothesis  $n \geq \max\{2(j_1 - i_1 + i_2 - j_2), 1\}$ . Hence, the assumption  $v_2 \leq n + j_1$  is wrong, and  $v_2 > j_1 + n$ , finishing the proof of (2).

For (3): We get  $i_1 \leq u_1 < i_2$  and  $j_1 + n < v_2 \leq j_2 + n$ , consequently  $j_1 < j_2$ . Hence,  $A_1$  and  $A_2$  are distinct. This concludes the proof.  $\square$

The next key lemma of this section shows that if in the situation of Lemma 3.5, fewer points coming from  $(u_1, v_1)$  and  $(u_2, v_2)$  are allowed to be in  $\Delta_{1,n} \cup \Delta_{2,n}$ , then  $(u_1, v_1)$  must be in the lower-left corner of  $\Delta_{1,n}$ , near  $(i_1, j_1)$  and  $(u_2, v_2)$  must be in the upper right corner of  $\Delta_{2,n}$ , near  $(i_2 + n, j_2 + n)$ . Roughly speaking, if  $(I_n)_{n \geq 0}$  in an Inc-invariant chain of edge ideals where  $I_n = I(G_n)$ ,  $u_1 v_1$  and  $u_2 v_2$  form an induced  $2K_2$  subgraph of  $G_n$ , where  $n$  is large and  $u_1 \leq u_2$ ,  $u_p < v_p$  for  $p = 1, 2$ , then  $v_1$  has to be small, while  $u_2$  has to be approximately  $n$  (see Fig. 3).

**Lemma 3.6** Assume that  $A_1 = (i_1, j_1), A_2 = (i_2, j_2) \in \mathbb{R}_{\geq 1}^2$  are points such that  $i_p < j_p, p = 1, 2$  and  $i_1 \leq i_2$ . Let  $n \geq 2 \max\{j_1, j_2\}$  be an integer. Let  $(u_p, v_p) \in \Delta_{p,n}, p = 1, 2$  be two points such that none of the points  $(u_1, u_2), (v_1, v_2), (u_1, v_2), (u_2, v_1)$  belongs to  $\bigcup_{p=1}^2 \Delta_{p,n}$ . Then, the following inequalities hold:

$$v_1 < i_2 < n + j_1 < u_2.$$

In particular, the closed intervals  $[u_1, v_1]$  and  $[u_2, v_2]$  are disjoint.

Conversely, let  $(u_p, v_p) \in \Delta_{p,n}, p = 1, 2$  be two points such that

$$v_1 < i_2 < n + j_1 < u_2.$$

Then none of the points  $(u_i, u_j), (u_i, v_j), (v_j, u_i), (v_i, v_j)$  where  $1 \leq i, j \leq 2, i \neq j$ , belongs to the set  $\bigcup_{p=1}^2 \Delta_{p,n}$ .

**Proof** Let  $\Delta_p = \Delta_{p,n}$ . Since  $(u_1, v_2), (u_2, v_1)$  do not belong to  $\Delta_1 \cup \Delta_2$ , by Lemma 3.5, we have  $u_1 < v_1, u_2 < v_2$ , and

$$u_1 < i_2, \tag{3.5}$$

$$v_2 > j_1 + n. \tag{3.6}$$

Assume the contrary that  $v_1 \geq i_2$ . By the assumption  $(v_1, v_2) \notin \Delta_2$ , so the chain  $0 \leq v_1 - i_2 \leq v_2 - j_2 \leq n$  is invalid. Since the first and the last inequalities hold, the middle one is false. Consequently,

$$v_1 - i_2 > v_2 - j_2 \geq u_2 - i_2,$$

using  $(u_2, v_2) \in \Delta_2$ . Together with (3.6), we deduce

$$v_1 > \max\{u_2, n + j_1 - j_2 + i_2\}. \tag{3.7}$$

By the hypothesis,  $(u_2, v_1) \notin \Delta_1$ , so the following chain is invalid

$$0 \leq u_2 - i_1 \leq v_1 - j_1 \leq n.$$

The first and the last inequalities are true, since  $u_2 \geq i_2 \geq i_1$  and  $(u_1, v_1) \in \Delta_1$ . Hence, the middle one is false, namely

$$u_2 > v_1 + i_1 - j_1.$$

Using (3.7), this yields

$$u_2 > n + i_1 - j_2 + i_2. \tag{3.8}$$

By the hypothesis,  $(u_1, u_2) \notin \Delta_1$ , so the following chain is false

$$0 \leq u_1 - i_1 \leq u_2 - j_1 \leq n.$$

Since  $(u_1, v_1) \in \Delta_1$ , the first inequality in the chain is valid. Per (3.7),  $u_2 < v_1$ , so  $u_2 - j_1 < v_1 - j_1 \leq n$ . Hence in the last chain only the middle inequality can be false, and consequently

$$u_1 > u_2 - j_1 + i_1 > n + 2i_1 - j_1 + i_2 - j_2.$$

The second inequality follows from (3.8).

Per (3.5),  $n + 2i_1 - j_1 + i_2 - j_2 < u_1 < i_2$ , so  $n < j_2 + j_1 - 2i_1 \leq 2 \max\{j_1, j_2\}$ , contradicting the hypothesis on  $n$ . Therefore, the assumption  $v_1 \geq i_2$  is wrong, so  $v_1 < i_2$ .

The last inequality implies  $v_1 < i_2 \leq u_2$ . Now  $0 \leq u_1 - i_1 \leq v_1 - j_1 < u_2 - j_1$ , but  $(u_1, u_2) \notin \Delta_1$ , so the chain  $0 \leq u_1 - i_1 \leq u_2 - j_1 \leq n$  is invalid. Only the last inequality can be wrong, so  $u_2 > n + j_1$ , as desired.

Finally, the remaining inequality  $i_2 < n + j_1$  is clear since  $n + j_1 \geq 2j_2 + j_1 > i_2$ . Hence  $v_1 < i_2 < n + j_1 < u_2$ .

The second assertion holds since  $u_1 < v_1 < u_2 < v_2$ .

Finally, we prove the third assertion. As  $i_p < j_p$ , points in  $\bigcup_{p=1}^2 \Delta_{p,n}$  are of the form  $(x, y)$  with  $x < y$ . Hence using the hypothesis

$$v_1 < i_2 < n + j_1 < u_2,$$

we only need to consider the points  $(u_1, u_2), (u_1, v_2), (v_1, u_2), (v_1, v_2)$ . Each of these points does not lie in  $\Delta_{2,n}$  since  $u_1 < v_1 < i_2$ . Also none of them lies in  $\Delta_{1,n}$  since  $n + j_1 < u_2 < v_2$ . The proof is completed.  $\square$

**Corollary 3.7** *Let  $A_1 = (i_1, j_1), A_2 = (i_2, j_2), A_3 = (i_3, j_3) \in \mathbb{R}_{\geq 1}^2$  be three given points such that  $i_p < j_p, p = 1, 2, 3$ . Assume that  $i_1 \leq i_2 \leq i_3$ . Then, for all  $n \geq 2 \max\{j_1, j_2, j_3\}$  and for every three points  $(u_p, v_p) \in \Delta_{p,n}, p = 1, 2, 3$ , one of the twelve points  $(u_i, u_j), (v_i, v_j), (u_s, v_t)$  (where  $1 \leq i < j \leq 3, 1 \leq s, t \leq 3, s \neq t$ ), belongs to  $\bigcup_{p=1}^3 \Delta_{p,n}$ .*

**Proof** Assume the contrary, that for some  $n \geq 2 \max\{j_1, j_2, j_3\}$ , and some three points  $(u_p, v_p) \in \Delta_{p,n}, p = 1, 2, 3$ , none of the twelve points  $(u_i, u_j), (v_i, v_j), (u_s, v_t)$  (where  $1 \leq i < j \leq 3, 1 \leq s, t \leq 3, s \neq t$ ), belongs to  $\bigcup_{p=1}^3 \Delta_{p,n}$ .

Using Lemma 3.6 for  $(A_1, A_2, (u_1, v_1), (u_2, v_2))$  and  $(A_2, A_3, (u_2, v_2), (u_3, v_3))$ , we get the following inequalities

$$\begin{aligned} u_2 &> n + j_1 \\ u_2 &\leq v_2 < i_3. \end{aligned}$$

But then  $n < i_3 < \max\{j_1, j_2, j_3\}$ . This contradiction finishes the proof.  $\square$

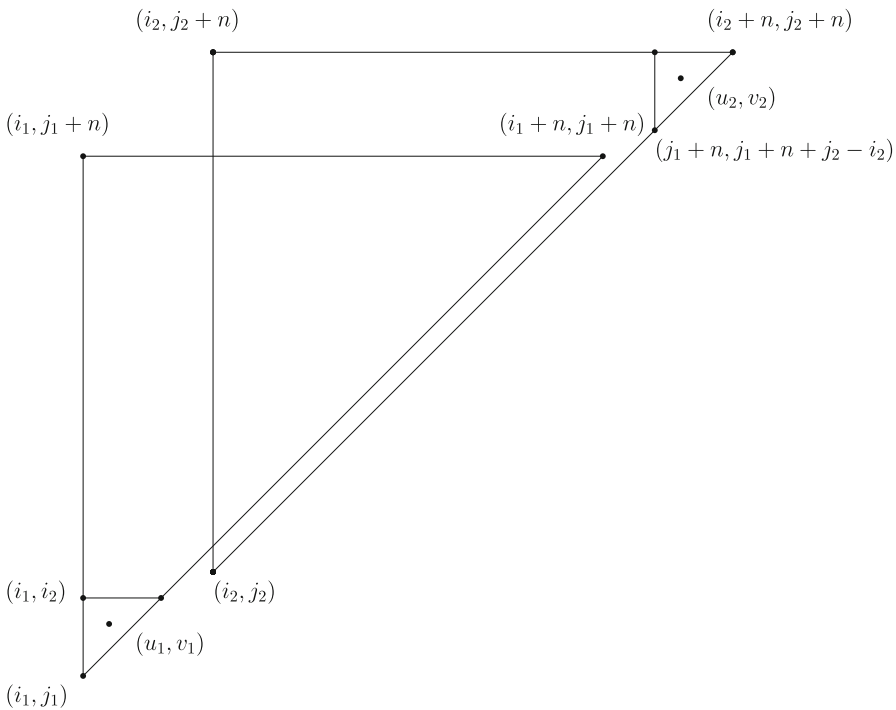


Fig. 3 Two points  $(u_1, v_1), (u_2, v_2)$  satisfying the hypotheses of Lemma 3.6

Now we are ready for the

**Proof of Theorem 3.1** Assume the contrary, for some  $n \geq 3r$ ,  $G_n$  contains an induced  $3K_2$  subgraph, say  $\{u_1v_1, u_2v_2, u_3v_3\}$ , where  $u_p < v_p$ . Since  $r = \text{ind } \mathcal{I}$ , there exist  $1 \leq i_p < j_p \leq r, p = 1, 2, 3$  such that  $x_{i_p}x_{j_p} \in I_r$  and  $x_{u_p}x_{v_p}$  belongs to the Inc-orbit of  $x_{i_p}x_{j_p}$ . Again denote  $\Delta_{p,n-r} = \Delta((i_p, j_p), n-r)$ . By Lemma 3.3, that  $x_{u_p}x_{v_p} \in \text{Inc}_{r,n}(x_{i_p}x_{j_p})$  yields  $(u_p, v_p) \in \Delta_{p,n-r}$ . We may assume  $i_1 \leq i_2 \leq i_3$ .

Now as  $n \geq 3r, n-r \geq 2r \geq 2 \max\{j_1, j_2, j_3\}$ , hence by Corollary 3.7, the  $3K_2$  graph  $\{u_1v_1, u_2v_2, u_3v_3\}$  is not an induced subgraph of  $G_n$ . This contradiction shows that  $G_n$  does not contain any induced  $3K_2$  subgraph for all  $n \geq 3r$ .

Take  $n \geq 2r$ . We prove that  $\text{indmatch}(G_{n+r}) = \text{indmatch}(G_{n+r+1})$ .

Step 1: First assuming  $\text{indmatch}(G_{n+r}) = 2$  and  $n \geq 2r$ , we show the equality  $\text{indmatch}(G_{n+r+1}) = 2$ . Since  $\text{indmatch}(G_{n+r}) = 2$ , there exists an induced  $2K_2$  in  $G_{n+r}$ , say  $u_1v_1, u_2v_2$  where  $u_p < v_p$ . We may assume that  $u_1 < u_2$ . Assume that  $\{i_1, j_1\}, \{i_2, j_2\} \in G_r$  where  $1 \leq i_p < j_p \leq r$  and  $(u_p, v_p) \in \Delta((i_p, j_p), n)$ . Since  $n \geq 2r$ , by Lemma 3.6, we have

$$v_1 < i_2 < n + j_1 < u_2. \tag{3.9}$$

Assume that  $E(G_r) = \{\{i_1, j_1\}, \dots, \{i_s, j_s\}\}$  where  $1 \leq i_p < j_p \leq r$  for  $p = 1, \dots, s$ .

Now  $u_1 v_1, u_2 v_2$  is an induced  $2K_2$  in  $G_{n+r}$ , hence we have

$$(u_1, u_2), (u_1, v_2), (v_1, u_2), (v_1, v_2) \notin \Delta((i_p, j_p), n) \text{ for any } 1 \leq p \leq s. \tag{3.10}$$

**Claim 1:** The conditions (3.10) imply that  $u_1 v_1, (u_2 + 1)(v_2 + 1)$  form a  $2K_2$  in  $G_{n+r+1}$ .

Assume the contrary, then from Lemma 3.3, for some  $1 \leq p \leq s$ , one of the points  $(u_1, u_2 + 1), (u_1, v_2 + 1), (v_1, u_2 + 1), (v_1, v_2 + 1)$  belongs to  $\Delta((i_p, j_p), n + 1)$ . We treat the case  $(u_1, u_2 + 1) \in \Delta((i_p, j_p), n + 1)$ , the remaining cases are similar. The following chain is valid

$$0 \leq u_1 - i_p \leq u_2 + 1 - j_p \leq n + 1.$$

In particular,  $i_p \leq u_1, u_2 \leq n + j_p$ . Per (3.9) and  $n \geq 2r$ , we also have

$$u_2 - u_1 \geq u_2 - v_1 > n + j_1 - i_2 \geq 2r + j_1 - i_2 \geq r > j_p - i_p.$$

Combining these inequalities yields  $(u_1, u_2) \in \Delta((i_p, j_p), n)$ , contradicting (3.10). Hence,  $u_1 v_1, (u_2 + 1)(v_2 + 1)$  form a  $2K_2$  in  $G_{n+r+1}$ , whence the desired conclusion  $\text{indmatch}(G_{n+r+1}) = 2$ .

**Step 2:** Next we show that if  $\text{indmatch}(G_{n+r}) = 2$  and  $n \geq 2r + 1$ , then also  $\text{indmatch}(G_{n+r-1}) = 2$ .

Choose  $u_1 v_1, u_2 v_2, i_1, j_1, i_2, j_2$  as in Step 1. We claim that  $u_1 v_1, (u_2 - 1)(v_2 - 1)$  form a  $2K_2$  in  $G_{n+r-1}$ .

Since  $(u_1, v_1) \in \Delta((i_1, j_1), n)$ , we get  $(u_1, v_1) \in \Delta((i_1, j_1), v_1 - j_1)$ . But by (3.9),  $v_1 - j_1 < i_2 - j_1 < r - 1$ , so as  $n \geq 2r + 1$ ,  $(u_1, v_1) \in \Delta((i_1, j_1), n - 1)$ , namely  $u_1 v_1 \in E(G_{n+r-1})$ . As per (3.9),

$$u_2 - 1 > n + j_1 - 1 \geq 2r + j_1 > i_2,$$

we deduce  $(u_2 - 1, v_2 - 1) \in \Delta((i_2, j_2), n - 1)$ , so that  $(u_2 - 1)(v_2 - 1) \in E(G_{n+r-1})$ . Since  $n \geq 2r + 1$  and  $i_2 \leq r$ , together with (3.9),

$$v_1 < i_2 < n - 1 + j_1 < u_2 - 1.$$

Assume now that  $u_1 v_1$  and  $(u_2 - 1)(v_2 - 1)$  do not form a  $2K_2$  in  $G_{n+r-1}$ . Then as  $u_1 < v_1 < u_2 - 1 < v_2 - 1$ , for some  $1 \leq p \leq s$ ,  $\Delta((i_p, j_p), n - 1)$  contains one of the points  $(u_1, u_2 - 1), (u_1, v_2 - 1), (v_1, u_2 - 1), (v_1, v_2 - 1)$ . We treat the case  $(u_1, u_2 - 1) \in \Delta((i_p, j_p), n - 1)$ , the remaining cases are similar.

We have a chain

$$0 \leq u_1 - i_p \leq u_2 - 1 - j_p \leq n - 1,$$

which clearly implies

$$0 \leq u_1 - i_p \leq u_2 - j_p \leq n,$$



namely  $(u_1, u_2) \in \Delta((i_p, j_p), n)$ . This contradicts (3.10).

Hence,  $u_1 v_1, (u_2 - 1)(v_2 - 1)$  form a  $2K_2$  in  $G_{n+r-1}$  and  $\text{indmatch}(G_{n+r-1}) = 2$ , as desired.

From Step 1 and Step 2, for any  $n \geq 2r$ , the equality  $\text{indmatch}(G_{n+r}) = 2$  holds if and only if  $\text{indmatch}(G_{n+r+1}) = 2$ . Since  $I_n$  is a nonzero edge ideal,  $1 \leq \text{indmatch}(G_{n+r}) \leq 2$ , for all  $n \geq 2r$ , and we deduce that  $\text{indmatch}(G_{n+r}) = \text{indmatch}(G_{n+r+1})$  for all such  $n$ . The proof is concluded.  $\square$

### 4 Induced anticycles

Our main goal in this section is to show that  $G_n$  does not contain short induced anticycles of length at least 5 for all sufficiently large  $n$ .

**Proposition 4.1** *Let  $\mathcal{I} = (I_n)_{n \geq 1}$  be an Inc-invariant chain of eventually nonzero edge ideals with the stability index  $\text{ind } \mathcal{I} = r \geq 1$ . Let  $G_n$  be the graph corresponding to  $I_n$ . Let  $m \geq 5$  be an integer. Then for all  $n \geq mr$ , the complement graph  $G_n^c$  of  $G_n$  has no induced cycle  $C_m$ .*

*In particular, for all  $n \geq 5r$ , the complement graph  $G_n^c$  of  $G_n$  has no induced cycle  $C_m$  for any integer  $m$  such that  $5 \leq m \leq \frac{n}{r}$ .*

The main ingredients to the proof are Lemma 3.5 and the following lemma.

**Lemma 4.2** *With usual notations, let  $n \geq 2r$  be an integer. Let  $H$  be an induced subgraph of  $G_{n+r}$  such that  $H$  contains an induced anticycle  $C_m^c$  (where  $m \geq 5$ ) whose consecutive vertices are labeled as  $a_1, \dots, a_m \in \{1, \dots, n + r\}$  such that  $a_1 = \min\{a_1, \dots, a_m\}$ . Then  $\max\{a_1, a_2, \dots, a_m\} = \max\{a_2, a_m\}$ .*

**Proof** Again, denote  $E(G_r) = \{\{i_1, j_1\}, \dots, \{i_s, j_s\}\}$  where  $1 \leq i_p < j_p \leq r$  for  $p = 1, \dots, s$ . Denote  $\Delta_p := \Delta_{p,n} = \Delta((i_p, j_p), n)$  for  $1 \leq p \leq s$ .

It suffices to show that if  $a_2 \leq a_m$ , then  $a_m = \max\{a_1, a_2, \dots, a_m\}$ . Indeed, for the case  $a_m < a_2$ , reindex the vertices as follows:  $b_1 = a_1, b_i = a_{m+2-i}$  for  $2 \leq i \leq m$ . Then  $b_1 = \min\{b_1, b_2, \dots, b_m\}$  and  $b_2 = a_m < a_2 = b_m$ . Thus we may assume that  $a_2 \leq a_m$ , and we show that  $a_m = \max\{a_1, a_2, \dots, a_m\}$  (Fig. 4).

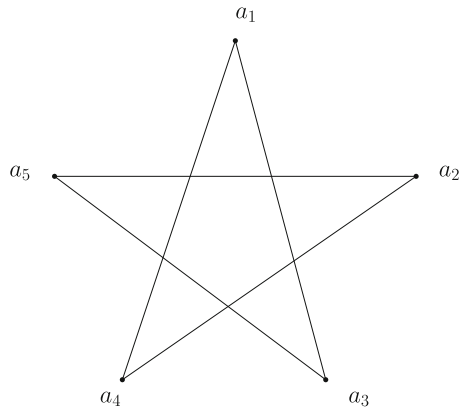
By the definition of the stability index, for each pair of integers  $(p, q)$  such that  $1 \leq p, q \leq m, q - p \geq 2$  and  $(p, q) \neq (1, m)$ , there exists  $(i_{pq}, j_{pq}) \in G_r$  where  $1 \leq i_{pq} < j_{pq} \leq r$  such that  $(a_p, a_q) \in \Delta((i_{pq}, j_{pq}), n)$ .

For simplicity, denote  $\Delta_{pq} = \Delta((i_{pq}, j_{pq}), n)$ . We let  $a_{m+1} = a_1, A_{pq} = (i_{pq}, j_{pq})$  for  $1 \leq p < q \leq m, q \geq p + 2, (p, q) \neq (1, m)$ .

**Claim 1:** For all  $1 \leq p \leq m$ , none of the points  $(a_p, a_{p+1}), (a_{p+1}, a_p)$  belongs to

$$\Gamma = \bigcup_{\substack{1 \leq p < q \leq m \\ q \geq p+2, (p,q) \neq (1,m)}} \Delta_{pq}.$$

Fig. 4 An induced anticycle  $C_5^c$



*Proof of Claim 1:* First assume that for some  $1 \leq p \leq m$ , we have

$$(a_p, a_{p+1}) \in \Gamma \subseteq \bigcup_{t=1}^s \Delta_t.$$

Therefore, there exists an integer  $t \in \{1, \dots, s\}$  such that  $(a_p, a_{p+1}) \in \Delta_t$ . Since  $i_t < j_t, a_p < a_{p+1}$ . Now

$$\{a_p, a_{p+1}\} \in E(G_{n+r}),$$

and  $H$  is an induced subgraph of  $G_{n+r}$ , thus  $\{a_p, a_{p+1}\} \in E(H)$ , a contradiction. Hence  $(a_p, a_{p+1}) \notin \Gamma$ . Similarly for  $(a_{p+1}, a_p)$ , we finish the proof of Claim 1.

Now consider the two points  $(a_1, a_3) = (a_1, a_3)^{\leq} \in \Delta_{13}, (a_2, a_m) = (a_2, a_m)^{\leq} \in \Delta_{2m}$ .

Observe that  $i_{13} < i_{2m}$ . Indeed, assume the contrary  $i_{2m} \leq i_{13}$ . By Claim 1,  $(a_2, a_3), (a_1, a_m) \notin \Delta_{2m} \cup \Delta_{13}$ . Note that  $n \geq 2r \geq 2 \max\{j_{2m}, j_{13}\}$ . Hence using Lemma 3.5 with  $(A_{2m}, A_{13}, (a_2, a_m), (a_1, a_3))$  in place of  $(A_1, A_2, (u_1, v_1), (u_2, v_2))$ , we get

$$a_2 < i_{13} \leq a_1,$$

contradicting the minimality of  $a_1$ .

This implies that  $i_{13} < i_{2m}$ . The last inequality puts us in a position to apply Lemma 3.5 for the data  $(A_{13}, A_{2m}, (a_1, a_3), (a_2, a_m))$ , as by Claim 1, it holds that  $(a_1, a_m), (a_2, a_3) \notin \Delta_{13} \cup \Delta_{2m}$ . The results are the following inequalities

$$a_3 \leq j_{13} + n < a_m.$$

In particular,  $a_3 < a_m$ . Continuing this argument inductively by looking at the points  $(a_1, a_{i+1})$  and  $(a_i, a_m)$  for  $3 \leq i \leq m - 2$ , we get  $a_m = \max\{a_p : 1 \leq p \leq m\}$ , as desired.  $\square$

We are ready now for the

**Proof of Proposition 4.1** It suffices to prove the first assertion.

Assume the contrary that there exists  $n \geq (m - 1)r$  such that  $G_{n+r}^c$  contains an induced cycle  $a_1, a_2, \dots, a_m, a_1$  where  $1 \leq a_i \leq n + r, a_i \neq a_j$  for  $i \neq j$ . We may assume that  $a_1 = \min\{a_1, a_2, \dots, a_m\}$ .

Hence  $(a_p, a_q) \in G_{n+r}$  for all  $1 \leq p < q \leq m$  such that  $q - p \geq 2$  and  $(p, q) \neq (1, m)$ . By the definition of the stability index, for each pair of integers  $(p, q)$  such that  $1 \leq p, q \leq m, q - p \geq 2$  and  $(p, q) \neq (1, m)$ , there exists  $(i_{pq}, j_{pq}) \in G_r$  where  $1 \leq i_{pq} < j_{pq} \leq r$  such that  $(a_p, a_q)^{\leq} \in \Delta((i_{pq}, j_{pq}), n)$ . As in the proof of Lemma 4.2, we have

**Claim 1:** For all  $1 \leq p \leq m$ , none of the points  $(a_p, a_{p+1}), (a_{p+1}, a_p)$  belongs to

$$\Gamma = \bigcup_{\substack{1 \leq p < q \leq m \\ q \geq p+2, (p,q) \neq (1,m)}} \Delta_{pq}.$$

For  $2 \leq p \leq m - 2$ , note that  $(a_p, a_{p+1}) \notin \Delta_{1(p+1)}$  by Claim 1. Hence the following chain is false

$$0 \leq a_p - i_{1(p+1)} \leq a_{p+1} - j_{1(p+1)} \leq n.$$

Since  $a_p \geq a_1$  and  $(a_1, a_{p+1}) \in \Delta_{1(p+1)}$ , the first inequality is true. The last inequality is also true, so the middle one is false. Equivalently, we get

$$a_{p+1} - a_p < j_{1(p+1)} - i_{1(p+1)} \quad \text{for } 2 \leq p \leq m - 2. \tag{4.1}$$

For  $3 \leq p \leq m - 1$ , note that  $(a_{p+1}, a_p) \notin \Delta_{1p}$  by Claim 1. Therefore, the following chain is invalid

$$0 \leq a_{p+1} - i_{1p} \leq a_p - j_{1p} \leq n.$$

As  $a_{p+1} \geq a_1$  and  $(a_1, a_p) \in \Delta_{1p}$ , the first and last inequalities are valid, and hence, the middle one is not. Thus

$$a_p - a_{p+1} < j_{1p} - i_{1p} \quad \text{for } 3 \leq p \leq m - 1. \tag{4.2}$$

Applying Lemma 3.5 with the data  $(A_{13}, A_{2m}, (a_1, a_3), (a_2, a_m))$  as in the proof of Lemma 4.2, we get

$$a_1 < i_{2m} \leq a_2, \tag{4.3}$$

$$a_3 \leq j_{13} + n < a_m. \tag{4.4}$$

We may reduce to the case where  $a_2 \leq a_m$  as explained in the proof of Lemma 4.2. In that situation,  $a_m = \max\{a_1, \dots, a_m\}$  by Lemma 4.2. We consider the following two cases.

**Case 1:**  $a_2 \leq a_{m-1}$ . By Claim 1,  $(a_1, a_2) \notin \Delta_{1(m-1)}$  hence the following chain is false

$$0 \leq a_1 - i_{1(m-1)} \leq a_2 - j_{1(m-1)} \leq n.$$

The first inequality in the chain holds since  $(a_1, a_{m-1}) \in \Delta_{1(m-1)}$ , and the last inequality also holds since  $a_2 \leq a_{m-1} \leq n + j_{1(m-1)}$ . Hence the middle inequality is false, namely

$$a_2 - a_1 < j_{1(m-1)} - i_{1(m-1)}. \quad (4.5)$$

Now  $(a_{m-1}, a_m) \notin \Delta_{2m}$  by Claim 1 so the following chain is false

$$0 \leq a_{m-1} - i_{2m} \leq a_m - j_{2m} \leq n.$$

Since  $(a_2, a_m) \in \Delta_{2m}$  and  $a_2 \leq a_{m-1}$  by the hypothesis of Case 1, we see that the first and last inequalities of the last chain are true. Thus the middle one is false, namely

$$a_m - a_{m-1} < j_{2m} - i_{2m}. \quad (4.6)$$

Using (4.5), (4.1) multiple times, and (4.6), we get

$$\begin{aligned} a_m - a_1 &< (j_{1(m-1)} - i_{1(m-1)}) + \sum_{p=2}^{m-2} (j_{1(p+1)} - i_{1(p+1)}) + (j_{2m} - i_{2m}) \\ &< (m-2) \max\{j_{pq} : 1 \leq p < q \leq m, q \geq p+2, (p, q) \neq (1, m)\} \\ &\leq (m-2)r. \end{aligned}$$

Together with (4.3) and (4.4), we get

$$n + j_{13} - i_{2m} < a_m - a_1 < (m-2)r,$$

so  $n < (m-2)r + i_{2m} \leq (m-1)r$ . This contradicts the hypothesis  $n \geq (m-1)r$  from above.

**Case 2:**  $a_2 > a_{m-1}$ .

Consider the points  $(a_{m-1}, a_2) \in \Delta_{2(m-1)}$  and  $(a_3, a_m) \in \Delta_{3m}$ . Since none of the points  $(a_3, a_2)$ ,  $(a_{m-1}, a_m)$  belongs to  $\Gamma$ , by Lemma 3.5 and the fact that  $a_m = \max\{a_i : 1 \leq i \leq m\}$ , we get  $i_{2(m-1)} < i_{3m}$ , and

$$a_{m-1} < i_{3m} \leq a_3. \quad (4.7)$$

Consider the points  $(a_1, a_{m-2}) \in \Delta_{1(m-2)}$  and  $(a_{m-1}, a_2) \in \Delta_{2(m-1)}$ . Arguing as above using Lemma 3.5 and the fact that  $a_1 \leq a_{m-1}$ , we get

$$a_2 > n + j_{1(m-2)} \geq a_{m-2}. \quad (4.8)$$

We have  $(a_3, a_2) \notin \Delta_{2(m-1)}$  per Claim 1, so the following chain is invalid

$$0 \leq a_3 - i_{2(m-1)} \leq a_2 - j_{2(m-1)} \leq n.$$

Thanks to (4.7),  $a_3 > a_{m-1} \geq i_{2(m-1)}$  and  $a_2 - j_{2(m-1)} \leq n$ , the middle inequality is false, namely

$$a_2 - a_3 < j_{2(m-1)} - i_{2(m-1)}. \tag{4.9}$$

Combining (4.7), (4.8), (4.2), and (4.9), we obtain

$$\begin{aligned} n + j_{1(m-2)} - i_{3m} &< a_2 - a_{m-1} \\ &= (a_2 - a_3) + \sum_{p=3}^{m-2} (a_p - a_{p+1}) \\ &< j_{2(m-1)} - i_{2(m-1)} + \sum_{p=3}^{m-2} (j_{1p} - i_{1p}) \quad (\text{by (4.9) and (4.2)}) \\ &< (m - 3) \max\{j_{pq} : 1 \leq p < q \leq m, q \geq p + 2, (p, q) \neq (1, m)\} \\ &\leq (m - 3)r. \end{aligned}$$

Hence  $n < (m - 3)r + i_{3m} \leq (m - 2)r$ , a contradiction. This finishes the proof.  $\square$

Recall that a graph  $G$  is *weakly chordal* if neither  $G$  nor  $G^c$  contains an induced cycle of length at least 5. The following example shows that one cannot improve the conclusion of Proposition 4.1 in general.

**Example 4.3** In general, it may happen that the graph  $G_n$  corresponding to  $I_n$  is not weakly chordal for all large  $n$ .

Take  $I_4 = (x_1x_3, x_2x_4)$  and  $I_n = \text{Inc}_{4,n}(I_4)$  for all  $n \geq 4$ . Explicitly, for all  $n \geq 4$ ,

$$I_n = (x_i x_j : 1 \leq i < j \leq n, j - i \geq 2, (i, j) \neq (1, n)).$$

For each  $n \geq 4$ ,  $G_n^c$  contains the induced cycle with vertices  $1, 2, 3, \dots, n$ , of length  $n$ , as one may easily check.

In particular, for all  $n \geq 5$ ,  $G_n$  is not weakly chordal. Moreover for each  $n \geq 6$ ,  $\text{reg } I_n \geq \text{reg } I(C_n^c) = 3$ . Together with Theorem 6.1 below, we conclude that  $\text{reg } I_n = 3$  for all  $n \geq 6$ .

On the other hand, using Lemma 3.6 for  $(i_1, j_1) = (1, 3), (i_2, j_2) = (2, 4)$ , we deduce that  $G_{n+4}$  has no induced  $2K_2$  for all  $n \geq 2 \max\{3, 4\} = 8$ . In other words,  $\text{indmatch}(G_n) = 1$  for all  $n \geq 12$ . Therefore, we get  $\text{reg}(I_n) = 3 > 2 = \text{indmatch}(G_n) + 1$  for all  $n \geq 12$ .

## 5 Long induced anticycles

In this section, we fix the following notations.

**Notation 5.1** Let  $\mathcal{I} = (I_n)_{n \geq 1}$  be an Inc-invariant chain of eventually nonzero edge ideals. Let  $r = \text{ind}(\mathcal{I})$  be the stability index of the chain. Denote  $E(G_r) = \{\{i_1, j_1\}, \dots, \{i_s, j_s\}\}$  where  $1 \leq i_t < j_t \leq r$ ,  $i_1 \leq i_2 \leq \dots \leq i_s$  and if  $i_t = i_{t+1}$  then  $j_t < j_{t+1}$ . Denote  $q = \max\{1 \leq t \leq s : i_t = i_1\}$ .

We have seen in Proposition 4.1 that  $G_n$  does not contain *short* induced anticycles, namely anticycle of the form  $C_m^c$ , for  $5 \leq m \leq \frac{n}{r}$ . The following result provides the complementary information, revealing that  $G_n$  does contain a *long* induced anticycles of length  $\geq \left\lceil \frac{n}{r-2} \right\rceil$  under some extra condition on the edges of  $G_r$ . This result is important later in Sect. 7 in the characterization of the case that  $\text{reg } I_n = 2$  for all  $n \gg 0$ .

**Proposition 5.2** *Keep using Notation 5.1. Assume  $\max\{j_1, \dots, j_s\} = j_q + 1$  and  $\min\{j_t - i_t : t = 1, \dots, s\} \geq 2$ . Then for all  $n \geq 3r$ ,  $G_n$  contains an induced anticycle of length at least  $\left\lceil \frac{n}{r-2} \right\rceil$ . In particular, there is an inequality  $\text{reg}(I_n) \geq 3$ .*

The significance of the two hypotheses  $\max\{j_1, \dots, j_s\} = j_q + 1$  and  $\min\{j_t - i_t : t = 1, \dots, s\} \geq 2$  will be apparent in the equivalence between statements (1) and (3) of Theorem 7.1. Note that the second hypothesis implies that  $r \geq 3$ , as  $1 \leq i_t < j_t \leq r$  for each  $1 \leq t \leq s$ .

The most difficult part of the proof is certain constructions of long induced anticycles in the graph  $G_n$ , specifically Construction 5.7. The following notations are used throughout the proof and the construction.

**Notation 5.3** For simplicity, denote  $\Delta_t = \Delta((i_t, j_t), n)$  for any  $t = 1, \dots, s$ . Set  $J = \{1, 2, \dots, s\}$ ,  $J_1 = \{t \in J \mid j_t - i_t = \min\{j_1 - i_1, \dots, j_s - i_s\}\}$ . Denote

$$b = \min\{t \mid j_t = \max\{j_1, \dots, j_s\}\}, \quad B = \max\{t \mid j_t = \max\{j_1, \dots, j_s\}\}, \\ h = \min J_1, \quad H = \max J_1.$$

The following observation is useful to clarify the sets and indices in Notation 5.3.

**Lemma 5.4** *With Notation 5.3, we have  $j_H \leq j_B$ . Moreover, the following statements are equivalent:*

- (1)  $B = H$ ;
- (2)  $B \in J_1$ ;
- (3)  $j_H = j_B$ .

**Proof** Clearly,  $j_H \leq j_B = \max\{j_1, \dots, j_s\}$ . Since  $H = \max J_1$ , it is also clear that (1)  $\implies$  (2). We prove that (2)  $\implies$  (3) and (3)  $\implies$  (1).

(2)  $\implies$  (3): Assume that  $B \in J_1$ . Then  $B \leq H = \max J_1$  and  $j_B - i_B = j_H - i_H$ . From Notation 5.1,  $j_H - j_B = i_H - i_B \geq 0$  as  $H \geq B$ . But the definition of  $B$  yields  $j_H \leq j_B$ , hence  $j_H = j_B$ . Thus (3) is true.

(3)  $\implies$  (1): Assume that  $j_H = j_B$ . The definition of  $B$  implies  $H \leq B$ , so  $i_H \leq i_B$ . In particular,  $j_H - i_H \geq j_B - i_B$ . But  $j_H - i_H = \min\{j_1 - i_1, \dots, j_s - i_s\}$ , so  $j_H - i_H = j_B - i_B$ , and hence  $i_H = i_B$ . The only situation when  $i_H = i_B$  and  $j_H = j_B$  are both valid is  $B = H$ . Hence (1) is true. The proof is concluded.  $\square$

The key step in the proof of Proposition 5.2 is accomplished by the following constructions (5.5, 5.6 and 5.7). The last construction employs the first two, and we clarify all of them in Example 5.8.

**Construction 5.5** Keep using Notation 5.3. In this construction, we assume that  $i_b \leq i_h$ .

Note that in this case  $i_1 = i_q < i_b$ , since  $j_q = \max\{j_t : i_t = i_1\} < \max\{j_t : t = 1, \dots, s\} = j_b$ . Hence  $q \in J \setminus J_1$ . Indeed, if not, then  $j_q - i_q = \min\{j_t - i_t : 1 \leq t \leq s\}$ . This implies  $q = 1$ , and  $h = \min\{t : j_t - i_t = \min\{j_1 - i_1, \dots, j_s - i_s\}\} = 1$ , and by the assumption,  $i_b \leq i_h = i_1$ , a contradiction. Thus  $q \in J \setminus J_1$  and  $i_q < i_b \leq i_h$ . This justifies the first step of Algorithm 1, which is useful for sorting the triangles  $\Delta_p$ .

---

**Algorithm 1** Rearrange the positions of the triangles  $\Delta_p$  with  $i_b \leq i_p \leq i_h$

---

**Input:** The set of edges  $\{\{i_1, j_1\}, \dots, \{i_s, j_s\}\}$  of  $G_r$  given that  $i_b \leq i_h$ .

**Output:** A sequence of sets  $J_1, \dots, J_\beta$  and a sequence of integers  $u_1, \dots, u_\beta \in \{1, \dots, s\}$ .

```

t := 1
u1 := h(= min J1)
while Jt ≠ ∅ and iut ≥ ib do
    Jt+1 := {p | jp - ip = min{ja - ia | a ∈ J \ ∪ti=1 Ji and ia < iut}
    ut+1 := min Jt+1
    t := t + 1
end while
    
```

---

Apply Algorithm 1 and assume that  $(\beta - 1)$  is the number of repeating steps of this algorithm. By what said above,  $\beta \geq 2$ , as  $J_2 \neq \emptyset$ . The algorithm gives

$$J_\beta = \{p \mid j_p - i_p = \min\{j_a - i_a \mid a \in J \setminus \bigcup_{t=1}^{\beta-1} J_t \text{ and } i_a < i_{u_{\beta-1}}\}\},$$

and  $u_\beta = \min J_\beta$ . Note that  $q \in J \setminus \bigcup_{t=1}^{\beta-1} J_t$  since  $i_q < i_b$  and  $i_t \geq i_b$  for any  $t \in J_1 \cup \dots \cup J_{\beta-1}$ . Hence  $J_\beta \neq \emptyset$ , so because the algorithm stops, we get  $i_{u_\beta} < i_b$ .

It follows that we obtain a sequence  $h = u_1, u_2, \dots, u_\beta \in J$  such that

$$i_{u_\beta} < i_b \leq i_{u_{\beta-1}} < \dots < i_{u_2} < i_{u_1} = i_h,$$

$$j_{u_{t+1}} - i_{u_{t+1}} > j_{u_t} - i_{u_t} \geq 2 \quad \text{for } t = 1, \dots, \beta - 1.$$

**Construction 5.6** Keep using Notation 5.3. The following algorithm is used to rearrange the positions of the triangles  $\Delta_p$ .

In the case  $j_H = j_B$ , note that  $H = B$  by Lemma 5.4, and therefore Algorithm 2 does nothing. In other words,  $K_1 = J_1$ ,  $\gamma = 1$ , and  $v_1 = H = B$  in this case.

**Algorithm 2** Rearrange the positions of the triangles  $\Delta_p$  with  $j_H < j_p \leq j_B$

**Input:** The set of edges  $\{\{i_1, j_1\}, \dots, \{i_s, j_s\}\}$  of  $G_r$ .

**Output:** A sequence of sets  $J_1 = K_1, K_2, \dots, K_\gamma$  and a sequence of integers  $v_1, \dots, v_\gamma \in \{1, \dots, s\}$ .

```

 $K_1 := J_1$ 
 $t := 1$ 
 $v_1 := H (= \max K_1)$ 
while  $K_t \neq \emptyset$  and  $j_{v_t} < j_B$  do
     $K_{t+1} := \{p \mid j_p - i_p = \min\{j_a - i_a \mid a \in J \setminus \cup_{i=1}^t K_i \text{ and } j_a > j_{v_t}\}\}$ 
     $v_{t+1} := \max K_{t+1}$ 
     $t := t + 1$ 
end while
    
```

Consider the case  $j_H < j_B$ . Apply Algorithm 2 and assume that  $(\gamma - 1)$  is the number of repeating steps of this algorithm. Thanks to Lemma 5.4,  $B \notin K_1 = J_1$  and  $j_{v_1} = j_H < j_B$ . So  $K_2 \neq \emptyset$  and  $\gamma \geq 2$ .

**Claim 1:** If  $B \in K_t$  for some  $2 \leq t \leq \gamma$ , then  $B = \max K_t = v_t$ .

Indeed, since  $B \in K_t$ ,  $B \leq \max K_t = v_t$ . From the setting of Notation 5.1,  $i_B \leq i_{v_t}$ . Since  $B \in K_t$ ,  $j_B - i_B = j_{v_t} - i_{v_t}$ , so  $j_{v_t} - j_B = i_{v_t} - i_B \geq 0$ . The definition of  $B$  implies  $j_{v_t} - j_B \leq 0$ , so  $0 = j_{v_t} - j_B = i_{v_t} - i_B$ . This forces  $B = v_t$ .

**Claim 2:** It holds that  $B \in J \setminus \cup_{t=1}^{\gamma-1} K_t$ . In particular,  $B \in K_\gamma$  and

$$v_\gamma = B. \tag{5.1}$$

Indeed, we have seen that  $B \notin K_1$ . Assume that  $B \in K_t$  for some  $2 \leq t \leq \gamma - 1$ , then  $B = v_t$  by Claim 1. But then  $j_{v_t} = j_B$ , contradicting the description of Algorithm 2. Hence  $B \in J \setminus \cup_{t=1}^{\gamma-1} K_t$ . The description of the algorithm yields  $j_{v_{\gamma-1}} < j_B$ , so  $K_\gamma \neq \emptyset$ . Since the algorithm stops, we deduce  $j_{v_\gamma} = j_B$ . The definition of  $B$  in Notation 5.3 yields  $v_\gamma \leq B$ , so  $i_{v_\gamma} \leq i_B$ . In particular,  $j_B - i_B \leq j_{v_\gamma} - i_{v_\gamma}$ , so the definition of  $K_\gamma$ , that reads

$$K_\gamma = \{p \mid j_p - i_p = \min\{j_a - i_a \mid a \in J \setminus \cup_{i=1}^{\gamma-1} K_i \text{ and } j_a > j_{v_{\gamma-1}}\}\}$$

implies that  $B \in K_\gamma$ . Using Claim 1, we conclude that  $B = \max K_\gamma = v_\gamma$ .

So at the end of Algorithm 2, we obtain a sequence of indices  $H = v_1, v_2, \dots, v_\gamma = B$  such that

$$j_B = j_{v_\gamma} > \dots > j_{v_2} > j_{v_1} = j_H,$$

$$j_{v_{t+1}} - i_{v_{t+1}} > j_{v_t} - i_{v_t} \geq 2 \quad \text{for all } t = 1, \dots, \gamma - 1.$$

**Construction 5.7** Keep using Notation 5.3. Let  $h = u_1, u_2, \dots, u_\beta$  be as in Construction 5.5 and  $H = v_1, v_2, \dots, v_\gamma = B$  be as in Construction 5.6 (recalling Equation (5.1)). Now for  $n \geq 2r$ , we construct an induced anticycle  $C_m^c$  in  $G_{n+r}$  by considering the following two cases.



**Case I:**  $i_b \leq i_h$ . We employ Algorithm 3 to construct the initial vertices of  $C_m^c$  in  $G_{n+r}$ . Note that the numbers  $\beta, h = u_1, u_2, \dots, u_\beta$  in Algorithm 3 come from the output of Algorithm 1.

---

**Algorithm 3** Generate the initial vertices of an induced anticycle in  $G_{n+r}$

---

**Input:** The set of edges  $\{\{i_1, j_1\}, \dots, \{i_s, j_s\}\}$  of  $G_r$  given that  $\min\{j_t - i_t \mid t = 1, \dots, s\} \geq 2$  and  $i_b \leq i_h$ , and an integer  $n \geq 2r$ .

**Output:** A sequence of numbers  $a_1, \dots, a_d, a_{d+1} \in \{1, 2, \dots, n+r\}$ .

```

 $\epsilon := \max\{t \in \mathbb{Z}_{\geq 0} \mid t(j_{u_\beta} - i_{u_\beta} - 1) + i_{u_\beta} < i_b\}$ 
 $a_1 := \epsilon(j_{u_\beta} - i_{u_\beta} - 1) + i_{u_\beta}$ 
term :=  $a_1$ 
 $d := 1$ 
while term <  $i_h$  do
     $t = \min\{\alpha \in \{1, \dots, \beta\} \mid i_{u_\alpha} \leq \text{term}\}$ 
    term := term +  $j_{u_t} - i_{u_t} - 1$ 
     $a_{d+1} := \text{term}$ 
     $d := d + 1$ 
end while

```

---

Note that

$$a_2 = a_1 + j_{u_\beta} - i_{u_\beta} - 1. \tag{5.2}$$

Indeed, by Construction 5.5,  $i_{u_\beta} < i_b \leq i_{u_{\beta-1}}$ , so  $a_1 = \epsilon(j_{u_\beta} - i_{u_\beta} - 1) + i_{u_\beta} < i_b \leq i_{u_{\beta-1}}$ . Hence at the step  $d = 1$ ,

$$t = \min\{\alpha \in \{1, \dots, \beta\} \mid i_{u_\alpha} \leq a_1\} = \beta.$$

From this, (5.2) follows.

At the end of Algorithm 3, we get a sequence  $a_1, \dots, a_{d+1}$  satisfying

$$i_q \leq i_{u_\beta} \leq a_1 < i_b \leq a_2 < a_3 < \dots < a_d < i_h \leq a_{d+1} = a_d + j_{u_t} - i_{u_t} - 1,$$

where  $t = \min\{\alpha \in \{1, \dots, \beta\} \mid i_{u_\alpha} \leq a_d\}$ .

At this point, note that  $v_\gamma = B$  per (5.1), so

$$a_{d+1} = a_d + j_{u_t} - i_{u_t} - 1 < i_h + j_{u_t} - i_{u_t} - 1 < 2r < n + i_{v_\gamma} = n + i_B,$$

using the hypothesis on  $n$ . Thus we may apply Algorithm 4 with  $a_{\text{index}} := a_{d+1}$ . Note that the numbers  $\gamma, H = v_1, v_2, \dots, v_\gamma = B$  in Algorithm 4 come from the output of Algorithm 2.

At the end of Algorithm 4, we get a sequence  $a_{d+2}, \dots, a_m$  satisfying

$$a_{d+1} < a_{d+2} < \dots < a_{m-2} \leq n + i_B < a_{m-1} < a_m = n + j_B.$$

---

**Algorithm 4** Generate the final vertices of an induced anticycle in  $G_{n+r}$

---

**Input:** The set of edges  $\{\{i_1, j_1\}, \dots, \{i_s, j_s\}\}$  of  $G_r$  and integers  $n \geq 2r$  and  $i_h \leq a_{\text{index}} \leq n+r$ .

**Output:** A sequence of numbers  $a_{\text{index}}, a_{\text{index}+1}, \dots, a_m \in \{1, 2, \dots, n+r\}$ .

---

```

term := aindex
v := 1
while term ≤ n + iB do
  t = min{α ∈ {1, ..., γ} | term ≤ n + ivα}
  term := term + jvt - ivt - 1
  aindex+v := term
  v := v + 1
end while
m := v + 1
am := n + jB

```

---

**Case II:**  $i_h < i_b$ . Set  $\epsilon = \max\{t \in \mathbb{Z}_{\geq 0} \mid t(j_h - i_h - 1) + i_h < i_b\}$  and define

$$a_1 = \epsilon(j_h - i_h - 1) + i_h < i_b,$$

$$a_2 = a_1 + j_h - i_h - 1 \geq i_b.$$

We are in place to apply Algorithm 4 with  $a_{\text{index}} := a_2$  since

$$a_2 = a_1 + j_h - i_h - 1 < i_b + j_h - i_h - 1 < 2r < n + i_B.$$

From the last algorithm, we obtain a sequence  $a_1, a_2, \dots, a_m$  satisfying

$$i_h \leq a_1 < i_b \leq a_2 < a_3 < \dots < a_{m-2} \leq n + i_B < a_{m-1} < a_m = n + j_B.$$

With Lemmas 5.9, 5.10, 5.11, we show that  $a_1, \dots, a_m$  in Construction 5.7 form an induced anticycle in  $G_{n+r}$  of length  $\geq \left\lfloor \frac{n+r}{r-2} \right\rfloor$  for all  $n \geq 2r$ . But first, we consider an example of Constructions 5.5–5.7.

**Example 5.8** Consider the Inc-invariant chain  $\mathcal{I} = (I_n)_{n \geq 1}$  of edge ideals with stability index 9 and

$$E(G_9) = \{\{1, 5\}, \{1, 8\}, \{2, 9\}, \{3, 6\}, \{4, 7\}, \{5, 9\}\}.$$

Clearly the conditions  $j_q + 1 = \max\{j_1, \dots, j_s\}$  and  $\min\{j_t - i_t \mid t = 1, \dots, s\} \geq 2$  of Proposition 5.2 are satisfied. In this case  $i_1 = 1, q = 2, j_2 = 8, \max\{j_1, \dots, j_6\} = 9 = 8 + 1$ , and  $\min\{j_t - i_t \mid t = 1, \dots, 6\} = 3$ .

The plot of the corresponding points  $(i_t, j_t), t = 1, \dots, 6$  is as in Fig. 5.

Running Algorithm 1, we get

$$J_1 = \{4, 5\}, u_1 = 4, J_2 = \{1\}, u_2 = 1, \beta = 2.$$

Running Algorithm 2, we get

$$K_1 = \{4, 5\}, v_1 = 5, K_2 = \{6\}, v_2 = 6, \gamma = 2.$$

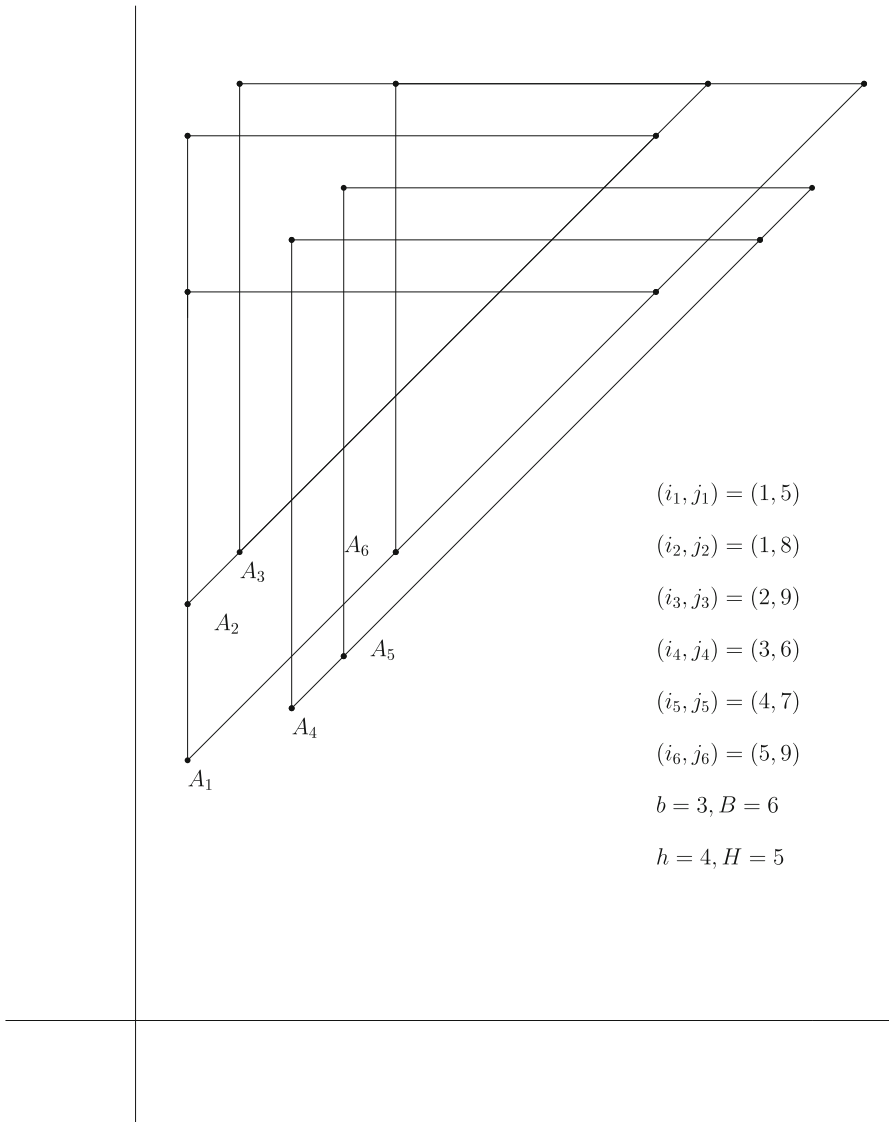


Fig. 5 An example with  $G_9$  having six edges

Note that  $b = 3, B = 6, h = 4, H = 5$  and

$$i_b = i_3 = 2 < i_h = i_4 = 3.$$

Let  $n = 18 = 2r$ . Running Algorithm 3 on Macaulay2 [8] with the input  $G_9$  and  $n = 18$ , we get the sequence

$$a_1 = 1, a_2 = 4.$$

Running Algorithm 4 with the input  $G_9$ ,  $n = 18$  and  $a_{\text{index}} = a_2 = 4$ , noting that  $i_B = i_6 = 5$ ,  $j_B = j_6 = 9$ , we get the sequence

$$a_2 = 4, 6, 8, \dots, 22, 24, 27.$$

Hence  $G_{27}$  has an induced anticycle  $C_{13}^c$  whose consecutive vertices are

$$1, 4, 6, 8, 10, \dots, 22, 24, 27.$$

Similarly,  $G_{28}$  has an induced  $C_{14}^c$  whose consecutive vertices are

$$1, 4, 6, 8, 10, \dots, 22, 24, 27, 28,$$

while  $G_{29}$  has an induced  $C_{14}^c$  with consecutive vertices

$$1, 4, 6, 8, 10, \dots, 22, 24, 26, 29.$$

**Lemma 5.9** *With the notations as in Construction 5.7, there are inequalities*

$$m \geq \left\lceil \frac{n+r}{r-2} \right\rceil \geq 4.$$

**Proof** From Construction 5.7, we see that the sequence  $a_1, \dots, a_m$  have the following properties:

$$\begin{aligned} 1 &\leq a_1 \leq i_b, \\ a_1 &< a_2 < \dots < a_{m-1} < a_m = n + j_B, \\ a_j - a_{j-1} &\leq r - 2 \quad \text{for each } 2 \leq j \leq m. \end{aligned}$$

The third property holds since for each such  $j$ , there exists some  $1 \leq u \leq s$  such that  $a_j - a_{j-1} \leq j_u - i_u - 1 \leq r - 1 - 1 = r - 2$ .

Since  $b \leq B$  by Notation 5.3, using the setup of Notation 5.1 and the hypothesis of Proposition 5.2 we see that  $i_b \leq i_B \leq j_B - 2$ . Therefore,

$$n + 2 \leq n + j_B - i_b \leq a_m - a_1 = \sum_{j=2}^m (a_j - a_{j-1}) \leq (m - 1)(r - 2).$$

Hence,  $m \geq \left\lceil \frac{n+r}{r-2} \right\rceil$ . The remaining inequality  $\left\lceil \frac{n+r}{r-2} \right\rceil \geq 4$  holds since  $n \geq 2r$  and  $r \geq 3$ .  $\square$

**Lemma 5.10** *With the notations as in Construction 5.7, we have  $\{a_p, a_{p+1}\} \notin E(G_{n+r})$  for all  $p = 1, \dots, m$  with  $a_{m+1} = a_1$ .*

**Proof** Since  $i_B + n < a_{m-1} < a_m = j_B + n$  and  $i_q \leq a_1 < i_b$ , we get  $\{a_{m-1}, a_m\}, \{a_1, a_m\} \notin E(G_{n+r})$ . Now assume that  $p \in \{1, \dots, m - 2\}$ .

CASE 1:  $a_p < i_h$ . In this case,  $a_{p+1}$  is determined from  $a_p$  by Algorithm 3. Hence,  $1 \leq p \leq d$  and  $a_p \leq a_d < i_h$ . Set

$$t = \min\{\alpha \in \{1, \dots, \beta\} \mid i_{u_\alpha} \leq a_p\}. \tag{5.3}$$

It follows that  $t \geq 2$  and  $i_{u_t} \leq a_p < i_{u_a}$  for all  $1 \leq a \leq t - 1$ . Moreover, we see that

$$a_{p+1} - a_p = j_{u_t} - i_{u_t} - 1 < j_{u_t} - i_{u_t}.$$

Now if there exists an integer  $\alpha \in \{1, \dots, s\}$  such that  $(a_p, a_{p+1}) \in \Delta_\alpha$  then  $i_\alpha \leq a_p < i_{u_{t-1}}$  and  $j_\alpha - i_\alpha \leq a_{p+1} - a_p < j_{u_t} - i_{u_t}$ . By the definitions of  $u_t$  and  $J_t$ , we must have

$$\alpha \in \cup_{i=1}^{t-1} J_i,$$

which implies  $\alpha \in J_\nu$  for some  $\nu = 1, \dots, t - 1$ . Thus,  $a_p \geq i_\alpha \geq i_{u_\nu}$ . By (5.3), we get  $\nu \geq t$ , a contradiction. This means  $(a_p, a_{p+1}) \notin \Delta_\alpha$  for all  $\alpha = 1, \dots, s$ .

CASE 2:  $a_p \geq i_h$ . In this case,  $a_{p+1}$  is determined from  $a_p$  by Algorithm 4. Set

$$t = \min\{\alpha \in \{1, \dots, \gamma\} \mid a_p \leq n + i_{v_\alpha}\}. \tag{5.4}$$

It follows that  $t \geq 1$  and  $a_{p+1} - a_p = j_{v_t} - i_{v_t} - 1 < j_{v_t} - i_{v_t}$ . If  $t = 1$  then

$$a_{p+1} - a_p < j_H - i_H \leq j_\alpha - i_\alpha$$

for all  $\alpha = 1, \dots, s$ . Hence  $(a_p, a_{p+1}) \notin \Delta_\alpha$  for all  $\alpha = 1, \dots, s$ . Now we consider the case where  $t \geq 2$ . By (5.4), we get  $n + i_{v_{t-1}} < a_p \leq n + i_{v_t}$  and

$$a_{p+1} = a_p + j_{v_t} - i_{v_t} - 1 \leq n + i_{v_t} + j_{v_t} - i_{v_t} - 1 = n + j_{v_t} - 1 < n + j_{v_t}.$$

Furthermore,

$$a_{p+1} = a_p + j_{v_t} - i_{v_t} - 1 > n + i_{v_{t-1}} + j_{v_t} - i_{v_t} - 1.$$

Since  $j_{v_t} - i_{v_t} > j_{v_{t-1}} - i_{v_{t-1}}$  per Construction 5.5, one has

$$a_{p+1} \geq n + i_{v_{t-1}} + j_{v_t} - i_{v_t} > n + j_{v_{t-1}}. \tag{5.5}$$

Now if there exists an integer  $\alpha \in \{1, \dots, s\}$  such that  $(a_p, a_{p+1}) \in \Delta_\alpha$  then

$$n + j_{v_{t-1}} < a_{p+1} \leq n + j_\alpha,$$

which implies  $j_\alpha > j_{v_{t-1}}$ . Combining with the fact that

$$j_{v_t} - i_{v_t} > a_{p+1} - a_p \geq j_\alpha - i_\alpha$$

and the definitions of  $K_t$  and  $v_t$ , we must have  $\alpha \in \cup_{i=1}^{t-1} K_i$ . This implies that  $\alpha \in K_\nu$  for some  $1 \leq \nu \leq t - 1$ . As a consequence,

$$a_{p+1} \leq n + j_\alpha \leq n + j_{v_\nu} \leq n + j_{v_{t-1}}. \quad (5.6)$$

The third inequality in the last chain holds because of Construction 5.5. For the second inequality, note that  $j_\alpha - i_\alpha = j_{v_\nu} - i_{v_\nu}$ , as  $\alpha, v_\nu \in K_\nu$ . By definition,  $v_\nu = \max K_\nu \geq \alpha$ , hence

$$j_{v_\nu} - j_\alpha = i_{v_\nu} - i_\alpha \geq 0,$$

as desired.

Now (5.6) contradicts the inequality (5.5). So  $(a_p, a_{p+1}) \notin \Delta_\alpha$  for all  $\alpha = 1, \dots, s$ .  $\square$

**Lemma 5.11** *With the notations as in Construction 5.7, we have  $\{a_p, a_z\} \in E(G_{n+r})$  for any  $1 \leq p < z \leq m$  such that  $z - p \geq 2$  and  $(p, z) \neq (1, m)$ .*

**Proof** We proceed through two steps.

Step 1:  $\{a_1, a_z\} \in E(G_{n+r})$  for all  $3 \leq z \leq m - 1$ .

Indeed,  $a_{m-1} < a_m = n + j_B = n + j_q + 1$ , thanks to the hypothesis, hence  $a_z \leq a_{m-1} \leq n + j_q$ .

We first consider the case where  $i_b \leq i_h$ . Then thanks to Algorithm 3,  $i_{u_\beta} \leq a_1$ . If  $a_z \leq j_{u_\beta} + n$ , then  $(a_1, a_z) \in \Delta_{u_\beta}$ , as the following chain holds

$$0 \leq a_1 - i_{u_\beta} \leq a_z - j_{u_\beta} \leq n.$$

In this chain, the middle inequality holds because thanks to Equation (5.2),

$$a_z - a_1 = a_z - a_2 + a_2 - a_1 \geq 1 + a_2 - a_1 = j_{u_\beta} - i_{u_\beta}.$$

If  $a_z > n + j_{u_\beta}$ , then

$$j_q + n \geq a_z > j_{u_\beta} + n \geq 2r \geq i_b + j_q - i_q > a_1 + j_q - i_q \geq j_q,$$

hence  $(a_1, a_z) \in \Delta_q$  because the following chains hold

$$0 \leq a_1 - i_q \leq a_z - j_q \leq n.$$

In other words,  $\{a_1, a_z\} \in E(G_{n+r})$  for all  $3 \leq z \leq m - 1$  in this case.

Next consider the case  $i_h < i_b$ . We show that  $\{a_1, a_z\} \in E(G_{n+r})$  for any  $3 \leq z \leq m - 1$  by going along the same lines by replacing  $u_\beta$  by  $u_1 = h$ . We leave the details to the interested reader. This completes the proof of Step 1.

Step 2:  $\{a_p, a_z\} \in E(G_{n+r})$  for all  $2 \leq p < z \leq m$  and  $z - p \geq 2$ .

Observe that  $a_p \geq i_b$ . We consider two cases.

CASE 1:  $a_p < i_h$ . Again let

$$t = \min\{\alpha \in \{1, \dots, \beta\} \mid i_{u_\alpha} \leq a_p\},$$

then  $t \geq 2$  as  $u_1 = h$ . We have  $i_{u_t} \leq a_p < i_{u_{t-1}}$  and  $a_{p+1} - a_p = j_{u_t} - i_{u_t} - 1$  by Algorithm 3. If  $a_z \leq j_{u_t} + n$  then

$$a_z - a_p = a_z - a_{p+1} + a_{p+1} - a_p \geq j_{u_t} - i_{u_t},$$

which gives  $0 \leq a_p - i_{u_t} \leq a_z - j_{u_t} \leq n$ , i.e.  $(a_p, a_z) \in \Delta_{u_t}$ . On the other hand, if  $a_z > j_{u_t} + n$ , then we claim  $(a_p, a_z) \in \Delta_b$ . Indeed, since  $a_z > 2r \geq i_{u_{t-1}} + j_b - i_b > a_p + j_b - i_b$  and  $a_z \leq j_b + n$ , the chain

$$0 \leq a_p - i_b \leq a_z - j_b \leq n$$

is valid.

CASE 2:  $a_p \geq i_h$ . If  $z = m$ , then we have that  $(a_p, a_m) \in \Delta_b \cup \Delta_B$ , as  $i_b \leq a_p \leq i_B + n$  for all  $2 \leq p \leq m - 2$ . More precisely, it can be checked that if  $a_p \geq i_B$  then  $(a_p, a_m) \in \Delta_B$ , while if  $a_p < i_B$  then  $(a_p, a_m) \in \Delta_b$ . If  $z \leq m - 1$ , then as

$$a_{z-1} \leq a_{m-2} \leq n + i_B = n + i_{v_\gamma} \quad (\text{per Construction 5.7 and Equation (5.1)}),$$

we may let

$$\mu = \min\{\alpha \in \{1, \dots, \gamma\} \mid a_{z-1} \leq i_{v_\alpha} + n\}.$$

Then  $a_{z-1} \leq i_{v_\mu} + n$  and by Algorithm 4,  $a_z - a_{z-1} = j_{v_\mu} - i_{v_\mu} - 1$ . Therefore,

$$a_z - a_p = a_z - a_{z-1} + a_{z-1} - a_p \geq j_{v_\mu} - i_{v_\mu}$$

and

$$a_z = a_{z-1} + j_{v_\mu} - i_{v_\mu} - 1 \leq j_{v_\mu} + n.$$

It follows that if  $a_p \geq i_{v_\mu}$ , then  $(a_p, a_z) \in \Delta_{v_\mu}$ , since the following chain holds

$$0 \leq a_p - i_{v_\mu} \leq a_z - j_{v_\mu} \leq n.$$

Finally, it remains to consider the case  $i_b \leq a_p < i_{v_\mu}$ . If  $a_z \geq j_b - i_b + i_{v_\mu}$ , then

$$a_z - a_p > a_z - i_{v_\mu} \geq j_b - i_b,$$

which yields

$$0 \leq a_p - i_b \leq a_z - j_b \leq n.$$

In other words,  $(a_p, a_z) \in \Delta_b$ .

Assume that  $a_z < j_b - i_b + i_{v_\mu}$ . In this case, the definition of  $\mu$  implies  $\mu = 1$ , hence

$$a_z - a_p = a_z - a_{z-1} + a_{z-1} - a_p \geq j_H - i_H = j_h - i_h.$$

Since  $n \geq 2r \geq j_b - i_b + i_{v_\mu} - j_h > a_z - j_h$ , the following chain holds

$$0 \leq a_p - i_h \leq a_z - j_h \leq n, \text{ i.e. } (a_p, a_z) \in \Delta_h.$$

This also finishes the proof of the lemma.  $\square$

**Proof of Proposition 5.2** It suffices to construct an induced anticycle  $C_m^c$  in  $G_{n+r}$ , for some  $m \geq \left\lceil \frac{n+r}{r-2} \right\rceil \geq 4$ . Keep using notations as in Construction 5.7. We wish to show that the vertices  $a_1, \dots, a_m$  in Construction 5.7 form an induced anticycle in the graph  $G_{n+r}$ .

This follows from combining Lemmas 5.9, 5.10, and 5.11. Hence  $G_{n+r}$  is not cochordal for all  $n \geq 2r$ . Therefore by Fröberg's theorem [7, Theorem 1],  $\text{reg } I_n \geq 3$  for all  $n \geq 3r$ .  $\square$

## 6 A sharp regularity bound

The next theorem gives our first major result on the regularity of invariant chains of edge ideals.

**Theorem 6.1** *Let  $\mathcal{I} = (I_n)_{n \geq 1}$  be an Inc-invariant chain of eventually nonzero edge ideals. Let  $r = \text{ind}(\mathcal{I}) \geq 1$ . Then for all  $n \geq 4r$ , there is an inequality  $\text{reg } I_n \leq 3$ .*

The key to this theorem is the following

**Lemma 6.2** *Keep using notations of Theorem 6.1. Let  $G_n$  be the graph corresponding to  $I_n$ . Let  $H$  be an induced subgraph of  $G_n$  with at least one edge. Set*

$$u_1 = \max\{u \in V(H) \mid \text{for some } u < v, v \in V(H), \text{ we have } \{u, v\} \in E(H)\}.$$

*Let  $v_1 \in V(H)$  be any vertex such that  $u_1 < v_1$  and  $\{u_1, v_1\} \in E(H)$ . Then for every  $n \geq 4r$  and every choice of  $H$  and  $v_1$  as above, the graph  $H \setminus N_H[v_1]$  is cochordal.*

**Proof** Assume the contrary that for some  $n \geq 3r$  and some induced subgraph  $H$  of  $G_{n+r}$  with  $E(H) \neq \emptyset$ , the graph  $H \setminus N_H[v_1]$  is not cochordal.

Assume that  $E(G_r) = \{\{i_1, j_1\}, \dots, \{i_s, j_s\}\}$  where  $s \geq 1$ ,  $1 \leq i_p < j_p \leq r$  for all  $p = 1, \dots, s$ . For each  $1 \leq t \leq s$ , denote  $\Delta_t = \Delta(\{i_t, j_t\}, n)$  for simplicity. Therefore,

$$E(H) \subseteq E(G_{n+r}) = \{(u, v) : (u, v) \in \cup_{t=1}^s \Delta_t\}.$$



Hence there exists an integer  $\ell \in \{1, \dots, s\}$  such that  $(u_1, v_1) \in \Delta_\ell$ . Observe that  $(i_\ell + j, v_1) \in \Delta_\ell$  for all  $0 \leq j \leq u_1 - i_\ell$ . Hence

$$\{v_1, i_\ell, i_\ell + 1, \dots, u_1\} \subseteq N_{G_{n+r}}[v_1].$$

Since  $H$  is an induced subgraph of  $G_{n+r}$ , it holds that

$$N_H[v_1] = V(H) \cap N_{G_{n+r}}[v_1].$$

Therefore,  $V(H \setminus N_H[v_1]) \subseteq \{1, 2, \dots, i_\ell - 1\} \cup V_1$ , where  $i_\ell + j \notin V_1$  for all  $0 \leq j \leq u_1 - i_\ell$ , in particular, if  $u \in V_1$  then  $u > u_1$ . It follows from the maximality of  $u_1$  in the hypothesis that:

$$\text{If } u < v \text{ and } \{u, v\} \in E(H \setminus N_H[v_1]) \text{ then } u \leq i_\ell - 1. \tag{6.1}$$

Step 1: We claim that  $H \setminus N_H[v_1]$  has no induced  $2K_2$ .

Note that  $H \setminus N_H[v_1]$  is also an induced subgraph of  $G_{n+r}$ . By Lemma 3.6, the graph  $H \setminus N_H[v_1]$  has no induced  $2K_2$ . Indeed, assume that  $\{u_2, v_2\}, \{u_3, v_3\}$  is an induced  $2K_2$  of  $H \setminus N_H[v_1]$  where  $u_p < v_p$ . Then Lemma 3.6 implies that for some non-negative number  $j_{2s}, u_3 > n + j_{2s} > n$ . This contradicts (6.1) and the fact that  $n \geq 3r > i_\ell$ .

Step 2: Since  $H \setminus N_H[v_1]$  is not cochordal, and yet does not contain any induced  $2K_2 = C_4^c$ , it necessarily contains an induced anticycle  $C_m^c$ , where  $m \geq 5$ . The consecutive vertices of  $C_m^c$  are labeled as  $a_1, \dots, a_m$  such that  $a_1 = \min\{a_p : 1 \leq p \leq m\}$  as in Lemma 4.2. Let  $a_{m+1} = a_1$ .

For each  $1 \leq p < q \leq m$  such that  $q - p \geq 2, (p, q) \neq (1, m)$ , it holds that  $\{a_p, a_q\} \in E(H \setminus N_H[v_1])$ , so  $(a_p, a_q) \in \Delta_{pq}$ , where  $\Delta_{pq} = \Delta((i_{pq}, j_{pq}), n)$  is one of the sets  $\Delta_1, \dots, \Delta_s$ . As in the proof of Lemma 4.2, we have

**Claim 1:** For all  $1 \leq t \leq m$ , none of the points  $(a_t, a_{t+1}), (a_{t+1}, a_t)$  belongs to

$$\Gamma = \bigcup_{\substack{1 \leq p < q \leq m \\ q - p \geq 2, (p, q) \neq (1, m)}} \Delta_{pq}.$$

We may reduce to the case where  $a_2 \leq a_m$  as explained in the proof of Lemma 4.2. In that situation, thanks to Lemma 4.2, we get  $a_m = \max\{a_p : 1 \leq p \leq m\}$ .

Apply Lemma 3.5 for the points  $(a_1, a_3) \in \Delta_{13}$  and  $(a_2, a_m) \in \Delta_{2m}$ . The conclusion is that  $a_1 < i_{2m}$  and

$$a_3 \leq j_{13} + n < a_m. \tag{6.2}$$

**Case 1:**  $a_2 \leq a_{m-1}$ . By Claim 1,  $(a_{m-1}, a_m) \notin \Delta_{2m}$ , hence the following chain is false

$$0 \leq a_{m-1} - i_{2m} \leq a_m - j_{2m} \leq n.$$

The first inequality in the chain holds since  $a_{m-1} \geq a_2 \geq i_{2m}$ , and the last inequality also holds since  $a_m \leq j_{2m} + n$ . Hence the middle inequality is false, namely

$$a_{m-1} > a_m + i_{2m} - j_{2m}. \quad (6.3)$$

By Claim 1,  $(a_{m-2}, a_{m-1}) \notin \Delta_{1(m-1)}$ , hence the following chain is false

$$0 \leq a_{m-2} - i_{1(m-1)} \leq a_{m-1} - j_{1(m-1)} \leq n.$$

The first inequality in the chain holds since  $a_{m-2} \geq a_1 \geq i_{1(m-1)}$ , and the last inequality also holds since  $a_{m-1} \leq j_{1(m-1)} + n$ . Hence the middle inequality is false, namely

$$a_{m-2} > a_{m-1} + i_{1(m-1)} - j_{1(m-1)}. \quad (6.4)$$

Combining (6.2), (6.3), and (6.4), we obtain

$$a_{m-2} > n + j_{13} + i_{2m} - j_{2m} + i_{1(m-1)} - j_{1(m-1)}.$$

Since  $a_{m-2} \leq a_m$ ,  $a_{m-2} \neq a_m$ , and  $\{a_{m-2}, a_m\} \in E(H \setminus N_H[v_1])$ , we obtain  $a_{m-2} \leq i_\ell - 1$  by (6.1). Thus

$$i_\ell - 1 \geq a_{m-2} > n + j_{13} + i_{2m} - j_{2m} + i_{1(m-1)} - j_{1(m-1)},$$

so that

$$n < i_\ell - 1 - j_{13} - i_{2m} + j_{2m} - i_{1(m-1)} + j_{1(m-1)} \leq 3r,$$

where the last inequality follows from  $i_\ell < j_\ell \leq \max\{j_1, \dots, j_s\} \leq r$ . This contradicts the hypothesis  $n \geq 3r$ .

**Case 2:**  $a_2 > a_{m-1}$ . Looking at the points  $(a_1, a_{m-2}) = (a_1, a_{m-2})^{\leq} \in \Delta_{1(m-2)}$  and  $(a_{m-1}, a_2) = (a_2, a_{m-1})^{\leq} \in \Delta_{2(m-1)}$ . Note that  $a_1 \leq a_{m-1}$ , and  $(a_1, a_2), (a_{m-1}, a_{m-2}) \notin \Delta_{1(m-2)} \cup \Delta_{2(m-1)}$  thanks to Claim 1, so arguing as above using Lemma 3.5, we deduce  $i_{1(m-2)} < i_{2(m-1)}$ . Moreover, the same result yields

$$a_2 > n + j_{1(m-2)}. \quad (6.5)$$

Since  $\{a_2, a_m\} \in E(H \setminus N_H[v_1])$ ,  $a_2 \leq a_m$  and  $a_2 \neq a_m$ , we get from (6.1) that  $a_2 \leq i_\ell - 1$ . Combining with (6.5), we obtain

$$i_\ell - 1 \geq a_2 > n + j_{1(m-2)},$$

hence

$$n < i_\ell - 1 - j_{1(m-2)} < r.$$

This contradicts the hypothesis  $n \geq 3r$ . We conclude that  $H \setminus N_H[v_1]$  is cochordal. The proof is completed.  $\square$

**Proof of Theorem 6.1** For a graph  $H$ , by abuse of notation, write  $\text{reg } H$  for the regularity of the ideal  $I(H)$ . Let  $n \geq 3r$  be an integer. We prove that if  $H$  is an induced subgraph of  $G_{n+r}$  then  $\text{reg } H \leq 3$ , by induction on the cardinality of  $E(H)$ .

If  $E(H) = \emptyset$  then  $\text{reg } H = \text{reg}(0) = -\infty$ . Assume that  $E(H) \neq \emptyset$ . Let  $u_1$  be the maximal value of  $u \in V(H)$  such that for some  $u < v, v \in V(H), \{u, v\}$  belongs to  $E(H)$ . Let  $v_1 \in V(H)$  be a neighbor of  $u_1$  in  $H$  such that  $v_1 > u_1$ . By Lemma 6.2 and Fröberg’s theorem,  $\text{reg}(H \setminus N_H[v_1]) \leq 2$ . This, together with Lemma 2.10, yields the chain

$$\text{reg}(H) \leq \max\{\text{reg}(H \setminus N_H[v_1]) + 1, \text{reg}(H \setminus v_1)\} \leq \max\{3, \text{reg}(H \setminus v_1)\}.$$

Clearly  $\{u_1, v_1\} \in E(H) \setminus E(H \setminus v_1)$ . Also  $H \setminus v_1$  is an induced subgraph of  $G_{n+r}$ , hence by the induction hypothesis  $\text{reg}(H \setminus v_1) \leq 3$ . Therefore  $\text{reg}(H) \leq 3$ . This finishes the induction and choosing  $H = G_{n+r}$ , we get the desired conclusion.  $\square$

### 7 Convergence

Now we state our main accomplishment in this paper. In the following statement, we employ the  $q$ -invariant introduced in Definition 2.5.

**Theorem 7.1** *Let  $\mathcal{I} = (I_n)_{n \geq 1}$  be an Inc-invariant chain of eventually nonzero edge ideals. Keep using Notation 5.1. Then for all  $n \geq N = \max\{5r, 2r(r - 2), 4(r + q(I_r))\}$ , there is an equality*

$$\text{reg } I_n = \text{reg } I_{n+1},$$

and moreover  $\lim_{n \rightarrow \infty} \text{reg } I_n \in \{2, 3\}$ .

In addition, the following statements are equivalent

- (1)  $\text{reg } I_n = 2$  for all  $n \gg 0$ ;
- (2)  $\text{reg } I_N = 2$ ;
- (3) Either  $j_q = \max\{j_1, \dots, j_s\}$ , or it holds that  $\min\{j_t - i_t : t = 1, \dots, s\} = 1$  and  $\text{indmatch}(G_{3r}) = 1$ .

**Remark 7.2** We note that the number  $N = \max\{5r, 2r(r - 2), 4(r + q(I_r))\}$  in Theorem 7.1 can be replaced by a constant depending only on  $r$ . Indeed, let  $p = \text{Msupp } I_r \leq r$ , since  $I_r$  is quadratic and nonzero, by Definition 2.5,

$$q(I_r) = \sum_{i=0}^2 \dim_{\mathbb{k}} \left( \frac{R_p}{I_r \cap R_p} \right)_i \leq 1 + p + \binom{p+1}{2} - 1 \leq \frac{r^2 + 3r}{2}.$$

Therefore  $4(r + q(I_r)) \leq 2(r^2 + 5r)$ , and thus  $N \leq 2(r^2 + 5r)$ . Hence we can replace  $N$  by  $2(r^2 + 5r)$  in the statement of Theorem 7.1. While we do not know the optimal

(minimal) value for the number  $N$  such that  $\text{reg } I_n = \text{reg } I_{n+1}$  for all  $n \geq N$ , Remark 7.5 below shows that this value has to be at least  $3r - 9$ .

For the proof of Theorem 7.1, we start in Propositions 7.3 and 7.4 with the sufficient conditions for the equality  $\text{reg } I_n = 2$  to hold for all  $n \gg 0$ .

**Proposition 7.3** *Keep using Notation 5.1. If  $\text{indmatch}(G_{3r}) = 1$  and  $\min\{j_p - i_p : p = 1, \dots, s\} = 1$ , then  $\text{reg } I_n = 2$  for all  $n \geq \max\{5r, 2r(r - 2)\}$ .*

**Proof** By Fröberg’s theorem, we have to show that  $G_n$  is cochordal for all  $n \geq \max\{5r, 2r(r - 2)\}$ .

By the hypothesis,  $I_n$  is a nonzero edge ideal for all  $n \geq r$ . Thanks to Theorem 3.1,  $\text{indmatch}(G_n) = 1$  for all  $n \geq 3r$ . And hence  $G_n$  does not contain any induced  $2K_2$  for all  $n \geq 3r$ . Take  $n \geq \max\{5r, 2r(r - 2)\}$ . We show that  $G_n$  is cochordal.

It suffices to prove that  $G_n$  does not contain any induced  $C_m^c$  for  $m \geq 5$ . Assume the contrary, that for some  $m \geq 5$  and some  $n \geq \max\{5r, 2r(r - 2)\}$ ,  $G_n$  contain an induced  $C_m^c$ . By Proposition 4.1,

$$m > \frac{n}{r} \geq \max\{5, 2(r - 2)\}. \tag{7.1}$$

By the hypothesis, for some  $1 \leq p \leq s$ , we have  $j_p = i_p + 1$ . Since

$$\Delta((i_p, i_p + 1), n - r) = \{(u, v) : i_p \leq u < v \leq i_p + 1 + n - r\},$$

we deduce that  $A = \{i_p, i_p + 1, \dots, i_p + n - r + 1\}$  is a clique in  $G_n$ . Note that  $|V(G_n \setminus A)| \leq n - (n - r + 2) = r - 2$ .

Denote the consecutive vertices of the induced  $C_m^c$  as a subgraph of  $G_n$  by  $a_1, a_2, \dots, a_m$ . Denote  $a_{m+1} = a_1$ . So  $a_i, a_{i+1}$  are not adjacent for all  $1 \leq i \leq m$ . Since  $A$  is a clique and  $C_m^c$  is an induced subgraph of  $G_n$ , this implies  $\{a_1, \dots, a_m\} \cap A$  has at most  $m/2$  elements. Hence

$$m = |\{a_1, \dots, a_m\} \cap A| + |\{a_1, \dots, a_m\} \setminus A| \leq \frac{m}{2} + |V(G_n \setminus A)| \leq \frac{m}{2} + (r - 2).$$

In particular,  $m \leq 2(r - 2)$ , contradicting (7.1). Therefore, the above assumption is wrong, and  $G_n$  is cochordal for all  $n \geq \max\{5r, 2r(r - 2)\}$ . This concludes the proof.  $\square$

**Proposition 7.4** *Keep using Notation 5.1. If  $j_q = \max\{j_1, \dots, j_s\}$  then  $\text{reg } I_n = 2$  for all  $n \geq 3r$ .*

**Proof** By Fröberg’s theorem, we have to show that  $G_{n+r}$  is cochordal for all  $n \geq 2r$ . Assume the contrary, that for some  $m \geq 4$  and some  $n \geq 2r$ ,  $G_{n+r}$  contains an induced  $C_m^c$ .

Label the consecutive vertices of the induced  $C_m^c$  by  $a_1, a_2, \dots, a_m$  in such a way that  $a_1 = \min\{a_1, \dots, a_m\}$ .

We may assume that  $a_2 \leq a_m$  (when  $a_2 < a_m$ , relabel the vertices as  $b_1 = a_1, b_i = a_{m+2-i}$  for  $2 \leq i \leq m$ ).

By the hypothesis  $j_q = \max\{j_1, \dots, j_s\}$  and  $i_1 = \dots = i_q = \min\{i_1, \dots, i_s\}$ , one has  $i_q \leq a_i \leq j_q + n$  for  $i = 1, \dots, m$ . Since  $(a_1, a_m) \notin \Delta_q = \Delta((i_q, j_q), n)$ , the following chain is false

$$0 \leq a_1 - i_q \leq a_m - j_q \leq n.$$

It follows that  $a_1 > a_m + i_q - j_q$ . Assume that  $(a_1, a_3) \in \Delta_b, (a_2, a_m) = (a_2, a_m)^{\leq} \in \Delta_c$  and  $(a_1, a_{m-1}) \in \Delta_d$ , with  $b, c, d \in \{1, 2, \dots, s\}$  (note that  $b = d$  if  $m = 4$ ).

Consider the points  $(a_1, a_3) \in \Delta_b$  and  $(a_2, a_m) \in \Delta_c$ . Since  $\{a_1, a_m\}, \{a_2, a_3\} \notin E(G_{n+r})$ , we have  $(a_1, a_m), (a_2, a_3) \notin \Delta_b \cup \Delta_c$ . Arguing as in Lemma 6.2, using Lemma 3.5 for the two points  $(a_1, a_3), (a_2, a_m)$  and the fact that  $a_1 < a_2$ , we get  $i_b < i_c$ , moreover  $a_1 < i_c \leq r$  and  $a_m > n + j_b$ . Thus

$$r > a_1 > a_m + i_q - j_q > n + j_b + i_q - j_q,$$

which implies  $n < r + j_q - j_b - i_q < 2r$ . This contradicts the hypothesis  $n \geq 2r$ . Therefore the above assumption is wrong, and  $G_{n+r}$  is cochordal for all  $n \geq 2r$ . This concludes the proof.  $\square$

**Remark 7.5** The lower bound  $3r$  in the statement of Proposition 7.4 is nearly sharp. Let  $r \geq 5$  be an integer,  $(I_n)_{n \geq 1}$  be the invariant chain with stability index  $r$  and  $I_r = (x_1 x_r, x_{r-3} x_{r-1})$ . The hypothesis of Proposition 7.4 is clearly satisfied. We claim that  $\text{reg } I_{3r-10} \geq 3$ . Indeed, it is not hard to check that  $\{2r - 8, 2r - 6\}, \{r - 4, 3r - 10\}$  form a  $2K_2$  in  $G_{3r-10}$ , so Lemma 2.9 implies the desired inequality. Thus the lower bound  $3r$  in the statement “ $\text{reg } I_n = 2$  for all  $n \geq 3r$ ” cannot be improved to  $3r - 10$ .

Using the results in Sect. 5, we complement Propositions 7.3 and 7.4 by giving a condition guaranteeing that  $\text{reg } I_n = 3$  for  $n \gg 0$ .

**Theorem 7.6** *Keep using Notation 5.1. Assume  $\max\{j_1, \dots, j_s\} \geq j_q + 1$  and  $\min\{j_p - i_p : p = 1, \dots, s\} \geq 2$ . Then for all  $n \geq 4(r + q(I_r))$ , there is an equality  $\text{reg}(I_n) = 3$ .*

**Proof** The hypothesis implies that  $r \geq 4$ . We proceed by induction on  $q(I_r) \geq 0$ . The case  $q(I_r) = 0$  is vacuous: in that case  $I_r = (1)$  and the hypothesis is not satisfied.

Assume that  $q(I_r) > 0$ . Let  $p = \text{Msupp}(I_r) = \max\{j_1, \dots, j_s\}$ . If  $p = j_q + 1$ , then by Proposition 5.2, we deduce that  $\text{reg } I_n \geq 3$  for all  $n \geq 3r$ . Together with Theorem 6.1, we get  $\text{reg } I_n = 3$  for all  $n \geq 4(r + q(I_r))$ .

Now assume that  $p \geq j_q + 2$ .

Let  $\mathcal{J} = (J_n)_{n \geq 1}$  be the chain defined in Proposition 2.4. By the last result,  $\text{ind}(\mathcal{J}) = r + 1$  and since  $\langle I_r \rangle_{R_{r+1}} \subseteq J_{r+1}$  are quadratic monomial ideals with

$$\text{Msupp}(J_{r+1}) = \text{Msupp}(\langle I_r \rangle_{R_{r+1}}) = p,$$

Lemma 2.7 implies that  $q(J_{r+1}) \leq q(\langle I_r \rangle_{R_{r+1}}) = q(I_r)$ .

We claim that  $q(J_{r+1}) < q(I_r)$ . Assume the contrary, then Lemma 2.7 implies that  $J_{r+1} = \langle I_r \rangle_{R_{r+1}}$ . By Proposition 2.4, we get  $\mathcal{I}$  is a quasi-saturated chain.

But now  $x_{i_1}x_{j_q} \in I_r$ , so  $x_{i_1}x_{j_q+1} \in I_{r+1}$  by Inc-invariance. Since  $\text{Msupp}(I_r) = \max\{j_1, \dots, j_s\} \geq j_q + 1$  and  $\mathcal{I}$  is quasi-saturated, we deduce

$$x_{i_1}x_{j_q+1} \in I_{r+1} \cap \mathbb{k}[x_1, \dots, x_p] = I_r \cap \mathbb{k}[x_1, \dots, x_p].$$

Thus  $x_{i_1}x_{j_q+1} \in I_r$ , namely  $\{i_1, j_q + 1\} \in E(G_r)$ . This contradicts the fact that  $j_q = \max\{j_t \mid i_t = i_1\}$ . Therefore  $q(J_{r+1}) < q(I_r)$ .

Let  $H_n$  be the graph corresponding to  $J_n$  for each  $n$ .

**Observation:** There is an equality

$$E(H_{r+1}) = \{\{i_t, j_t\}, \{i_t, j_t + 1\}, \{i_t + 1, j_t + 1\} \mid 1 \leq t \leq s, j_t < p\} \cup \{\{i_t, j_t\} \mid 1 \leq t \leq s, j_t = p\}.$$

This follows from inspecting the definition

$$J_{r+1} = \langle I_{r+1} \cap R_p \rangle_{R_{r+1}}$$

and the fact that  $I_{r+1} = \langle \text{Inc}_{r,r+1}(I_r) \rangle_{R_{r+1}}$ .

From the Observation and the assumption that  $p \geq j_q + 2$ , we see that the chain  $\mathcal{J}$  also satisfies the hypothesis of Theorem 7.6. In detail,

$$\begin{aligned} \min\{v - u \mid u < v, \{u, v\} \in E(H_{r+1})\} &= \min\{v - u \mid u < v, \{u, v\} \in E(G_r)\} \geq 2, \\ \max\{v \mid \{i_1, v\} \in E(H_{r+1})\} &= j_q + 1, \\ \max\{v \mid \text{for some } u < v \text{ we have } \{u, v\} \in E(H_{r+1})\} &= p. \end{aligned}$$

By the induction hypothesis for the chain  $\mathcal{J}$ , we get for all  $n \geq 4(r + 1 + q(J_{r+1}))$  that

$$\text{reg } J_n = 3.$$

Since  $q(J_{r+1}) < q(I_r)$ , we have  $4(r + q(I_r)) \geq 4(r + 1 + q(J_{r+1}))$ . For each  $n \geq 4(r + q(I_r)) \geq \max\{4(r + 1 + q(J_{r+1})), r + 1\}$ , we get

$$\text{reg } I_n \geq \text{reg}(I_n + (x_{n-r+p})) = \text{reg } J_n = 3.$$

The inequality in the chain follows from Lemma 2.8. The first equality in the last chain is due to Proposition 2.4. Together with Theorem 6.1, we get

$$\text{reg } I_n = 3 \quad \text{for each } n \geq 4(r + q(I_r)).$$

This finishes the induction and the proof. □

We are ready for the

**Proof of Theorem 7.1** First assume that  $\text{indmatch}(G_{3r}) = 2$ . Then by Theorem 3.1, we deduce  $\text{indmatch}(G_n) = 2$  for all  $n \geq 3r$ . Hence using Lemma 2.9, for all such  $n$ ,

$$\text{reg } I_n \geq 1 + \text{indmatch}(G_n) = 3.$$

Together with Theorem 6.1, we get  $\text{reg } I_n = 3$  for all  $n \geq 4r$ .

Next assume that  $\text{indmatch}(G_{3r}) = 1$ . Then by Theorem 3.1,  $\text{indmatch}(G_n) = 1$  for all  $n \geq 3r$ .

If  $\min\{j_t - i_t : t = 1, \dots, s\} = 1$  then by Proposition 7.3,  $\text{reg } I_n = 2$  for all  $n \geq \max\{5r, 2r(r-2)\}$ .

If  $j_q = \max\{j_1, \dots, j_s\}$  then  $\text{reg } I_n = 2$  for all  $n \geq 3r$  by Proposition 7.4.

It remains to consider the case  $\max\{j_1, \dots, j_s\} \geq j_q + 1$  and  $\min\{j_t - i_t : t = 1, \dots, s\} \geq 2$ . In this case, by Theorem 7.6, we deduce that  $\text{reg } I_n = 3$  for all  $n \geq 4(r + q(I_r))$ . All the desired assertions follow from the above arguments.  $\square$

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**Data availability** Authors can confirm that all relevant data are included in the article and/or its supplementary information files.

## Declarations

**Conflict of interest** To our best knowledge, no conflict of interests, whether of financial or personal nature, has influenced the work presented in this article.

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## References

1. Aschenbrenner, M., Hillar, C.J.: Finite generation of symmetric ideals. *Trans. Amer. Math. Soc.* **359**(11), 5171–5192 (2007)
2. Caviglia, G., Hà, H.T., Herzog, J., Kummini, M., Terai, N., Trung, N.V.: Depth and regularity modulo a principal ideal. *J. Algebr. Comb.* **49**, 1–20 (2019)
3. Church, T., Ellenberg, J.S., Farb, B.: FI-modules and stability for representations of symmetric groups. *Duke Math. J.* **164**(9), 1833–1910 (2015)
4. Cohen, D.E.: Closure relations, Buchberger’s algorithm, and polynomials in infinitely many variables. In: *Computation Theory and Logic. Lecture Notes in Computer Science*, vol. 270. pp. 78–87, Springer, Berlin (1987)
5. Dao, H.L., Huneke, C., Schweig, J.: Bounds on the regularity and projective dimension of ideals associated to graphs. *J. Algebraic Combin.* **38**(1), 37–55 (2013)
6. Draisma, J., Eggermont, R.H., Farooq, A.: Components of symmetric wide-matrix varieties. *J. Reine Angew. Math.* **793**, 143–184 (2022)

7. Fröberg, R.: On Stanley-Reisner rings. In: Topics in Algebra, vol. 26. Part 2, pp. 57–70. Banach Center Publications (1990)
8. Grayson, D., Stillman, M.: Macaulay2, a software system for research in algebraic geometry. Available at <http://www.math.uiuc.edu/Macaulay2/>
9. Güntürkün, S., Nagel, U.: Equivariant Hilbert series of monomial orbits. Proc. Amer. Math. Soc. **146**(6), 2381–2393 (2018)
10. Hillar, C.J., Sullivant, S.: Finite Gröbner bases in infinite dimensional polynomial rings and applications. Adv. Math. **229**(1), 1–25 (2012)
11. Kahle, T., Le, D.V., Römer, T.: Invariant chains in algebra and discrete geometry. SIAM J. Discrete Math. **36**(2), 975–999 (2022)
12. Katzman, M.: Characteristic-independence of Betti numbers of graph ideals. J. Combin. Theory Ser. A **113**, 435–454 (2006)
13. Le, D.V., Nagel, U., Nguyen, H.D., Römer, T.: Castelnuovo–Mumford regularity up to symmetry. Int. Math. Res. Not. **14**, 11010–11049 (2021)
14. Le, D.V., Nguyen, H.D.: On regularity and projective dimension up to symmetry. Preprint (2022), [arXiv:2206.15141](https://arxiv.org/abs/2206.15141) [math.AC]
15. Morey, S., Villarreal, R.: Edge ideals: algebraic and combinatorial properties. In: Francisco, C., Klingler, L.C., Sather-Wagstaff, S., Vassilev, J.C. (eds.) Progress in Commutative Algebra, Combinatorics and Homology, vol. 1, pp. 85–126. De Gruyter, Berlin (2012)
16. Murai, S.: Betti tables of monomial ideals fixed by permutations of the variables. Trans. Amer. Math. Soc. **373**, 7087–7107 (2020)
17. Murai, S., Raicu, C.: An equivariant Hochster’s formula for  $\mathfrak{S}_n$ -invariant monomial ideals. J. London Math. Soc. **105**, 1974–2010 (2022)
18. Nagel, U., Römer, T.: Equivariant Hilbert series in non-Noetherian polynomial rings. J. Algebra **486**, 204–245 (2017)
19. Raicu, C.: Regularity of  $\mathfrak{S}_n$ -invariant monomial ideals. J. Combin. Theory Ser. A **177**, 105307 (2021)
20. Ramos, E.: An application of the theory of FI-algebras to graph configuration spaces. Math. Z. **294**, 1–15 (2020)
21. Ramos, E., White, G.: Families of nested graphs with compatible symmetric-group actions. Sel. Math. New Ser. **25**, 70 (2019)
22. Sam, S., Snowden, A.: Gröbner methods for representations of combinatorial categories. J. Amer. Math. Soc. **30**(1), 159–203 (2017)
23. Woodroffe, R.: Matchings, coverings, and Castelnuovo–Mumford regularity. J. Commut. Algebra **6**(2), 287–304 (2014)

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