Pacific Journal of Mathematics

ON THE GAUSS MAPS OF COMPLETE MINIMAL SURFACES IN $\mathbb{R}^n$

Dinh Tuan Huynh

Volume 330 No. 1 May 2024

PACIFIC JOURNAL OF MATHEMATICS
Vol. 330, No. 1, 2024
https://doi.org/10.2140/pjm.2024.330.157

ON THE GAUSS MAPS OF COMPLETE MINIMAL SURFACES IN $\mathbb{R}^n$

DINH TUAN HUYNH

Dedicated to Professor Doan The Hieu

We prove that the generalized Gauss map of a nonflat complete minimal surface immersed in $\mathbb{R}^n$ can omit a generic hypersurface D of degree at most $n^{n+2}(n+1)^{n+2}$.

1. Introduction

Let $f = (x_1, x_2, \dots, x_n) : M \to \mathbb{R}^n$ be an oriented surface immersed in $\mathbb{R}^n$. Using systems of isothermal coordinates (x, y), one can consider M as a Riemann surface. We are interested in the class of minimal surfaces, namely, those which have minimal areas for all small perturbations. It is a well-known fact that if M is minimal, then its generalized Gauss map $g : M \to \mathbb{CP}^{n-1}$, defined as

$$g(z) := [\partial f / \partial z],$$

where $z = x + iy$ is a holomorphic chart on M, is a holomorphic map.

In the particular case where $n = 3$, recalling that the classical Gauss map of M is the map sending each point $p \in M$ to the point in the unit sphere corresponding to the unit normal vector of M at p. By identifying the unit sphere with the complex projective line via the stereographic projection, one can view the classical Gauss map as a map of M into $\mathbb{CP}^1$. Osserman [18] proved that if M is a nonflat complete minimal surface immersed in $\mathbb{R}^3$, then the complement of the image of its Gauss map is of logarithmic capacity zero in $\mathbb{CP}^1$. This interesting result could be regarded as a significant improvement of the classical Bernstein's theorem. Strengthening this result, Xavier [22] proved that in this situation, the Gauss map of M can avoid at most 6 points. Sharp result was obtained by Fujimoto [10], where he proved that indeed, the Gauss map of M can avoid at most 4 points.

Passing to higher-dimensional case, first step was made by Fujimoto [9], where the intersection between the generalized Gauss maps of a complete minimal surface immersed in $\mathbb{R}^n$ and family of hyperplanes in $\mathbb{CP}^{n-1}$ was considered. Precisely, Fujimoto established that:

MSC2020: primary 53A10; secondary 32H30.
Keywords: value distribution theory, Gauss map, minimal surface, hyperbolicity.

Theorem 1.1. *If the generalized Gauss map of a nonflat complete minimal surface in $\mathbb{R}^n$ is nondegenerate, it can omit at most $q = n^2$ hyperplanes in $\mathbb{CP}^{n-1}$ in general position.*

Later, Fujimoto himself [11] decreased the number of hyperplanes in the above statement to $q = \frac{1}{2}n(n+1)$ and it turns out that this number is sharp. Ru [19] was able to remove the nondegenerate assumption of the generalized Gauss map in Fujimoto's result. Since then, by adapting tools and techniques from value distribution theory of holomorphic curves to study generalized Gauss maps, many generalizations of the above works of Fujimoto–Ru were made. Note that in these results, it is required the presence of many hypersurfaces.

In this paper, based on recent progresses towards the hyperbolicity problem [1; 2; 4; 6; 7; 8; 13; 21], we consider the case when there is only one hypersurface of high enough degree.

Theorem 1.2 (Main Theorem). *Let M be a nonflat complete minimal surface immersed in $\mathbb{R}^n$ and let $G : M \to \mathbb{CP}^{n-1}$ be its generalized Gauss map. Then G can avoid a generic hypersurface $D \subset \mathbb{CP}^{n-1}$ of degree at most*

$$d = n^{n+2}(n+1)^{n+2}.$$

2. Logarithmic jet differentials

Let X be a complex projective variety of dimension n. For a point $x \in X$, consider the holomorphic germs $(\mathbb{C}, 0) \to (X, x)$. Two such germs are said to be equivalent if they have the same Taylor expansion up to order k in some local coordinates around x. The equivalence class of an analytic germ $f : (\mathbb{C}, 0) \to (X, x)$ is called the *k-jet of f*, denoted by $j_k(f)$, which is independent of the choice of local coordinates. A k-jet $j_k(f)$ is said to be *regular* if $\mathrm{d}f(0) \neq 0$. For a given point $x \in X$, denote by $j_k(X)_x$ the vector space of all k-jets of analytic germs $(\mathbb{C}, 0) \to (X, x)$, set

$$J_k(X) := \bigcup_{x \in X} J_k(X)_x,$$

and consider the natural projection

$$\pi_k : J_k(X) \to X.$$

Then $J_k(X)$ carries the structure of a holomorphic fiber bundle over X, which is called the *k-jet bundle over X*. Note that in general, $J_k(X)$ is not a vector bundle. When $k = 1$, the 1-jet bundle $J_1(X)$ is canonically isomorphic to the tangent bundle T_X of X.

For an open subset $U \subset X$, for $\omega \in H^0(U, T_X^*)$, for a k-jet $j_k(f) \in J_k(X)|_U$, the pullback $f^*\omega$ is of the form $A(z)\,\mathrm{d}z$ for some analytic function A, where z is the global coordinate of $\mathbb{C}$. Since each derivative $A^{(j)}$ $(0 \leq j \leq k-1)$ is well defined, independent of the representation of f in the class $j_k(f)$, the analytic 1-form ω induces the holomorphic map

$$\tilde{\omega} : J_k(X)|_U \to \mathbb{C}^k, \quad j_k(f) \to \big(A(z), A(z)^{(1)}, \dots, A(z)^{(k-1)}\big). \tag{2-1}$$

Hence on an open subset U, a given local holomorphic coframe $\omega_1 \wedge \cdots \wedge \omega_n \neq 0$ yields a trivialization

$$H^0(U, J_k(X)) \to U \times (\mathbb{C}^k)^n$$

by providing the following new nk independent coordinates:

$$\sigma \to (\pi_k \circ \sigma; \tilde{\omega}_1 \circ \sigma, \dots, \tilde{\omega}_n \circ \sigma),$$

where $\tilde{\omega}_i$ are defined as in (2-1). The components $x_i^{(j)}$ $(1 \le i \le n,\ 1 \le j \le k)$ of $\tilde{\omega}_i \circ \sigma$ are called the *jet-coordinates*. In a more general setting, where ω is a section over U of the sheaf of meromorphic 1-forms, the induced map $\tilde{\omega}$ is meromorphic.

Now, suppose that $D \subset X$ is a normal crossing divisor on X. This means that at each point $x \in X$, there exist some local coordinates $z_1, \dots, z_\ell, z_{\ell+1}, \dots, z_n$ $(\ell = \ell(x))$ centered at x in which D is defined by

$$D = \{z_1 \cdots z_\ell = 0\}.$$

Following Iitaka [14], the *logarithmic cotangent bundle of X along D*, denoted by $T_X^*(\log D)$, corresponds to the locally free sheaf generated by

$$\frac{dz_1}{z_1}, \dots, \frac{dz_\ell}{z_\ell}, z_{\ell+1}, \dots, z_n$$

in the above local coordinates around x.

A holomorphic section $s \in H^0(U, J_k(X))$ over an open subset $U \subset X$ is said to be a *logarithmic k-jet field* if $\tilde{\omega} \circ s$ are analytic for all sections $\omega \in H^0(U', T_X^*(\log D))$, for all open subsets $U' \subset U$, where $\tilde{\omega}$ are induced maps defined as in (2-1). Such logarithmic k-jet fields define a subsheaf of $J_k(X)$, and this subsheaf is itself a sheaf of sections of a holomorphic fiber bundle over X, called the *logarithmic k-jet bundle over X along D*, denoted by $J_k(X, -\log D)$ (see [16]).

The group $\mathbb{C}^*$ admits a natural fiberwise action defined as follows. For local coordinates

$$z_1, \dots, z_\ell, z_{\ell+1}, \dots, z_n \quad (\ell = \ell(x))$$

centered at x in which $D = \{z_1 \dots z_\ell = 0\}$, for any logarithmic k-jet field along D represented by some germ $f = (f_1, \dots, f_n)$, if $\varphi_\lambda(z) = \lambda z$ is the homothety with ratio $\lambda \in \mathbb{C}^*$, the action is given by

$$\begin{cases} (\log(f_i \circ \varphi_\lambda))^{(j)} = \lambda^j (\log f_i)^{(j)} \circ \varphi_\lambda & (1 \le i \le \ell), \\ (f_i \circ \varphi_\lambda)^{(j)} = \lambda^j f_i^{(j)} \circ \varphi_\lambda & (\ell + 1 \le i \le n). \end{cases}$$

A *logarithmic jet differential of* **order** k *and* **degree** m at a point $x \in X$ is a polynomial $Q(f^{(1)}, \dots, f^{(k)})$ on the fiber over x of $J_k(X, -\log D)$ enjoying weighted homogeneity:

$$Q(j_k(f \circ \varphi_\lambda)) = \lambda^m Q(j_k(f)) \quad (\lambda \in \mathbb{C}^*).$$

Consider the symbols

$$\mathrm{d}^j \log z_i \quad (1 \le j \le k,\ 1 \le i \le \ell) \qquad \text{and} \qquad \mathrm{d}^j z_i \quad (1 \le j \le k,\ \ell+1 \le i \le n).$$

Set the weight of $\mathrm{d}^j \log z_i$ or $\mathrm{d}^j z_i$ to be j. Then a logarithmic jet differential of order k and weight k along D at x is a weighted homogeneous polynomial of degree m whose variables are these symbols. Denote by $E_{k,m}^{GG}\, T_X^*(\log D)_x$ be the vector space spanned by such polynomials and set

$$E_{k,m}^{GG}\, T_X^*(\log D) := \bigcup_{x \in X} E_{k,m}^{GG}\, T_X^*(\log D)_x.$$

By Faà di Bruno's formula [3; 15], one can check that $E_{k,m}^{GG}\, T_X^*(\log D)$ carries the structure of a vector bundle over X, called *logarithmic Green–Griffiths vector bundle* [12]. A global section of $E_{k,m}^{GG}\, T_X^*(\log D)$ is called a *logarithmic jet differential* of order k and weight m along D. Locally, a logarithmic jet differential form can be written as

$$\sum_{\substack{\alpha_1,\dots,\alpha_k \in \mathbb{N}^n \\ |\alpha_1|+2|\alpha_2|+\cdots+k|\alpha_k|=m}} A_{\alpha_1,\dots,\alpha_k} \left(\prod_{i=1}^{\ell} (\mathrm{d}\log z_i)^{\alpha_{1,i}} \prod_{i=\ell+1}^{n} (\mathrm{d}z_i)^{\alpha_{1,i}} \right) \cdots \left(\prod_{i=1}^{\ell} (\mathrm{d}^k \log z_i)^{\alpha_{k,i}} \prod_{i=\ell+1}^{n} (\mathrm{d}^k z_i)^{\alpha_{k,i}} \right), \tag{2-2}$$

where

$$\alpha_\lambda = (\alpha_{\lambda,1}, \dots, \alpha_{\lambda,n}) \in \mathbb{N}^n \quad (1 \le \lambda \le k)$$

are multiindices of length

$$|\alpha_\lambda| = \sum_{1 \le i \le n} \alpha_{\lambda,i},$$

and where $A_{\alpha_1,\dots,\alpha_k}$ are locally defined holomorphic functions.

Assigning the weight s for $(\mathrm{d}^s z_i)/z_i$, then one can rewritten $\mathrm{d}^j \log z_i$ as an isobaric polynomial of weight j of variables $(\mathrm{d}^s z_i)/z_i$ $(1 \le s \le j)$ with integer coefficients, namely

$$\mathrm{d}^j \log z_i = \sum_{\substack{\beta=(\beta_1,\dots,\beta_j) \in \mathbb{N}^j \\ \beta_1+2\beta_2+\cdots+j\beta_j=j}} b_{j\beta} \left(\frac{\mathrm{d}z_i}{z_i} \right)^{\beta_1} \cdots \left(\frac{\mathrm{d}^j z_i}{z_i} \right)^{\beta_j},$$

where $b_{j\beta} \in \mathbb{Z}$. Conversely, one can also express $(\mathrm{d}^j z_i)/z_i$ as an isobaric polynomial of weight j of variables $\mathrm{d}^s \log z_i$ $(1 \le s \le j)$ with integer coefficients [2]. Thus one can also use the following trivialization of logarithmic jet differentials:

$$\sum_{\substack{\beta_1,\dots,\beta_k\in\mathbb{N}^n\\ |\beta_1|+2|\beta_2|+\cdots+k|\beta_k|=m}} B_{\beta_1,\dots,\beta_k}\left(\prod_{i=1}^{\ell}\left(\frac{\mathrm{d}z_i}{z_i}\right)^{\beta_{1,i}}\prod_{i=\ell+1}^{n}(\mathrm{d}z_i)^{\beta_{1,i}}\right) \cdots\left(\prod_{i=1}^{\ell}\left(\frac{\mathrm{d}^k z_i}{z_i}\right)^{\beta_{k,i}}\prod_{i=\ell+1}^{n}(\mathrm{d}^k z_i)^{\beta_{k,i}}\right), \tag{2-3}$$

where

$$\beta_\lambda=(\beta_{\lambda,1},\dots,\beta_{\lambda,n})\in\mathbb{N}^n \quad (1\le\lambda\le k)$$

are multiindices of length

$$|\beta_\lambda|=\sum_{1\le i\le n}\beta_{\lambda,i},$$

and where $B_{\beta_1,\dots,\beta_k}$ are locally defined holomorphic functions.

Demailly [5] refined the Green–Griffiths' theory and considered the subbundle $E_{k,m}T_X^*(\log D)$ of $E_{k,m}^{GG}T_X^*(\log D)$, whose sections are logarithmic jet differentials that are invariant under arbitrary reparametrization of the source $\mathbb{C}$. Let

$$(X, D, V)$$

be a *log-direct manifold*, i.e., a triple consisting of a projective manifold X, a simple normal crossing divisor D on X and a holomorphic subbundle V of the logarithmic tangent bundle $T_X(-\log D)$. Starting with a log-direct manifold $(X_0, D_0, V_0) := (X, D, T_X(-\log D))$, one then defines $X_1 := \mathbb{P}(V_0)$ together with the natural projection $\pi_1 : X_1 \to X_0$. Setting $D_1 := \pi_1^* D_0$, so that π_1 becomes a log-morphism, and defines the subbundle $V_1 \subset T_{X_1}(-\log D_1)$ as

$$V_{1,(x,[v])} := \left\{\xi\in T_{X_1,(x,[v])}(-\log D_1) : \pi_*\xi\in\mathbb{C}\cdot v\right\},$$

one obtains the log-direct manifold (X_1, D_1, V_1) from the initial one. Any germ of a holomorphic map $f : (\mathbb{C}, 0) \to (X\setminus D, x)$ can be lifted to $f^{[1]} : \mathbb{C}\to X_1\setminus D_1$. Inductively, one can construct on $X = X_0$ the *Demailly–Semple tower*:

$$(X_k, D_k, V_k)\to\cdots\to(X_1, D_1, V_1)\to(X_0, D_0, V_0),$$

together with the projections $\pi_k : X_k \to X_0$. Denote by $\mathcal{O}_{X_k}(1)$ the tautological line bundle on X_k. Then the direct image $(\pi_k)_*\mathcal{O}_{X_k}(m)$ of $\mathcal{O}_{X_k}(m)=\mathcal{O}_{X_k}(1)^{\otimes m}$, denoted by $E_{k,m}T_X^*(\log D)$, is a locally free subsheaf of $E_{k,m}^{GG}T_X^*(\log D)$ generated by all polynomial operators in the derivatives up to order k, which are furthermore invariant under any change of parametrization $(\mathbb{C}, 0)\to(\mathbb{C}, 0)$. From the construction, one can immediately check that:

Theorem 2.1 (direct image formula). *For any ample line bundle $\mathcal{A}$ on X, one has*

$$H^0(X, E_{k,m}T_X^*(\log D)\otimes\mathcal{A}^{-1})\cong H^0(X_k, \mathcal{O}_{X_k}(m)\otimes\pi_k^*\mathcal{A}^{-1}). \tag{2-4}$$

The bundles $E_{k,m}^{GG} T_X^*(\log D)$, $E_{k,m} T_X^*(\log D)$ are fundamental tools in studying the degeneracy of holomorphic curves into $\mathbb{C} \setminus D$. By the fundamental vanishing theorem of entire curves [5; 21], for any ample line bundle $\mathcal{A}$ on X, a nontrivial global section of $E_{k,m}^{GG} T_X^*(\log D) \otimes \mathcal{A}^{-1}$ gives a corresponding algebraic differential equation that all entire holomorphic function $f : \mathbb{C} \to X \setminus D$ must satisfy. The existence of these sections was proved recently [6; 15], provided that the order of jet is high enough. However, despite many efforts, the problem of controlling the base locus of these bundles can be only handled under the condition that the degree of D must be very large compared with the dimension of the variety [1; 2; 7; 8; 21].

Now we consider the case where D is a generic hypersurface of degree d in $\mathbb{CP}^n$. To guarantee the existence of logarithmic jet differentials along D, we consider the order jet $k = n + 1$ and put

$$k' = \tfrac{1}{2}k(k+1), \quad \delta = (k+1)\,n + k.$$

Fixing two positive integers $\epsilon > 0$ and $r > \delta^{k-1} k(\epsilon + k\delta)$. For a smooth hypersurface D, denote by $Y_k(D)$ the log-Demailly–Semple k-jet tower associated to $(\mathbb{CP}^n, D, T_{\mathbb{CP}^n}(-\log D))$. For a line bundle L on $\mathcal{O}_{Y_k(D)}$, denote by $\mathrm{Bs}(\mathcal{O}_{Y_k(D)} L)$ the base locus of the line bundle L. We will employ the following key result in [2].

Proposition 2.2 [2, Corollary 4.5]. *There exist $\beta, \tilde{\beta} \in \mathbb{N}$ such that for any $\alpha \geq 0$ and for any generic hypersurface $D \in |\mathcal{O}_{\mathbb{CP}^n(1)}^{\epsilon+(r+k)\,\delta}|$, one has*

$$\mathrm{Bs}\big(\mathcal{O}_{Y_k(D)}(\beta + \alpha\,\delta^{k-1} k') \otimes \pi_{0,k}^* \, \mathcal{O}_{\mathbb{CP}^n(1)}^{\tilde{\beta}+\alpha(\delta^{k-1} k(\epsilon+k\delta)-r)}\big) \subset Y_k(D)^{\mathrm{sing}} \cup \pi_{0,k}^{-1}(D).$$

Using this result, Brotbek–Deng confirmed the logarithmic Kobayashi conjecture in the case where the degree of D is large enough. We extract from their proof that:

Theorem 2.3. *Let $D \subset \mathbb{P}^n(\mathbb{C})$ be a generic smooth hypersurface in $\mathbb{P}^n(\mathbb{C})$ having large enough degree*

$$d \geq (n+1)^{n+3}(n+2)^{n+3}.$$

Let $f : \Delta \to \mathbb{CP}^n$ be a nonconstant holomorphic disk. If $f(\Delta) \not\subset D$, then for jet order $k = n+1$, there exist some weighted degree m, vanishing order $\widetilde{m}$ with $\widetilde{m} > 2m$ and some global logarithmic jet differential

$$\mathscr{P} \in H^0\big(\mathbb{CP}^n, E_{k,m}^{GG} T_{\mathbb{CP}^n}^*(\log D) \otimes \mathcal{O}_{\mathbb{CP}^n}(1)^{-\widetilde{m}}\big)$$

such that

$$\mathscr{P}(j_k(f)) \not\equiv 0. \tag{2-5}$$

Proof. We follow the arguments in [2, Corollary 4.9], with a slightly modification to get higher vanishing order. First, putting

$$r_0 = 2\delta^{k-1} k' + \delta^{k-1}(\delta+1)^2 = \delta^{k-1}(\delta+1)(\delta+2).$$

Since

$$k(k+\delta-1+k\delta) < (\delta+1)^2,$$

any integer number $d \geq (r_0+k)\,\delta+2\delta$ can be written as

$$d = \epsilon + (r+k)\,\delta,$$

where $k \leq \epsilon \leq k+\delta-1$ and $r > 2\delta^{k-1}k' + \delta^{k-1}k(\epsilon+k\delta)$. For such d, since

$$\lim_{\alpha\to\infty} \frac{\beta+\alpha\,\delta^{k-1}k'}{-\tilde{\beta}-\alpha(\delta^{k-1}k(\epsilon+k\delta)-r)} = \frac{\alpha\,\delta^{k-1}k'}{r-\delta^{k-1}k(\epsilon+k\delta)} < \tfrac{1}{2},$$

using Proposition 2.2, for $\alpha \gg 1$ large enough, there exists some global logarithmic jet differential

$$\mathscr{P} \in H^0\big(\mathbb{CP}^n,\, E^{GG}_{k,m}\, T^*_{\mathbb{CP}^n}(\log D) \otimes \mathscr{O}_{\mathbb{CP}^n}(1)^{-\widetilde{m}}\big)$$

satisfying (2-5) with $m = \beta+\alpha\,\delta^{k-1}k'$, $\widetilde{m} = -\tilde{\beta}-\alpha(\delta^{k-1}k(\epsilon+k\delta)-r)$ and $\widetilde{m} > 2m$. Hence it remains to giving a lower bound for $(r_0+k)\,\delta+2\delta$. This could be done by a straightforward computation:

$$(r_0+k)\,\delta+2\delta = \big(\delta^{k-1}(\delta+1)(\delta+2)+k+2\big)\,\delta < (n+1)^{n+3}(n+2)^{n+3}. \quad \square$$

3. Value distribution theory for holomorphic maps from unit disc into projective spaces

Let $E = \sum_i \alpha_i\, a_i$ be a divisor on the unit disc Δ where $\alpha_i \geq 0$, $a_i \in \Delta$ and let $k \in \mathbb{N}\cup\{\infty\}$. For each $0<t<1$, denote by Δ_t the disk $\{z\in\mathbb{C},\, |z|<t\}$. Summing the k-truncated degrees of the divisor on disks by

$$n^{[k]}(t, E) := \sum_{a_i\in\Delta_t} \min\{k, \alpha_i\} \quad (0<t<1),$$

the *truncated counting function* at level k of E is then defined by taking the logarithmic average

$$N^{[k]}(r, E) := \int_0^r \frac{n^{[k]}(t, E)}{t}\,\mathrm{d}t \quad (0<r<1).$$

When $k=\infty$, we write $n(t,E)$, $N(r,E)$ instead of $n^{[\infty]}(t,E)$, $N^{[\infty]}(r,E)$. Let $f:\Delta\to\mathbb{CP}^n$ be an entire curve having a reduced representation $f=[f_0:\cdots:f_n]$ in the homogeneous coordinates $[z_0:\cdots:z_n]$ of $\mathbb{CP}^n$. Let $D=\{Q=0\}$ be a divisor in $\mathbb{CP}^n$ defined by a homogeneous polynomial $Q\in\mathbb{C}[z_0,\ldots,z_n]$ of degree $d\geq 1$. If $f(\Delta)\not\subset D$, we define the *truncated counting function* of f with respect to D as

$$N^{[k]}_f(r, D) := N^{[k]}(r, (Q\circ f)_0),$$

where $(Q\circ f)_0$ denotes the zero divisor of $Q\circ f$.

The *proximity function* of f for the divisor D is defined as

$$m_f(r, D) := \int_0^{2\pi} \log \frac{\|f(re^{i\theta})\|^d \|Q\|}{|Q(f)(re^{i\theta})|} \frac{\mathrm{d}\theta}{2\pi},$$

where $\|Q\|$ is the maximum absolute value of the coefficients of Q and

$$\|f(z)\| = \max\{|f_0(z)|, \dots, |f_n(z)|\}.$$

Since $|Q(f)| \leq \|Q\| \cdot \|f\|^d$, one has $m_f(r, D) \geq 0$.

Lastly, the *Cartan order function* of f is defined by

$$T_f(r) := \frac{1}{2\pi} \int_0^{2\pi} \log\|f(re^{i\theta})\| \,\mathrm{d}\theta.$$

With the above notations, the Nevanlinna theory consists of two fundamental theorems (for comprehensive presentations, see [17; 20]).

Theorem 3.1 (First Main Theorem). *Let $f : \Delta \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve and let D be a hypersurface of degree d in $\mathbb{CP}^n$ such that $f(\Delta) \not\subset D$. Then for every $r > 1$, the following holds*:

$$m_f(r, D) + N_f(r, D) = d\, T_f(r) + O(1),$$

whence

$$N_f(r, D) \leq d\, T_f(r) + O(1). \tag{3-1}$$

On the other side, in the harder part, so-called Second Main Theorem, one tries to bound the order function from above by some sum of certain counting functions. Such types of results were given in several situations, and most of them were relied on the following key estimate.

Theorem 3.2 (logarithmic derivative lemma). *Let g be a nonconstant meromorphic function on the unit disc and let $k \geq 1$ be a positive integer number. Then for any $0 < r < 1$, the following estimate holds*:

$$m_{g^{(k)}/g}(r) := m_{g^{(k)}/g}(r, \infty) = O\Big(\log \frac{1}{1-r}\Big) + O(\log T_g(r)) \qquad \|,$$

where the notation $\|$ means that the above estimate holds true for all $0 < r < 1$ outside a subset $E \subset (0, 1)$ with

$$\int_E \frac{dr}{1-r} < \infty.$$

4. An application of the logarithmic derivative lemma

It is a well-known fact that the growth of the order function of an entire holomorphic curve could be used to determine its rationality. Replacing the source of the curve by the unit disc Δ, one has:

Definition 4.1. A holomorphic map $f : \Delta \to \mathbb{CP}^n$ is said to be transcendental if

$$\limsup_{r\to 1} \frac{T_f(r)}{\log \frac{1}{1-r}} = \infty.$$

Theorem 4.2. *Let $f : \Delta \to \mathbb{CP}^n$ be a holomorphic map and $D \subset \mathbb{CP}^n$ be a generic hypersurface having large enough degree*:

$$d \geq (n+1)^{n+3}(n+2)^{n+3}.$$

If f avoids D, then it is not transcendental.

Proof. Employing the logarithmic jet differentials supplied by Theorem 2.3, following the arguments as in [13] and using the logarithmic derivative lemma for meromorphic functions on unit disc, one gets

$$T_f(r) \leq N_f^{[1]}(r, D) + O\Big(\log \frac{1}{1-r}\Big) + O(\log T_f(r)) = O\Big(\frac{1}{1-r}\Big) + O(\log T_f(r)) \quad \|,$$

whence concludes the proof. □

We will also need the following results due to Fujimoto [9].

Proposition 4.3 [9, Proposition 2.5]. *Let φ be a nowhere zero holomorphic function on Δ which is not transcendental. Then, for each positive integer number λ, the following estimate holds*:

$$\int_0^{2\pi} \left| \frac{d^{\lambda-1}}{dz^{\lambda-1}}\Big(\frac{\varphi'}{\varphi}\Big)(re^{i\theta}) \right| d\theta \leq \frac{\text{Const.}}{(1-r)^{\lambda}} \log \frac{1}{1-r} \quad (0 < r < 1).$$

Corollary 4.4 [9, Lemma 3.4]. *Let $\varphi_1, \ldots, \varphi_n$ be nowhere zero holomorphic functions on Δ which are not transcendental. Then, for any n-tuple of positive integer numbers $(\lambda_1, \ldots, \lambda_n)$ and for any positive real number t with $tn < 1$, the following estimate holds*:

$$\int_0^{2\pi} \left| \prod_{j=1}^{n} \Big(\frac{\varphi_j'}{\varphi_j}\Big)^{(\lambda_j-1)}(re^{i\theta}) \right|^t d\theta \leq \frac{\text{Const.}}{(1-r)^s} \Big(\log \frac{1}{1-r}\Big)^s \quad (0 < r < 1),$$

where $s = t\big(\sum_{j=1}^n \lambda_j\big)$.

5. Proof of the Main result

Proposition 5.1. *Let $D \subset \mathbb{CP}^n$ be a generic smooth hypersurface of degree d and let $f : \Delta \to \mathbb{CP}^n \setminus D$ be a nondegenerate holomorphic curve. Suppose that there exists a global logarithmic jet differential*

$$\mathscr{P} \in H^0\big(\mathbb{P}^n(\mathbb{C}),\, E^{GG}_{n,m}\, T^*_{\mathbb{CP}^n}(\log D) \otimes \mathcal{O}_{\mathbb{CP}^n}(1)^{-\widetilde{m}}\big)$$

such that

$$\mathscr{P}(j_n(f)) \not\equiv 0. \tag{5-1}$$

Then, there exists a positive constant K such that

$$\int_0^{2\pi} \big|\mathscr{P}(j_n(f))(re^{i\theta})\big|^{2/\widetilde{m}}\, \|f(re^{i\theta})\|^2\, d\theta \le \frac{K}{(1-r)^{2m/\widetilde{m}}}\left(\log \frac{1}{1-r}\right)^{2m/\widetilde{m}}$$

for $(0 < r < 1)$.

Proof. Let s be the canonical section of the ample line bundle $\mathcal{E} := \mathcal{O}_{\mathbb{CP}^n}(1)$. Since $\mathscr{P}$ vanishes on $\mathcal{E}$ with vanishing order $\widetilde{m}$, in any local chat U_α of $\mathbb{CP}^n$, one can represent $\mathscr{P}s^{\widetilde{m}}$ as an isobaric polynomial $\mathscr{P}^\alpha_s$ of weight m of variables

$$\frac{d^\lambda u^\alpha_{j,\lambda}}{u^\alpha_{j,\lambda}} \quad (1 \le \lambda \le k,\ 1 \le j \le n),$$

with local holomorphic coefficients, where $u_{j,\lambda}$ are rational functions on $\mathbb{CP}^n$. Consequently, we get that

$$|\mathscr{P}(j_k(f))| \cdot \|f\|^{\widetilde{m}} \le \sum_\alpha \left|\mathscr{P}^\alpha_s\left(\frac{d^\lambda(u^\alpha_{j,\lambda} \circ f)}{u^\alpha_{j,\lambda} \circ f}\right)\right|.$$

Since $0 < \frac{2}{\widetilde{m}} < 1$, using the elementary inequality

$$(x_1 + \cdots + x_r)^{2/\widetilde{m}} < x_1^{2/\widetilde{m}} + \cdots + x_r^{2/\widetilde{m}} \quad (x_i > 0),$$

the above estimate yields

$$\big|\mathscr{P}(j_n(f))(re^{i\theta})\big|^{2/\widetilde{m}}\, \|f(re^{i\theta})\|^2 < \sum_\alpha \left|\mathscr{P}^\alpha_s\left(\frac{d^\lambda(u^\alpha_{j,\lambda} \circ f)}{u^\alpha_{j,\lambda} \circ f}\right)\right|^{2/\widetilde{m}}.$$

Hence it suffices to prove

$$\int_0^{2\pi} \left|\mathscr{P}^{2/\widetilde{m}}_s\left(\frac{d^\lambda(u^\alpha_{j,\lambda} \circ f)}{u^\alpha_{j,\lambda} \circ f}\right)\right|^{2m/\widetilde{m}} d\theta \le \frac{\text{Const.}}{(1-r)^{2m/\widetilde{m}}}\left(\log \frac{1}{1-r}\right)^{2m/\widetilde{m}} \quad (0 < r < 1).$$

By assumption, f avoids D, hence it is not transcendental by Theorem 4.2. Since each function $u^\alpha_{j,\lambda}$ is rational, it follows that $u^\alpha_{j,\lambda} \circ f$ is also not transcendental.

Now, observing that each term

$$\frac{d^\lambda(u^\alpha_{j,\lambda}\circ f)}{u^\alpha_{j,\lambda}\circ f}$$

can be represented as a polynomial $\mathscr{P}^\alpha_{j,\lambda}$ of variables

$$\frac{(u^\alpha_{j,\lambda}\circ f)'}{u^\alpha_{j,\lambda}\circ f},\dots,\left(\frac{(u^\alpha_{j,\lambda}\circ f)'}{u^\alpha_{j,\lambda}\circ f}\right)^{\lambda-1},$$

which is isobaric of weight λ, using Corollary 4.4, one gets the desired result. □

We will also need the following result of Yau [23] in the sequence.

Theorem 5.2 [23]. *Let M be a complete Riemann manifold equipped with a volume form $d\sigma$. Let h be a nonnegative and nonconstant smooth function on M such that* $\Delta\log h = 0$ *almost everywhere. Then $\int_M h^p\,d\sigma=\infty$ for any $p>0$.*

Now we enter the details of the proof of the Main Theorem. Let f be the conjugate of G, which is a holomorphic map. It suffices to prove that f is constant. Suppose on the contrary that this is not the case. Let $\pi:\widetilde{M}\to M$ be the universal covering of M. Then $\widetilde{M}$ is also considered as a minimal surface in $\mathbb{R}^n$. Hence without lost of generality, we may assume $M=\widetilde{M}$. Since there is no compact minimal surface in $\mathbb{R}^n$, it follows that M is biholomorphic to either $\mathbb{C}$ or Δ. Thus we may assume $M=\mathbb{C}$ or $M=\Delta$. The first case was excluded by recent work towards Kobayashi's conjecture (see [2]). Hence it suffices to work in the case where $M=\Delta$. The area form of the metric on M induced from the flat metric on $\mathbb{R}^n$ is given by

$$d\sigma = 2\|f\|^2\,du\wedge dv.$$

Let $\mathscr{P}$ be a global logarithmic jet differential supplied by Theorem 2.3. Then it is clear that $h=|\mathscr{P}(j_k(f)|\not\equiv 0$ and $\Delta\log h=0$ for any z out side the zero set of h. Since Δ is complete, simply connected and of nonpositive curvature, it has the infinite area with respect to the metric induced from $\mathbb{R}^n$. Using Theorem 5.2, one obtains that

$$\int_\Delta h^{2/\widetilde{m}}\,d\sigma=\infty. \tag{5-2}$$

On the other hand, using Proposition 5.1, one has

$$\begin{aligned}\int_\Delta h^{2/\widetilde{m}}\,d\sigma &= 2\int_\Delta h^{2/\widetilde{m}}\,\|f\|^2\,du\,dv\\ &= 2\int_0^1 r\,dr\left(\int_0^{2\pi} h(re^{i\theta})^{2/\widetilde{m}}\,\|f(re^{i\theta})\|^2\,d\theta\right)\\ &\le K\int_0^1\frac{r}{(1-r)^{2m/\widetilde{m}}}\left(\log\frac{1}{1-r}\right)^{2m/\widetilde{m}}dr.\end{aligned}$$

The last integral in the above estimate is finite since $2m < \widetilde{m}$. This contradicts (5-2). Therefore, the map f must be constant, whence concludes the proof of the Main Theorem.

6. Some discussions

Theorem 1.1 can be recovered via the above jet method. Indeed, according to Siu [21], the Wronskian can be employed to build a suitable logarithmic jet differentials. Precisely, let us consider the inhomogeneous coordinates $x_1, x_2, \ldots, x_n$ of $\mathbb{CP}^n$. Let $\{H_i\}_{1\leq i\leq q}$ be the family of hyperplanes in general position in $\mathbb{CP}^n$. For each $1 \leq i \leq q$, denote by F_i the linear form of variables $x_1, \ldots, x_n$ defining the hyperplane H_i. Put

$$\omega = \frac{\mathrm{Wron}(dx_1, \ldots, dx_n)}{F_1 \ldots F_q},$$

where Wron denotes the Wronskian. The point is that by the assumption of general position, at any point $x = (x_1, \ldots, x_n)$, there exists a set $I = \{i_1, \ldots, i_n\}$ having cardinality n such that F_j are nowhere zero in a neighborhood U of x for all $j \notin I$. Locally on U, one can write ω as

$$\omega = \text{Const.}\ \frac{\mathrm{Wron}(d\log F_{i_1}(x), \ldots, d\log F_{i_n}(x))}{\prod_{j\notin I} F_j(x)},$$

and hence, ω gives rise to a logarithmic jet differentials along the divisor $\sum_{i=1}^{q} H_i$. The denominator $F_1 \ldots F_q$ in ω gives the vanishing order q at the infinity hyperplane, hence direct computation yields immediately that ω is of weight $m = \frac{1}{2}n(n+1)$ and vanishes on the infinity hyperplane with the vanishing order $\widetilde{m} = q - (n+1)$.

Finally, in view of the result of Fujimoto–Ru, one can expect that the optimal degree bound in the statement of our Main Theorem should be $\frac{1}{2}n(n+1)$.

Conjecture 6.1. *Let M be a nonflat complete minimal surface in $\mathbb{R}^n$ and let $G : M \to \mathbb{CP}^{n-1}$ be its generalized Gauss map. Then G could avoid a generic hypersurface $D \subset \mathbb{CP}^{n-1}$ of degree at most*

$$d = \tfrac{1}{2}n(n+1).$$

Acknowledgements

This work is supported by the Vietnam Ministry of Education and Training under the grant number B2024-DHH-14. I would like to thank Prof. Doan The Hieu for his encouragements. I want to thank Song-Yan Xie for helpful suggestions which improved the exposition. I warmly thank the referee for the careful reading of the first version of the manuscript and for the comments that helped me clarify the presentation.

References

[1] G. Bérczi, "Towards the Green–Griffiths–Lang conjecture via equivariant localisation", *Proc. Lond. Math. Soc.* (3) **118**:5 (2019), 1057–1083. MR Zbl

[2] D. Brotbek and Y. Deng, "Kobayashi hyperbolicity of the complements of general hypersurfaces of high degree", *Geom. Funct. Anal.* **29**:3 (2019), 690–750. MR Zbl

[3] G. M. Constantine and T. H. Savits, "A multivariate Faà di Bruno formula with applications", *Trans. Amer. Math. Soc.* **348**:2 (1996), 503–520. MR Zbl

[4] L. Darondeau, "On the logarithmic Green–Griffiths conjecture", *Int. Math. Res. Not.* **2016**:6 (2016), 1871–1923. MR Zbl

[5] J.-P. Demailly, "Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials", pp. 285–360 in *Algebraic geometry* (Santa Cruz, 1995), Proc. Sympos. Pure Math. **62(2)**, Amer. Math. Soc., Providence, RI, 1997. MR Zbl

[6] J.-P. Demailly, "Hyperbolic algebraic varieties and holomorphic differential equations", *Acta Math. Vietnam.* **37**:4 (2012), 441–512. MR Zbl

[7] J.-P. Demailly, "Recent results on the Kobayashi and Green–Griffiths–Lang conjectures", *Jpn. J. Math.* **15**:1 (2020), 1–120. MR Zbl

[8] S. Diverio, J. Merker, and E. Rousseau, "Effective algebraic degeneracy", *Invent. Math.* **180**:1 (2010), 161–223. MR Zbl

[9] H. Fujimoto, "On the Gauss map of a complete minimal surface in $\mathbf{R}^m$", *J. Math. Soc. Japan* **35**:2 (1983), 279–288. MR Zbl

[10] H. Fujimoto, "On the number of exceptional values of the Gauss maps of minimal surfaces", *J. Math. Soc. Japan* **40**:2 (1988), 235–247. MR Zbl

[11] H. Fujimoto, "Modified defect relations for the Gauss map of minimal surfaces, II", *J. Differential Geom.* **31**:2 (1990), 365–385. MR Zbl

[12] M. Green and P. Griffiths, "Two applications of algebraic geometry to entire holomorphic mappings", pp. 41–74 in *The Chern Symposium* 1979: *Proc. Internat. Sympos.* (Berkeley, Calif., 1979), Springer, New York, 1980. MR Zbl

[13] D. T. Huynh, D.-V. Vu, and S.-Y. Xie, "Entire holomorphic curves into projective spaces intersecting a generic hypersurface of high degree", *Ann. Inst. Fourier* (*Grenoble*) **69**:2 (2019), 653–671. MR Zbl

[14] S. Iitaka, *Algebraic geometry*: *an introduction to birational geometry of algebraic varieties*, Graduate Texts in Mathematics **76**, Springer, New York, 1982. MR Zbl

[15] J. Merker, "Algebraic differential equations for entire holomorphic curves in projective hypersurfaces of general type: optimal lower degree bound", pp. 41–142 in *Geometry and analysis on manifolds*, Progr. Math. **308**, Birkhäuser, Cham, 2015. MR Zbl

[16] J. Noguchi, "Logarithmic jet spaces and extensions of de Franchis' theorem", pp. 227–249 in *Contributions to several complex variables*, Aspects Math. **E9**, Friedr. Vieweg, Braunschweig, 1986. MR Zbl

[17] J. Noguchi and J. Winkelmann, *Nevanlinna theory in several complex variables and Diophantine approximation*, Fundamental Principles of Mathematical Sciences **350**, Springer, Tokyo, 2014. MR Zbl

[18] R. Osserman, "Global properties of minimal surfaces in E^3 and E^n", *Ann. of Math.* (2) **80** (1964), 340–364. MR Zbl

[19] M. Ru, "On the Gauss map of minimal surfaces immersed in $\mathbf{R}^n$", *J. Differential Geom.* **34**:2 (1991), 411–423. MR Zbl

[20] M. Ru, *Nevanlinna theory and its relation to Diophantine approximation*, 2nd ed., World Scientific Publishing, Hackensack, NJ, 2021. MR Zbl

[21] Y.-T. Siu, "Hyperbolicity of generic high-degree hypersurfaces in complex projective space", *Invent. Math.* **202**:3 (2015), 1069–1166. MR Zbl

[22] F. Xavier, "The Gauss map of a complete nonflat minimal surface cannot omit 7 points of the sphere", *Ann. of Math.* (2) **113**:1 (1981), 211–214. MR Zbl

[23] S. T. Yau, "Some function-theoretic properties of complete Riemannian manifold and their applications to geometry", *Indiana Univ. Math. J.* **25**:7 (1976), 659–670. MR Zbl

Received January 13, 2024. Revised March 14, 2024.

DINH TUAN HUYNH
DEPARTMENT OF MATHEMATICS
HUE UNIVERSITY OF EDUCATION
HUE UNIVERSITY
HUE CITY
VIETNAM

dinhtuanhuynh@hueuni.edu.vn

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2024 is US $645/year for the electronic version, and $875/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 330 No. 1 May 2024