Gorenstein singularities of 0-dimensional schemes in generic position

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Abstract

Given a 0-dimensional scheme X in generic position in $\mathbb{P}^n$ over a field K, we prove some characterizations of the Gorenstein property of X in terms of its colength and conductor.

KEYWORDS: 0-dimensional scheme, generic position, Gorenstein property, conductor

1 Introduction

Let K be an arbitrary infinite field, and let $\mathbb{P}^n$ be the projective n-space over K. We are interested in studying of a 0-dimensional scheme X in $\mathbb{P}^n$. In particular, we would like to examine closely the Gorenstein singularities of the scheme X. The Gorenstein property is very well-known and has been investigated in many decades (see e.g. [1], [3], [2], [6], [5], [4], [10]).

By I_X we denote the homogeneous vanishing ideal of X in the standard graded polynomial ring $P = K[x_0, \dots, x_n]$, where $\deg(x_0) = \dots = \deg(x_n) = 1$. Then the homogeneous coordinate ring of X is $R = P/I_X$. The ring R is a 1-dimensional Cohen-Macaulay ring. Because K is infinite, after a change of coordinates, we may assume that x_0 is a non-zerodivisor of R. Here the image of x_i in R is also denoted by x_i for $i = 0, \dots, n$. Note that the localization R_{x_0} of R at x_0 is also a graded ring. Set $\widetilde{R} = \bigoplus_{i \ge 0}(R_{x_0})_i$. The natural map $R \to R_{x_0}$ embeds R as a subring of the graded ring $\widetilde{R}$. The *conductor* of R in $\widetilde{R}$ is the ideal $\mathfrak{F}_{\widetilde{R}/R} = \{f \in R_{x_0} \mid f \cdot \widetilde{R} \subseteq R\}$. Under this terminology and Definition 2.3, we prove the following characterization of the Gorenstein property of X.

Theorem 1.1 (Theorem 3.3). *The scheme X is Gorenstein if and only if it is locally Gorenstein, $\mathfrak{F}_{\widetilde{R}/R} = \oplus_{i \ge r_X} R_i$, and $\ell(\widetilde{R}/R) = \ell(R/\mathfrak{F}_{\widetilde{R}/R})$, where "$\ell$" denotes length (or dimension) as K-vector space.*

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Based on this theorem, we may characterize the Gorenstein property of X when X is in generic position, as follows.

Theorem 1.2 (Theorem 3.4)**.** *Suppose X is locally Gorenstein and in generic position. Then the following conditions are equivalent.*

(a) X is Gorenstein.

(b) $\mathfrak{F}_{\widetilde{R}/R} = \oplus_{i \geq r_X} R_i$, $r_X d_X = 2\binom{n+r_X}{n+1}$.

(c) $\mathfrak{F}_{\widetilde{R}/R} = \oplus_{i \geq r_X} R_i$, $d_X = 2$ or $d_X = n+2$.

Theorem 1.3 (Theorem 3.5)**.** *Suppose that X is locally Gorenstein with minimal conductor, but not in generic position, and that there is a subset Y in generic position with $d_Y = d_X - 1$. Then X is Gorenstein if and only if*

$$(r_X - 2)d_X = 2\binom{n + r_X - 1}{n+1} - 2.$$

In the case that X is a finite set of points in $\mathbb{P}^n$, the two last theorems cover some of main results given in [5].

2 Preliminary

Our subjects of study are 0-dimensional schemes X in the projective n-space $\mathbb{P}^n$ over the field K. Let $\mathrm{Supp}(X) = \{p_1, ..., p_s\}$ be the set of all closed points of X, and let $I_X = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$ be the irredundant primary decomposition of I_X, where $\mathfrak{q}_j$ be the homogeneous primary ideal associated to the point p_i for $i = 1, ..., s$. The homogeneous coordinate ring of X is given by $R = P/(\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s)$. Since K is infinite, there is a linear form $\ell \in P$ such that $\ell \notin \sqrt{\mathfrak{q}_j}$ for all $j = 1, ..., s$ (see [9, Proposition 6.3.20]). By changing coordinates, we may assume that $\ell = x_0$. Set $S := R/\langle x_0 - 1 \rangle \cong P/(I_X + \langle x_0 - 1 \rangle)$. Then the ring S is a K-vector space of finite dimension. The dimension $d_X := \dim_K(S)$ is also called the *degree* of X.

The Hilbert function of X is a map $\mathrm{HF}_X : \mathbb{Z} \to \mathbb{N}$ given by $\mathrm{HF}_X(i) = \dim_K(R_i)$. We have $\mathrm{HF}_X(i) = 0$ if $i < 0$ and

$$1 = \mathrm{HF}_X(0) < \mathrm{HF}_X(1) < \cdots < d_X$$

and there exists a number r_X, called the *regularity index* of X, such that $\mathrm{HF}_X(r_X - 1) < d_X$ and $\mathrm{HF}_X(i) = d_X$ for all $i \geq r_X$. The Hilbert function of X is *symmetric* if

$$\mathrm{HF}_X(i-1) + \mathrm{HF}_X(r_X - i) = d_X$$

for all $i \in \mathbb{Z}$.

Definition 2.1. We say that the scheme X is *in generic position*, if we have

$$\mathrm{HF}_X(i) = \min\{\, d_X,\ \tbinom{n+i}{n} \,\}$$

for all $i \in \mathbb{Z}$.

When $n = 1$ or $d_X = 1$, it is easy to see that X is in generic position. In the following, we omit these cases by assuming that $n, d_X \geq 2$. Let

$$\alpha_X := \min\{i \in \mathbb{N} \mid (I_X)_i \neq 0\}$$

be the initial degree of I_X. Due on [4, Proposition 1.1], we have the following lemma.

Lemma 2.2. *The scheme X is in generic position if and only if $\alpha_X = r_X$. In this case I_X can be generated by polynomials of degree α_X and $\alpha_X + 1$, and α_X is the unique integer such that*

$$\binom{n+\alpha_X-1}{n} \leq d_X < \binom{n+\alpha_X}{n}.$$

Next, let us introduce briefly to the Gorenstein property of X. For $j = 1, ..., s$, let $\overline{\mathfrak{q}}_j$ be the image of $\mathfrak{q}_j$ under the canonical map $\pi : R \to S = R/\langle\, x_0 - 1 \,\rangle$. The ring $S/\overline{\mathfrak{q}}_j$ is a 0-dimensional local ring for all $j = 1, ..., s$. Moreover, the graded ring $\overline{R} := R/\langle\, x_0 \,\rangle$ is also a 0-dimensional local ring and it is isomorphic to the associated graded ring $\mathrm{gr}(S)$ of S with respect to the maximal ideal $\langle\, x_1, ..., x_n \,\rangle$.

Definition 2.3. (a) A 0-dimensional local ring $(T, \mathfrak{n})$ is called a *Gorenstein local ring* if $\dim_{T/\mathfrak{n}}(0 : \mathfrak{n}) = 1$.

(b) X is called *locally Gorenstein* if $S/\overline{\mathfrak{q}}_j$ is a Gorenstein local ring for $j = 1, ..., s$.

(c) X is called *(arithmetically) Gorenstein* if $\overline{R}$ is a Gorenstein local ring.

(d) X is called a *complete intersection* if I_X can be generated by n homogeneous polynomials.

It is well-known (see e.g. [8]) that any complete intersection is Gorenstein and any Gorenstein scheme is locally Gorenstein. Moreover, a Gorenstein scheme in $\mathbb{P}^2$ is also a complete intersection (see []).

Lemma 2.4. *There does not exist a Gorenstein set X in $\mathbb{P}^2$ such that:*

(a) $17 \leq d_X \leq 24$;

(b) X is not in generic position;

(c) there is a subset $Y \subseteq X$ in generic position with $d_Y = d_X - 1$.

Proof. Suppose X is a Gorenstein scheme with properties (a)-(c). Then X is also a complete intersection, and so I_X is generated by two homogeneous polynomials of degrees d_1 and d_2 with $1 \leq d_1 \leq d_2$. Then $d_X = d_1 d_2$. Since $d_Y = d_X - 1 > 15$ and Y is in generic position, we have $\alpha_Y > 4$. So, we have $(I_X)_4 \subseteq (I_Y)_4 = 0$. The condition $17 \leq d_X = d_1 d_2 \leq 24$ implies that $d_1 \leq 4$, and consequently we get $(I_X)_4 \neq 0$, a contradiction. □

Notice that $K[x_0]$ is a Noetherian normalization of the ring R.

Definition 2.5. The graded R-module

$$\omega_R = \mathrm{Hom}_{K[x_0]}(R, K[x_0])(-1)$$

is called the *canonical module* of R (or of X).

The canonical module ω_R is finitely generated and its Hilbert function satisfies

$$\mathrm{HF}_{\omega_R}(i) = d_X - \mathrm{HF}_X(-i)$$

for all $i \in \mathbb{Z}$. The following characterization of the Gorenstein property of X can be found in [6, Proposition 2.1.3].

Proposition 2.6. *The scheme X is Gorenstein if and only if $\omega_R \cong R(r_X - 1)$. In this case HF_X is symmetric.*

Next, let R_{x_0} be the graded localization of R at x_0 and let $\widetilde{R} = \bigoplus_{i\geq 0}(R_{x_0})_i$. Note that $R_{x_0}, \widetilde{R}$ are graded rings and $\imath : R \to R_{x_0}, f \mapsto f/1$, is an injection with $\mathrm{Im}(\imath) \subseteq \widetilde{R}$. In particular, we can identify R with its image in R_{x_0}.

Definition 2.7. The *conductor* of R in $\widetilde{R}$ is the ideal

$$\mathfrak{F}_{\widetilde{R}/R} = \{f \in R_{x_0} \mid f \cdot \widetilde{R} \subseteq R\}.$$

The conductor $\mathfrak{F}_{\widetilde{R}/R}$ is a homogeneous ideal of both R and $\widetilde{R}$. When $\mathfrak{q}_j$ is generated by linear forms (i.e., p_j is K-rational) for $j = 1, ..., s$, the ring $\widetilde{R}$ is exactly the integral closure of R in its full quotient ring (see [5, Thm. 2]).

Proposition 2.8. *(a) We have $\mathrm{HF}_{\widetilde{R}}(i) = d_X$ for all $i \geq 0$.*

(b) We have $\mathrm{HF}_{\mathfrak{F}_{\widetilde{R}/R}}(i) \leq \mathrm{HF}_X(i)$ for all $i \in \mathbb{Z}$ and $\mathrm{HF}_{\mathfrak{F}_{\widetilde{R}/R}}(i) = d_X$ for all $i \geq r_X$.

(c) $\mathfrak{F}_{\widetilde{R}/R} = \oplus_{i\geq r_X} R_i$ if and only if $\mathrm{Ann}_R(\omega_R)_{-r_X+1} = 0$.

Proof. (a) This follows from the fact that $x_0^{-k} R_{i+k} \subseteq \widetilde{R}_i$, $\mathrm{HF}_X(i) = d_X$ for $i \geq r_X$ and x_0 is a nonzero-divisor of R_{x_0}.
(b) Since $\mathfrak{F}_{\widetilde{R}/R}$ is a homogeneous ideal of R, the first part of (b) holds true. For the second part of (b), it suffices to show that $R_{r_X} \subseteq \mathfrak{F}_{\widetilde{R}/R}$. Let $f \in R_{r_X}$ and $g \in (\widetilde{R})_i$ be nonzero elements with $i \geq 0$. We write $g = h/x_0^k$ with $h \in R_{i+k}$ with $k \geq 0$. Then $fh \in R_{r_X+i+k} = x_0^{i+k} R_{r_X}$, and so there is $f' \in R_{r_X}$ such that $fh = x_0^{i+k} f'$. This implies that $fg = fh/x_0^k = x_0^i f' \in R_{r_X+i}$. Hence $f \in \mathfrak{F}_{\widetilde{R}/R}$, as wanted.
(c) This follows from [7, Thm. 5.4] and [8, Thm. 5.6]. □

Remark 2.9. If X satisfies (c) of Proposition 2.8, then X attains the minimal conductor and it is also known that X has the CB-property (see [8]).

3 Main Results

In this section we continue using the notation introduced in the previous section.

Definition 3.1. The number $\ell(R/\mathfrak{F}_{\widetilde{R}/R})$ is called the *conductor colength* of X, where "ℓ" denotes length (or dimension) as K-vector space.

The lengths $\ell(R/\mathfrak{F}_{\widetilde{R}/R})$ and $\ell(\widetilde{R}/R)$ are finite, since $R_{r_X} = \widetilde{R}_{r_X} = (\mathfrak{F}_{\widetilde{R}/R})_{r_X}$ by Proposition 2.8. In particular, we have the following relation between them.

Lemma 3.2. *We have*

$$\ell(R/\mathfrak{F}_{\widetilde{R}/R}) = \ell(\widetilde{R}/\mathfrak{F}_{\widetilde{R}/R}) - \ell(\widetilde{R}/R).$$

Proof. This follows from the exact sequence

$$0 \longrightarrow R/\mathfrak{F}_{\widetilde{R}/R} \longrightarrow \widetilde{R}/\mathfrak{F}_{\widetilde{R}/R} \longrightarrow \widetilde{R}/R \longrightarrow 0.$$

$\square$

Furthermore, we have the following characterization of the Gorenstein property of X.

Theorem 3.3. *X is Gorenstein if and only if the following conditions are satisfied:*

(a) X is locally Gorenstein;

(b) $\mathfrak{F}_{\widetilde{R}/R} = \oplus_{i\geq r_X} R_i$;

(c) $\ell(\widetilde{R}/R) = \ell(R/\mathfrak{F}_{\widetilde{R}/R})$.

Proof. Suppose that X is Gorenstein. Then X is clearly locally Gorenstein. By Proposition 2.6, there is $\varphi \in (\omega_R)_{-r_X+1}$ such that $\omega_R = \varphi \cdot R$ and $\mathrm{Ann}_R(\varphi) = 0$, and so $\mathrm{Ann}_R(\omega_R)_{-r_X+1} = 0$. Proposition 2.8.c yields that $\mathfrak{F}_{\widetilde{R}/R} = \oplus_{i\geq r_X} R_i$. It follows that $\ell(R/\mathfrak{F}_{\widetilde{R}/R}) = \sum_{i=0}^{r_X-1} \mathrm{HF}_X(i)$. We also have

$$\ell(\widetilde{R}/R) = \sum_{i=0}^{r_{\mathbb{X}}-1} (d_X - \mathrm{HF}_X(i)).$$

Since HF_X is symmetric by Proposition 2.6, Lemma 3.2 yields $\ell(R/\mathfrak{F}_{\widetilde{R}/R}) = \ell(\widetilde{R}/R)$.

Conversely, suppose that conditions (a)-(c) are satisfied. Condition (b) implies

$$\mathrm{Ann}_R(\omega_R)_{-r_X+1} = 0.$$

Since K is infinite, we find $\varphi \in (\omega_R)_{-r_X+1}$ such that $\mathrm{Ann}_R(\varphi) = 0$. In particular, we have an injection $\mu_\varphi : R(r_X - 1) \to \omega_R$ given by $\mu_\varphi(f) = f \cdot \varphi$. This shows that

$$\mathrm{HF}_X(i) \leq d_X - \mathrm{HF}_X(r_X - 1 - i)$$

for all $i \in \mathbb{Z}$. So, conditions (b) and (c) yields that HF_X is symmetric. Hence the map μ_φ is an isomorphism. Therefore X is Gorenstein by Proposition 2.6. $\square$

Now we apply previous results to prove the following theorems.

Theorem 3.4. *Suppose X is locally Gorenstein and in generic position. Then the following conditions are equivalent.*

(a) X is Gorenstein.

(b) $\mathfrak{F}_{\widetilde{R}/R} = \oplus_{i\geq r_X} R_i$, $r_X d_X = 2\binom{n+r_X}{n+1}$.

(c) $\mathfrak{F}_{\widetilde{R}/R} = \oplus_{i\geq r_X} R_i$, $d_X = 2$ or $d_X = n+2$.

Proof. Since X is in generic position, we have

$$\sum_{i=0}^{r_X-1} \mathrm{HF}_X(i) = \sum_{i=0}^{r_X-1} \binom{n+i}{n} = \binom{n+r_X}{n+1}.$$

The equivalence of (a) and (b) follows from Lemma 3.3 and the fact that $\ell(\widetilde{R}/R) = \ell(R/\mathfrak{F}_{\widetilde{R}/R})$ is equivalent to

$$r_X d_X = 2\binom{n+r_X}{n+1}.$$

For the equivalence of (b) and (c), we need to verify that $r_X d_X = 2\binom{n+r_X}{n+1}$ if and only if $d_X = 2$ or $d_X = n+1$. We distinguish two cases.

(c1) If $d_X = \binom{n+r_X}{n}$ then $r_X d_X = 2\binom{n+r_X}{n+1}$ if and only if $n = 1$. (We omit this case, since, for $n = 1$, X is always a complete intersection.)

(c2) If $\binom{n+r_X-1}{n} < d_X < \binom{n+r_X}{n}$, then $r_X d_X = 2\binom{n+r_X}{n+1}$ implies

$$\binom{n+r_X-1}{n} < \frac{2}{r_X}\binom{n+r_X}{n+1}$$

that is $r_X < \frac{2n}{n-1} = 2 + \frac{2}{n-1}$. Hence $r_X \leq 3$ and $r_X = 3$ if and only if $n = 2$. Using $r_X d_X = 2\binom{n+r_X}{n+1}$, we see that $(r_X, d_X) \in \{(1,2),(2,n+2)\}$ are satisfied. In the case $r_X = 3$ and $n = 2$, we get $3d_X = 2\binom{5}{3}$, which is impossible.

□

Next, we prove the following theorem.

Theorem 3.5. *Suppose that X is locally Gorenstein with minimal conductor, but not in generic position, and that there is a subset Y in generic position with $d_Y = d_X - 1$. Then X is Gorenstein if and only if*

$$(r_X - 2)d_X = 2\binom{n+r_X-1}{n+1} - 2.$$

Proof. Clearly, $I_X \subseteq I_Y$ and

$$\mathrm{HF}_Y(i) = \begin{cases} \mathrm{HF}_X(i) & \text{if } i < \alpha_{Y/X}, \\ \mathrm{HF}_X(i) - 1 & \text{if } i \geq \alpha_{Y/X}, \end{cases}$$

where $\alpha_{Y/X} = \min\{i \in \mathbb{N} \mid (I_Y/I_X)_i \neq 0\}$. In particular, $r_X - 1 \leq r_Y \leq r_X$. By the assumption, Y is in generic position, and so $\alpha_X \geq \alpha_Y \geq r_Y$. Since X is not in generic position, it must be the case $\alpha_X = \alpha_Y = r_Y = r_X - 1$. Subsequently, $\mathrm{HF}_X(r_X-1) = d_X - 1$ and

$$\begin{aligned} \ell(R/\mathfrak{F}_{\widetilde{R}/R}) &= \sum_{i=0}^{r_X-2} \binom{n+i}{n} + (d_X - 1) \\ &= \binom{n+r_X-1}{n+1} + d_X - 1. \end{aligned}$$

Also, X has the minimal conductor, i.e., $\mathfrak{F}_{\widetilde{R}/R} = \oplus_{i \geq r_X} R_i$, this implies

$$\begin{aligned} \ell(\widetilde{R}/R) &= r_X d_X - \sum_{i=0}^{r_X - 1} \mathrm{HF}_X(i) \\ &= (r_X - 1) d_X - \binom{n + r_X - 1}{n+1} + 1. \end{aligned}$$

Hence an application of Lemma 3.3 yields the claim. □

If $d_X \leq n$ then X is contained in a hyperplane $H \cong \mathbb{P}^{n-1}$, so in the following corollary it suffices to treat the case $d_X > n$.

Corollary 3.6. *In the setting of Theorem 3.5, if $d_X > n$ then X is Gorenstein if and only if $d_X = 2(n+2)$ or $d_X = (n+3)(n+2)/2 - 1$.*

Proof. This follows by Theorem 3.5 and from a similar calculation as in the proof of [5, Theorem 9] for sets of K-rational points. □

Remark 3.7. When $\mathfrak{q}_j$ is generated by linear forms (i.e., p_j is K-rational) for $j = 1, ..., s$, our theorems cover a result in [5, Sections 3-4].

Example 3.8. Let $X = \{p_1, p_2, p_3, p_4\} \subseteq \mathbb{P}^3$, where $p_1 = (1:0:0:0)$, $p_2 = (1:1:0:0)$, $p_3 = (1:0:1:0)$ and a non-reduced point p_4 corresponding to $\mathfrak{q}_4 = \langle x_1 - x_0, x_2 - x_0, (x_3 - x_0)^2 \rangle \subseteq K[x_0, ..., x_3]$. Clearly, X is locally Gorenstein with $d_X = 5$. We have $\mathrm{HF}_X : 1\ 4\ 5\ 5 \cdots$, and so $\alpha_X = r_X = 2$ and X is in generic position. Also, we have $\mathfrak{F}_{\widetilde{R}/R} = \langle R_2 \rangle$ and $r_X d_X = 10 = 2\binom{5}{4} = 2\binom{n+r_X}{n+1}$. Thus, Theorem 3.4 yields that $\mathbb{X}$ is a Gorenstein set.

Example 3.9. Consider the set $X' = \{p_1, p_2, p_3, p'_4\} \subseteq \mathbb{P}^3$ with $p_1 = (1:0:0:0)$, $p_2 = (1:1:0:0)$, $p_3 = (1:0:1:0)$ and a non-reduced point p'_4 corresponding to $\mathfrak{q}'_4 = \langle (x_1 - x_0)^2, x_2 - x_0, x_3 - x_0 \rangle \subseteq K[x_0, ..., x_3]$. Then $\mathrm{HF}_{X'} : 1\ 4\ 5\ 5 \cdots$ and $r_{X'} = \alpha_{X'} = 2$, and so X' is in generic position. However, we have $(\mathfrak{F}_{\widetilde{R}/R})_1 \neq 0$ and so $\langle R_2 \rangle \subsetneq \mathfrak{F}_{\widetilde{R}/R}$. Theorem 3.4 yields that X' is not a Gorenstein set.

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