Research Paper

Regularity of normal Rees algebras of edge ideals of graphs

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A R T I C L E   I N F O

A B S T R A C T

We classify all graphs for which the Rees algebras of their edge ideals are normal and have regularity equal to their matching numbers.

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1. Introduction

In March 2023, we were fortunate to participate in the lectures on binomial ideals by Professor Jürgen Herzog in the CoCoa school at Hue University. Motivated by his beautiful lectures and his recent joint work with Professor Takayuki Hibi on the regularity of Rees algebras of edge ideals we asked him about the regularity of Rees algebras of edge ideals of odd cycles. Professor Herzog said that it is a good research problem, so we started by trying to compute the regularity of Rees algebras of edge ideals of odd cycles. It turns out that a simple modification of the argument in [9] gives us the answer. Furthermore, it was known at that time that if $\mathcal{R}(G)$ is normal and G is a König graph or has a perfect matching then $\mathrm{reg}(\mathcal{R}(G)) = \mathrm{mat}(G)$, where $\mathcal{R}(G)$ is the Rees algebra of the edge ideal of G, reg denotes the Castelnuovo-Mumford regularity, and $\mathrm{mat}(G)$ is the matching number of G. It is not hard to find a non König graph which does not have a perfect matching but still has $\mathrm{reg}(\mathcal{R}(G)) = \mathrm{mat}(G)$. But they are essentially gluing of König graphs to perfect matching graphs. We will prove a more precise statement below in this work which grew out of influential lectures and the work of Professor Herzog. We humbly dedicate this to our admiral teacher, Professor Jürgen Herzog.

Let us now recall the notion of the Rees algebra of the edge ideal of a simple graph. Let G be a simple graph on the vertex set $V(G)$ and edge set $E(G) \subseteq V(G) \times V(G)$. Assume that $V(G) = [n] = \{1, \ldots, n\}$. The Rees algebra of the edge ideal of G over a field k, denoted by $\mathcal{R}(G)$ is the subalgebra of $\mathrm{k}[x_1, \ldots, x_n, t]$ generated by $x_1, \ldots, x_n$ and $x_i x_j t$ where $\{i, j\}$ is an edge of G. In [9], Herzog and Hibi proved that $\mathcal{R}(G)$ is normal if and only if G satisfies the odd cycle condition [13] and has at most one non-bipartite connected component. Furthermore, in this case, $\mathrm{mat}(G) \le \mathrm{reg}(\mathcal{R}(G)) \le \mathrm{mat}(G) + 1$. By [4, Theorem 4.2], [9, Corollary 2.3], and [12, Corollary 3.2], we have that $\mathrm{reg}(\mathcal{R}(G)) = \mathrm{mat}(G)$ when G is a bipartite graph, a perfect matching graph, or a connected König graph. But it was not known whether $\mathrm{reg}(\mathcal{R}(G)) = \mathrm{mat}(G)$ or $\mathrm{mat}(G) + 1$ when G is an odd cycle, which is the starting point of this work.

We will now introduce relevant graph concepts to state our main result. A subset $T \subseteq V(G)$ is an independent set of G if $E(G) \cap T \times T = \emptyset$. For a subset U of $V(G)$, we denote by $N_G(U)$ the set of neighbors of U in G.

Definition 1.1. A graph G is called a Tutte-Berge graph if there exists an independent set T of G such that

$$|T| = |N_G(T)| + |V(G)| - 2\,\mathrm{mat}(G). \tag{1}$$

The number $|V(G)| - 2\,\mathrm{mat}(G)$ is the number of uncovered vertices by a maximum matching, which was described by Tutte [T] and Berge [1]. When G has a perfect matching, $T = \emptyset$ satisfies Eq. (1). When G is König, then a maximum independent set of G satisfies Eq. (1). Hence, perfect matching graphs and König graphs are two extremes of Tutte-Berge graphs. Tutte-Berge graphs are precisely the graphs for which $\mathrm{reg}(\mathcal{R}(G)) = \mathrm{mat}(G)$ when $\mathcal{R}(G)$ is normal.

Theorem 1.2. *Let G be a simple graph. Assume that G has at least two edges and $\mathcal{R}(G)$ is normal. Then $\operatorname{reg}(\mathcal{R}(G)) = \operatorname{mat}(G)$ if and only if G is a Tutte-Berge graph.*

The main ingredients for the proof of Theorem 1.2 are the Gallai-Edmonds Structure Theorem and the description of the edge polytope of a graph of Ohsugi and Hibi [13]. To state a characterization of Tutte-Berge graphs, we recall the Gallai-Edmonds decomposition. Let G be a simple graph. Denote by $D(G)$ the set of all vertices in G which are not covered by at least one maximum matching of G. Let $A(G)$ be the set of vertices in $V(G) \backslash D(G)$ adjacent to at least one vertex in $D(G)$ and $C(G) = (V(G) \backslash D(G)) \backslash A(G)$. The decomposition $V(G) = D(G) \cup A(G) \cup C(G)$ is called the Gallai-Edmonds decomposition.

Theorem 1.3. *Let G be a simple graph. Then G is Tutte-Berge if and only if $D(G)$ consists of isolated vertices only.*

In the next section, we establish properties of Tutte-Berge graphs and prove Theorem 1.3. In Section 3, we prove Theorem 1.2.

2. Tutte-Berge graphs

In this section, we classify all Tutte-Berge graphs. We first introduce relevant concepts. We refer to the beautiful exposition [11] for unexplained terminology and further information.

Definition 2.1. Let G be a simple graph. A matching in G is a set of edges, no two of which share an endpoint. The matching number of G, denoted by $\operatorname{mat}(G)$, is the size of a maximum matching of G. A perfect matching is a matching that covers every vertex of the graph. A graph that has a perfect matching is also called a perfect matching graph.

For a subset $U \subset V(G)$, we denote by $G \backslash U$ the induced subgraph of G on $V(G) \setminus U$. When $U = \{u\}$, we use $G \backslash u$ instead of $G \backslash \{u\}$. We denote by $\alpha(G)$ the maximum size of an independent set of G.

Definition 2.2. Let G be a simple graph.

(1) G is factor-critical if for every vertex v of G, $G \backslash v$ has a perfect matching.
(2) G is König if $\alpha(G) + \operatorname{mat}(G) = |V(G)|$.

Gallai [7,8] and Edmonds [6] independently proved the following structure theorem.

Theorem 2.3 *(Gallai-Edmonds Structure Theorem). Let G be a simple graph, and $D(G), A(G)$ and $C(G)$ be defined as above. Then*

(1) *the components of the subgraph induced by $D(G)$ are factor-critical,*
(2) *the subgraph induced by $C(G)$ has a perfect matching,*
(3) *if M is any maximum matching of G, it contains a near-perfect matching of each component of $D(G)$, a perfect matching of each component of $C(G)$ and matches all vertices of $A(G)$ with vertices in distinct components of $D(G)$,*
(4) $2\operatorname{mat}(G) = |V(G)| - c(D(G)) + A(G)$, *where $c(D(G))$ denotes the number of components of the graph spanned by $D(G)$.*

We now have some preparation lemmas.

Lemma 2.4. *Let G be a factor-critical graph. Assume that $|V(G)| > 1$. Then for any independent set T of G we have $|T| \leq |N_G(T)|$. In particular, if $|V(G)| > 1$ then G is not Tutte-Berge.*

Proof. By definition, we have that $|V(G)|$ is odd. Let T be an independent set of G. Since $|V(G)| > 1$, there exists a vertex v of G such that $v \notin T$. Since $G\backslash v$ has a perfect matching and contains T, we deduce that

$$|T| \leq |N_{G\backslash v}(T)| \leq |N_G(T)|.$$

The conclusion follows.   □

Lemma 2.5. *Let G be a simple graph and $U \subseteq V(G)$ a subset of vertices of G. We denote by G_1 and G_2 the induced subgraphs of G on U and $V(G)\backslash U$, respectively. Assume that there exists a maximum matching M of G and a partition $M = M_1 \cup M_2$ such that M_1 is contained in G_1 and M_2 is contained in G_2. Then $\operatorname{mat}(G) = \operatorname{mat}(G_1) + \operatorname{mat}(G_2)$.*

Proof. Since a matching of G_1 and a matching of G_2 give a matching of G, we deduce that $\operatorname{mat}(G) \geq \operatorname{mat}(G_1) + \operatorname{mat}(G_2)$. The existence of a maximum matching M in the hypothesis implies that $\operatorname{mat}(G) \leq \operatorname{mat}(G_1) + \operatorname{mat}(G_2)$. The conclusion follows.   □

Lemma 2.6. *Let G be a simple graph and $T \subseteq V(G)$ an independent set of G. Then*

$$|T| \leq |N_G(T)| + |V(G)| - 2\operatorname{mat}(G).$$

Proof. Assume by contradiction that there exists a graph G and an independent set T of G such that $|T| > |N_G(T)| + |V(G)| - 2\operatorname{mat}(G)$. Let G be such a graph of the smallest size. In particular, G is connected. If G has a perfect matching, then $|T| \leq |N_G(T)|$ for any independent set T, so we must have G does not have a perfect matching. If $A(G) = \emptyset$ then $G = D(G)$ is factor-critical, which is a contradiction to Lemma 2.4. Hence, $A(G) \neq \emptyset$. Let v be any element of $A(G)$. Let M be any maximum matching of G. By the Edmonds-Gallai Structure Theorem, M contains an edge of the form vw with w in some component D_2 of $D(G)$. Let G_2 be the induced subgraph of G on $V(D_2) \cup \{v\}$ and G_1 be

the induced subgraph of G on $V(G) \setminus V(G_2)$. By Theorem 2.3 and Lemma 2.5, we have that G_2 has a perfect matching and $\mathrm{mat}(G) = \mathrm{mat}(G_1) + \mathrm{mat}(G_2)$. Let $T_1 = T \cap V(G_1)$ and $T_2 = T \cap V(G_2)$. Then we have $N_G(T) \geq N_{G_1}(T_1) + N_{G_2}(T_2)$. Thus, we have

$$|T_1| + |T_2| > |N_{G_1}(T_1)| + |N_{G_2}(T_2)| + |V(G_1)| - 2\,\mathrm{mat}(G_1).$$

Since G_2 has a perfect matching, $|T_2| \leq |N_{G_2}(T_2)|$. Therefore, we must have

$$|T_1| > |N_{G_1}(T_1)| + |V(G_1)| - 2\,\mathrm{mat}(G_1),$$

which is a contradiction, as G_1 is strictly smaller than G. The conclusion follows. □

Lemma 2.7. *Let G be a simple graph. Then G is Tutte-Berge if and only if each connected component of G is Tutte-Berge.*

Proof. The conclusion follows from the definition and Lemma 2.6. □

The following properties of Tutte-Berge graphs make it a natural class containing König graphs and perfect matching graphs.

Lemma 2.8. *Let G be a Tutte-Berge graph and $U \subseteq V(G)$ a subset of vertices of G. We denote by G_1 and G_2 the induced subgraph of G on U and $V(G) \setminus U$, respectively. Assume that there exists a maximum matching M of G and a partition $M = M_1 \cup M_2$ such that M_1 is contained in G_1 and M_2 is contained in G_2. Then G_1 and G_2 are Tutte-Berge.*

Proof. By Lemma 2.5, $\mathrm{mat}(G) = \mathrm{mat}(G_1) + \mathrm{mat}(G_2)$. Let T be an independent set of G such that $|T| = |N_G(T)| + |V(G)| - 2\,\mathrm{mat}(G)$. We denote by $T_1 = T \cap U$ and $T_2 = T \cap (V(G) \setminus U)$. We have that $|N_G(T)| \geq N_{G_1}(T_1) + N_{G_2}(T_2)$. Hence,

$$|T_1| + |T_2| \geq (|N_{G_1}(T_1)| + |V(G_1)| - 2\,\mathrm{mat}(G_1)) + (|N_{G_2}(T_2)| + |V(G_2)| - 2\,\mathrm{mat}(G_2)).$$

By Lemma 2.6, we deduce that $|T_1| = |N_{G_1}(T_1)| + |V(G_1)| - 2\,\mathrm{mat}(G_1)$ and $|T_2| = |N_{G_2}(T_2)| + |V(G_2)| - 2\,\mathrm{mat}(G_2)$. The conclusion follows. □

Lemma 2.9. *Assume that G is a Tutte-Berge graph and $D(G)$ has no isolated vertices. Then $D(G) = \emptyset$ and G has a perfect matching.*

Proof. By Lemma 2.7, we may assume that G is connected. We prove by induction on $|V(G)|$. If $|A(G)| = 0$ then $D(G) = \emptyset$ or $G = D(G)$. By Lemma 2.4, we must have $D(G) = \emptyset$ and G has a perfect matching. Assume by contradiction that $|A(G)| \geq 1$. Let u be an element of $A(G)$. As in the proof of Lemma 2.6, let M be any maximum matching of G. Then M contains an edge that connects u to a connected component D_2 of $D(G)$. Let $G_1 = G \setminus (D_2 \cup \{u\})$ and G_2 be the induced subgraph of G on $D_2 \cup \{u\}$.

By Theorem 2.3 and Lemma 2.8, G_1 is Tutte-Berge and G_2 has a perfect matching. We will prove that $D(G_1)$ has no isolated vertices. Let w be any element of $D(G_1)$. Then there exists a maximum matching M_1 of G_1 such that $w \notin M_1$. M_1 and a maximum matching of G_2 form a maximum matching of G. Hence, $w \in D(G)$. In other words, $D(G_1) \subseteq D(G) \setminus D_2$ and $A(G_1) \subseteq A \setminus u$. Assume by contradiction that $w \in D(G_1)$ is an isolated vertex. Then, the connected component of $D(G)$ containing w must have vertices in $A(G_1) \subseteq A(G) \setminus u$. In other words, $D(G) \cap A(G) \setminus u \neq \emptyset$. This is a contradiction. Thus, $D(G_1)$ has no isolated vertices. By induction, G_1 has a perfect matching. Hence, G itself has a perfect matching. In other words, $D(G) = \emptyset$. The conclusion follows. □

Lemma 2.10. *Assume that G is a Tutte-Berge graph. Then, $D(G)$ consists of isolated vertices only.*

Proof. By Lemma 2.7, we may assume that G is connected. Assume by contradiction that $D(G)$ has a connected component G_1 that is not an isolated vertex. By Theorem 2.3, G_1 is factor-critical. Let $G_2 = G \setminus V(G_1)$. Since $G_1 \subseteq D(G)$, there exists a maximum matching M of G that uncovers a vertex of G_1. This implies that $\mathrm{mat}(G) = \mathrm{mat}(G_1) + \mathrm{mat}(G_2)$. By Lemma 2.8, we deduce that G_1 is Tutte-Berge, which is a contradiction to Lemma 2.4. The conclusion follows. □

We are now ready for the proof of a characterization of Tutte-Berge graphs.

Proof of Theorem 1.3. By Lemma 2.10, it remains to prove the sufficiency condition. Let G_1 be the induced subgraph of G on $D(G) \cup A(G)$. By Theorem 2.3, $\mathrm{mat}(G_1) = |A(G)|$ and $\mathrm{mat}(G) = \mathrm{mat}(G_1) + |C(G)|/2$. We have that $D(G)$ is an independent set of G and

$$|D(G)| = |V(G_1)| - \mathrm{mat}(G_1) = |A(G)| + |V(G_1)| - 2\,\mathrm{mat}(G_1)$$
$$= |N_G(D(G))| + |V(G)| - 2\,\mathrm{mat}(G).$$

Hence, G is Tutte-Berge. □

Example 2.11. The following graph is Tutte-Berge but is not König nor has a perfect matching.
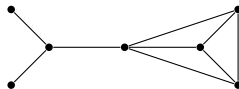
Remark 2.12.

(1) The Edmonds' blossom algorithm [6] gives a polynomial time algorithm for the Gallai-Edmonds structure decomposition. Hence, it also yields a polynomial time algorithm for determining if a graph is Tutte-Berge.

(2) When $D(G)$ consists of isolated vertices only the induced subgraph of G on $D(G) \cup A(G)$ is König. Hence, a Tutte-Berge graph decomposes into a König graph and a perfect matching graph.
(3) Deming [5] and Sterboul [S] independently gave the first characterization for König graphs. Theorem 1.3 is a natural analog of the characterization of König graphs given by Lovász [10, Lemma 3.3].

3. Regularity of normal Rees algebras of edge ideals

In this section, we compute the regularity of normal Rees algebras of edge ideals of graphs. First, we recall the description of the edge polytope of a graph by Ohsugi and Hibi [13].

Let G be a simple graph on n vertices. We denote by $\mathbf{e}_1, \ldots, \mathbf{e}_n$ the standard basis vectors of $\mathbb{R}^n$. The edge polytope of G, denoted by P_G, is the convex hull of $\{\mathbf{e}_i + \mathbf{e}_j \mid \{i, j\}$ is an edge of $G\}$. Let L be the hyperplane $L = \{x \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 2\}$. For each $i = 1, \ldots, n$, we denote by H_i^+ the half-space $H_i^+ = \{x \in \mathbb{R}^n \mid x_i \geq 0\}$. For each independent set T of G, we denote by $H_{G,T}^-$ the half-space

$$H_{G,T}^- = \left\{ x \in \mathbb{R}^n \mid \sum_{i \in T} x_i \leq \sum_{j \in N_G(T)} x_j \right\}.$$

Let T be an independent set of G, the bipartite graph induced by T, denoted by $B_G(T)$, is the graph with vertex set $V(B_G(T)) = T \cup N_G(T)$ and edge set $E(B_G(T)) = \{\{v, w\} \mid v \in T, w \in N_G(T)\}$.

Definition 3.1. A vertex v of G is said to be regular in G if each connected component of $G \setminus v$ has at least one odd cycle.

Definition 3.2. An independent set T of G is said to be fundamental in G if it satisfies the following conditions

(1) the bipartite graph $B_G(T)$ induced by T is connected;
(2) if $T \cup N_G(T) \neq V(G)$, then each connected component of $G \setminus V(B_G(T))$ has at least one odd cycle.

We have the following description of P_G [13, Theorem 1.7].

Theorem 3.3. *Assume that G has at least one odd cycle. Let R be the set of regular vertices of G and F the set of nonempty fundamental independent sets of G. Then*

$$P_G = L \cap \bigcap_{i \in R} H_i^+ \bigcap_{T \in F} H_{G,T}^-.$$

Let G^* be the cone graph over G, i.e., $V(G^*) = V(G) \cup \{n+1\}$ and

$$E(G^*) = E(G) \cup \{\{i, n+1\} \mid i = 1, \ldots, n\}.$$

We now have some preparation lemmas.

Lemma 3.4. *Assume that G has more than one edge. Then each vertex $v \in V(G)$ is a regular vertex of G^*.*

Proof. Since $G^* \backslash v$ is the cone graph over $G \backslash v$, it is connected. Furthermore, since G has more than one edge, $G \backslash v$ has at least one edge; this edge and the new vertex in G^* form a triangle in $G^* \backslash v$. The conclusion follows. $\square$

Lemma 3.5. *Let G be a simple graph on n vertices. Assume that G has at least two edges and $\mathcal{R}(G)$ is normal. Let $q_0 = \min\{q \geq 1 \mid q(P_{G^*} \setminus \partial P_{G^*}) \cap \mathbb{Z}^{n+1} \neq \emptyset\}$, where ∂P_{G^*} is the boundary of P_{G^*}. Then*

$$\mathrm{reg}(\mathcal{R}(G)) = n + 1 - q_0.$$

Proof. The conclusion follows from the proof of [9, Theorem 2.2]. $\square$

Lemma 3.6. *Assume that $q < n$ and $\mathbf{a} = (a_1, \ldots, a_{n+1}) \in q(P_{G^*} \backslash \partial P_{G^*})$. If $a_i > 1$ then $\mathbf{b} \in q(P_{G^*} \backslash \partial P_{G^*})$ where $\mathbf{b} = \mathbf{a} + \mathbf{e}_{n+1} - \mathbf{e}_i$.*

Proof. Since $a_i > 1$, we have that $b_i \geq 1$. Thus, $\mathbf{b} \in H_j^+ \backslash \partial H_j^+$ for all $j = 1, \ldots, n+1$. Now, let T be an independent set of G^*. If $T = \{n+1\}$, then we have $b_{n+1} = 2q - (b_1 + \cdots + b_n)$. Since $q < n$ and $b_i > 0$ for all $i = 1, \ldots, n$, we deduce that $b_{n+1} < n \leq \sum_{i=1}^{n} b_i$. Hence, $\mathbf{b} \in H_{G^*, T}^- \backslash \partial H_{G^*, T}^-$. Now, assume that T is an independent set of G. If $i \in T$ then

$$\sum_{j \in T} b_j = \sum_{j \in T} a_j - 1 < \sum_{j \in N_G(T)} a_j + a_{n+1} - 1 = \sum_{j \in N_G(T)} b_j + b_{n+1} - 2.$$

Now, assume that $i \notin T$. Then

$$\sum_{j \in T} b_j = \sum_{j \in T} a_j < \sum_{j \in N_G(T)} a_j + a_{n+1} - 1 \leq \sum_{j \in N_G(T)} b_j + b_{n+1} - 1.$$

Hence, $\mathbf{b} \in H_{G^*, T}^-$ for all independent sets T of G. The conclusion follows. $\square$

Lemma 3.7. *Let G be a simple graph. Let $\mathbf{p} = (1, 1, \ldots, 1, n - 2\,\mathrm{mat}(G))$ be a point in $\mathbb{R}^{n+1}$. Then $\mathbf{p} \in qP_{G^*}$, where $q = n - \mathrm{mat}(G)$.*

Proof. Clearly, $\mathbf{p} \in H_i^+$ for all $i = 1, \ldots, n+1$. Let T be an independent set of G^*. Then either $T = \{n+1\}$ or T is an independent set of G. If $T = \{n+1\}$, then $n - 2\operatorname{mat}(G) < n$, so $\mathbf{p} \in H_{G^*,T}^-$. If T is an independent set of G, the conclusion follows from the definition and Lemma 2.6. $\quad\square$

Lemma 3.8. *Let G be a simple graph with connected components $G_1, \ldots, G_c$. Assume that $T_1, \ldots, T_c$ are fundamental independent sets of $G_1, \ldots, G_c$. Then $T = T_1 \cup \cdots \cup T_c$ is a fundamental independent set of G^*.*

Proof. Since $B_{G^*}(T) = B_{G_1}(T_1) \cup \cdots \cup B_{G_c}(T_c) \cup \{n+1\}$, $B_{G^*}(T)$ is connected. Furthermore, $G^* \setminus B_{G^*}(T) = (G_1 \setminus B_{G_1}(T_1)) \cup \cdots \cup (G_c \setminus B_{G_c}(T_c))$. The conclusion follows from the definition of fundamental independent sets. $\quad\square$

Lemma 3.9. *Assume that G is a Tutte-Berge graph. Then there exists a fundamental independent set T of G^* such that $|T| = |N_G(T)| + |V(G)| - 2\operatorname{mat}(G)$.*

Proof. By Lemma 3.8, we may assume that G is connected. If G is bipartite, we can take T to be the maximum independent set of G. Thus, we may assume that G is not bipartite. If G has a perfect matching, we may take $T = \emptyset$. Thus, we may assume that G does not have a perfect matching. By Theorem 1.3, we have that $|P| = |N_G(P)| + |V(G)| - 2\operatorname{mat}(G)$, where $P = D(G)$. By definition $V(B_G(P)) = D(G) \cup A(G)$. Let $H_1, \ldots, H_c$ be the connected components of $C(G)$. Since $C(G)$ has a perfect matching, $H_1, \ldots, H_c$ have a perfect matching. For each $i = 1, \ldots, c$, we set

$$T_i = \begin{cases} \emptyset & \text{if } H_i \text{ is non-bipartite,} \\ \text{a maximum independent set of } H_i & \text{if } H_i \text{ is bipartite.} \end{cases}$$

Let $T = P \cup T_1 \cup \cdots \cup T_c$. Then T is an independent set of G and have $|T| = |N_G(T)| + |V(G)| - 2\operatorname{mat}(G)$. Furthermore, $G \setminus B_G(T) = \bigcup_j H_j$, where the union is taken over the indices j such that H_j is non-bipartite. Note that $B_{G^*}(T) = B_G(T) \cup \{n+1\}$, hence $B_{G^*}(T)$ is connected. By definition, T is a fundamental set of G^*. The conclusion follows. $\quad\square$

We are now ready for the proof of Theorem 1.2.

Proof of Theorem 1.2. We may assume that G does not have a perfect matching. Note that $\mathcal{R}(G)$ is normal by assumption.

First, assume that G is not Tutte-Berge. In particular, $n - 2\operatorname{mat}(G) > 0$ and for any independent set T of G we have $|T| < |N_G(T)| + n - 2\operatorname{mat}(G)$. Let $\mathbf{p} = (1, \ldots, 1, n - 2\operatorname{mat}(G))$ be a point in $\mathbb{Z}^{n+1}$. Hence, $\mathbf{p} \in q(P_{G^*} \setminus \partial P_{G^*})$ where $q = n - \operatorname{mat}(G)$. By Lemma 3.5, we have

$$\operatorname{reg}(\mathcal{R}(G)) = n + 1 - q_0 \geq n + 1 - q = n + 1 - (n - \operatorname{mat}(G)) = \operatorname{mat}(G) + 1.$$

By [9, Theorem 2.2], $\mathrm{reg}(\mathcal{R}(G)) = \mathrm{mat}(G) + 1$.

Now, assume that G is Tutte-Berge. By [4, Theorem 4.2] and [9, Corollary 2.3], we may assume that G is not bipartite and does not have a perfect matching. Since $\mathcal{R}(G)$ is normal, G can have at most one non-bipartite connected component. Let G_1 be the unique non-bipartite connected component of G. By Lemma 2.7, G_1 is Tutte-Berge. By Lemma 3.9, there exists a fundamental set T_1 of G_1^* such that $|T_1| = |N_{G_1}(T_1)| + |V(G_1)| - 2\,\mathrm{mat}(G_1)$. Let $G_2, \ldots, G_c$ be bipartite connected components of G. For each $i = 2, \ldots, c$, let T_i be a maximum independent set of G_i. Then $T = T_1 \cup \cdots \cup T_c$ is a fundamental set of G^* such that $|T| = |N_G(T)| + |V(G)| - 2\,\mathrm{mat}(G)$. Since G does not have a perfect matching, $|T| \geq 1$.

We will now prove that $\mathrm{reg}(\mathcal{R}(G)) = \mathrm{mat}(G)$. By Lemma 3.5, it suffices to prove that $q_0 \geq n - \mathrm{mat}(G) + 1$. Indeed, let $\mathbf{a} = (a_1, \ldots, a_{n+1})$ be a point in $q_0(P_{G^*} \backslash \partial P_{G^*})$. By Lemma 3.4, each $i \in [n]$ is a regular vertex of G^*. Hence, $a_i \geq 1$ for all $i = 1, \ldots, n$. Now, assume by contradiction that $q_0 \leq n - \mathrm{mat}(G) < n$. By Lemma 3.6, we may assume that $a_i = 1$ for all $i = 1, \ldots, n$. Then $a_{n+1} = 2q_0 - (a_1 + \cdots + a_n) = 2q_0 - n \leq n - 2\,\mathrm{mat}(G)$. In other words, $\mathbf{a} \notin q_0(H_{G^*, T} \backslash \partial H^-_{G^*, T})$. This is a contradiction. The conclusion follows. $\quad\square$

Corollary 3.10. *Let $G = C_{2n+1}$ be an odd cycle of length $2n + 1 \geq 3$. Then*

$$\mathrm{reg}(\mathcal{R}(G)) = \mathrm{mat}(G) + 1 = n + 1.$$

Proof. Since G is not Tutte-Berge, the conclusion follows from Theorem 1.2. $\quad\square$

Remark 3.11. Let $G = K_n$ be a complete graph on n vertices. Assume that $n \geq 3$ is odd. Then G is not Tutte-Berge. Hence, $\mathrm{reg}(\mathcal{R}(G)) = \mathrm{mat}(G) + 1$. This is also a special case of [3, Corollary 2.12].

Remark 3.12. The assumption that $\mathcal{R}(G)$ is normal is crucial in Theorem 1.2. As pointed out by Herzog and Hibi [9, Example 2.4], there is a perfect matching graph G such that $\mathrm{reg}(\mathcal{R}(G)) > \mathrm{mat}(G)$. The reason is that when $\mathcal{R}(G)$ is not normal, one cannot use Danilov-Stanley Theorem [2, Theorem 6.3.5]; hence, Lemma 3.5 is no longer valid.

Declaration of competing interest

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Data availability

Data sharing does not apply to this article as no datasets were generated or analyzed during the current study.

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