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A look at FPF rings

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Dedicated to the memory of Carl Faith

An error in Corollary 9.32 of [C. Faith, *Rings and Things and a Fine Array of Twentieth Century Associative Algebra*, Mathematical Surveys and Monographs, Vol. 65 (American Mathematical Society, Providence, RI, 2004)], motivated us to consider again FPF rings which were initiated by Faith in the 1970s. In this paper, it is shown that a commutative ring R is reduced FPF if and only if it is Π-semihereditary. We show that when a semiperfect ring with a strongly right bounded basic ring with right and left Ore conditions, is an FPF ring. After some general results, the article focuses on rings of continuous functions. We give some algebraic characterizations for a $C(X)$ to be FPF

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and retrieve a result of Jorge Martinez. Also, we show that a space X is fraction-dense if and only if $Q_{\mathrm{cl}}(X)$ is a continuous ring.

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1. Introduction

Throughout this paper, R will be an associative ring with identity. The Jacobson radical of a ring R is denoted by $J(R)$. The right and left annihilators of a nonempty subset Y of a ring R are denoted by $\mathrm{Ann}_r(Y)$ and $\mathrm{Ann}_l(Y)$, respectively. For a ring R, $Q^r_{\max}(R)$ (respectively, $Q^l_{\max}(R)$) will denote the right (respectively, left) maximal quotient ring of R; $Q^r_{\mathrm{cl}}(R)$ (respectively, $Q^l_{\mathrm{cl}}(R)$) will denote the right (respectively, left) classical ring of quotients of R. When $Q^r_{\max}(R) = Q^l_{\max}(R)$ (respectively, $Q^r_{\mathrm{cl}}(R) = Q^l_{\mathrm{cl}}(R)$), we write $Q_{\max}(R)$ for $Q^r_{\max}(R) = Q^l_{\max}(R)$ (respectively, $Q_{\mathrm{cl}}(R)$ for $Q^r_{\mathrm{cl}}(R) = Q^l_{\mathrm{cl}}(R)$) for a ring R.

We denote the ring of $n \times n$ matrices over R by $\mathrm{M}_n(R)$ for integer $n \geq 1$. Let $C(X)$ be the ring of all continuous real-valued functions on a Tychonoff (i.e. completely regular Hausdorff) space X. Also, as $C(X)$ is a commutative ring, we use $Q_{\mathrm{cl}}(X)$ and $Q_{\max}(X)$ to denote the classical quotient ring and the maximal quotient ring of $C(X)$, respectively. The reader is referred to [20] and [32] for undefined terms and notations. In order to give a relatively self-contained exposition of our results, we recall some definitions and facts.

(1) A ring R is called *von Neumann regular* if for every $a \in R$ there is an $x \in R$ for which $a = axa$. We refer the reader to [22] for the general theory of von Neumann regular rings.

(2) A ring R is called *Baer* if every right annihilator in R is of the form eR for some idempotent $e \in R$. This condition is left-right symmetric by [32, Proposition 7.46, p. 260].

(3) A ring R is called *right semihereditary* if every finitely generated right ideal of R is projective. A left semihereditary ring is defined similarly. The definition of semihereditary rings is not left–right symmetric, see [32, §2F]. A ring which is both right and left semihereditary is called semihereditary.

(4) A ring R is said to be a *right p.p. ring* (also known as *right Rickart* [32]) if every right principal ideal of R is projective. A left p.p. ring is defined similarly. The definition of p.p. rings is not left–right symmetric, see [32, p. 261]. A ring which is both right and left p.p. is called a p.p. ring.

(5) A ring R is *right self-injective* if every R-homomorphism $f : I \to R$, where I is a right ideal of R, can be extended to an endomorphism of R, i.e. there exists $r \in R$ such that $f(x) = rx$ for all $x \in I$. A left self-injective ring is defined similarly. The definition of self-injective rings is not left-right symmetric, see

[32, 3.74B, p. 98]. A ring which is both right and left self-injective is called self-injective. The reader is referred to [32, §3] for more details.

(6) We recall that a module is called *torsionless* if it can be embedded in a direct product of copies of the base ring.

(7) Consider the following conditions on a ring R:

(C1) Every nonzero right ideal is essential in a direct summand of R_R.

(C2) Every right ideal that is isomorphic to a direct summand of R_R is also a direct summand of R_R.

(C3) If $eR \cap fR = 0$ where e and f are idempotents in R, then $eR \oplus fR$ is a direct summand of R_R.

R is called *right continuous* if it satisfies conditions (C1) and (C2) (and hence C3), *right quasi-continuous* (also known as *right π-injective* [21]) if it satisfies (C1) and (C3), and a *right* CS-*ring* (also known as *right extending* [7]) if it satisfies condition (C1) only. A left (quasi-)continuous ring and a left CS ring are defined similarly. Similar to self-injectivity, none of the conditions, continuous, quasi-continuous, or CS is left–right symmetric for rings, see [7, Examples 2.1.33 and 2.1.36]. A (quasi-) continuous (respectively, CS) ring is a ring which is both right and left (quasi-) continuous (respectively, CS). Note that if R is a right CS ring and every idempotent is central, then R is right quasi-continuous. For future reference, let us make the following observation. The following implications hold true for rings:

Right self-injective $\Rightarrow$ Right continuous $\Rightarrow$ Right quasi-continuous $\Rightarrow$ Right CS.

However, neither implication is reversible, see [7, Example 2.1.14].

2. Π-Semihereditary Rings versus B-Rings

We begin this section with the property of interest to us, which was studied first by Jones [26], who showed that it is left-right symmetric. In [26], Jones proved that every finitely generated torsionless right R-module is projective if and only if R is left semihereditary and every finitely generated torsionless right R-module is finitely presented. A ring with the property that every finitely generated torsionless right (left) R-module is projective is called Π-*semihereditary* by Ara and Dicks [2].

It is clear that every Π-semihereditary ring is right (left) semihereditary but the converse is not true. For example, let F be a field and $F_n = F$ for any positive integer n. Let

$$R = \left\{ (r_n) \in \prod_{n=1}^{\infty} F_n \mid r_n \text{ is eventually constant} \right\},$$

which is a subring of $\prod_{n=1}^{\infty} F_n$. Then R is a commutative von Neumann regular ring (and hence is a semihereditary ring) which is not Π-semihereditary, see [44, Example 4.4]; in fact, R is not Baer, see [32, Example 7.54, p. 263].

In [42], Rizvi and Roman proved that every finitely generated torsionless right R-module is projective if and only if and only if $\mathrm{M}_n(R)$ is a Baer ring for every $n \geq 1$. By Lenzing [33], a ring R is called a B-*ring* if $\mathrm{M}_n(R)$ is a Baer ring for every positive integer n. Let us remind the reader that while the class of Baer rings has many noteworthy properties, it is not closed under matrix rings, even for commutative reduced rings. For the sake of completeness, see the following example that is essentially due to Cohn.

Example 2.1. Let $\mathbb{Z}$ be the ring of integer numbers. Obviously $\mathbb{Z}[x]$ is a Baer ring. Take $\alpha = \begin{bmatrix} 2 & 0 \\ x & 0 \end{bmatrix}$ in $\mathrm{M}_2(\mathbb{Z}[x])$. In [11], it is shown that $\mathrm{Ann}_l(\alpha)$ is not generated by an idempotent in $\mathrm{M}_2(\mathbb{Z}[x])$. Thus, $\mathrm{M}_2(\mathbb{Z}[x])$ is not a Baer ring.

In general, a commutative domain R is a B-ring if and only if $\mathrm{M}_2(R)$ is a Baer ring if and only if R is semihereditary, see [7, Theorem 6.1.4].

The following result is related to Example 2.1.

Theorem 2.2 ([42, Theorem 3.5]). *Let R be a ring. The following conditions are equivalent*:

(1) *R is a Π-semihereditary ring.*
(2) *R is a B-ring.*

A ring R is called *right coherent* if for each index set I, $\prod_I R$ is a flat right R-module, or equivalently, if every finitely generated right ideal J of R is finitely presented; this means that there exists a short exact sequence of R-modules

$$0 \to K \to R^{(n)} \to J \to 0,$$

where n is a positive integer and K is a finitely generated R-module. A left coherent ring is defined similarly. The notion of coherent rings is not left–right symmetric, see [32, 4.46(e), p. 139]. Examples of right coherent rings are right noetherian rings and right semihereditary rings. According to [8], a ring R is called *right* Π-*coherent* (also known as *right strong coherence* [26]) if for each index set I, every finitely generated submodule of $\prod_I R_R$ is finitely presented. A left Π-coherent ring is defined similarly. We do not know whether there exist examples of right Π-coherent rings that are not left Π-coherent rings. For future reference, let us make the following observation.

Corollary 2.3. *For any ring, we have*

$$\Pi\text{-}\textit{semihereditary} \Rightarrow \textit{right } \Pi\text{-}\textit{coherent} \Rightarrow \textit{right coherent}.$$

However, neither implication is reversible.

Proof. The implications are clear. To check the last statement, let us recall some facts. A ring R is a *right star-ring* (shortly, a right $*$-ring), if the R-dual module $M^* = \mathrm{Hom}_R(M, R)$ of any finitely generated right R-module M is finitely generated as a canonical left R-module. Kobayashi [30] proved that a commutative von Neumann regular ring R is self-injective if and only if R is an $*$-ring. Camillo [8] proved that a ring R is a right $*$-ring if and only if R is left Π-coherent. Hence, a commutative von Neumann regular ring R is self-injective if and only if it is Π-coherent. Thus, the nonself-injective von Neumann regular commutative rings provide a class of coherent rings which are not Π-coherent.

Let F be a field which has a proper subfield K, set $F_n = F$ and $K_n = K$ for $n = 1, 2, 3, \ldots$. Suppose that $R = \{x \in \prod F_n | x_n \in K_n$ for all but finitely many $n\}$. By [22, 13.8], R is a commutative von Neumann regular ring which is not self-injective. Hence R is a semihereditary ring and so R is coherent. Hence, we infer that R is not Π-coherent.

Let $R = F[x, y]$ be the ring of polynomials in two indeterminates over a field F. Since R is a commutative noetherian ring, R is a Π-coherent ring. As the weak dimension of R is equal to 2, i.e. $\mathrm{w.dim}R = 2$, we get R is not a Π-semihereditary ring. In fact, $R = F[x, y]$ is not semihereditary (note that semihereditary rings R are precisely those rings R with $\mathrm{w.dim}R \leq 1$ which are coherent, see [2, Theorem 4.7]). We leave the details of weak dimension to the interested reader. □

Let M be a right R-module. The *trace* of M, $t(M)$, is the sum of all images of morphisms $M \to R_R$. A right R-module M is called *generator* if $t(M) = R$. A ring R is called a *right* FPF *ring* if every finitely generated faithful right R-module is a generator of Mod-R, the category of all right R-modules. A left FPF ring is defined similarly. The definition of FPF rings is not left–right symmetric, see [40, p. 259]. A ring which is both right and left FPF is called FPF. Examples of FPF rings are QF (quasi-Frobenius) rings, right PF (pseudo Frobenius) rings and commutative self-injective rings. For more information on FPF rings, see [16, 17].

Recall that a commutative ring R is *integrally closed in* $Q_{\max}(R)$ if every $q \in Q_{\max}(R)$ that satisfies a monic polynomial in $R[x]$ belongs to R. By Eggert [13], a commutative ring R is called an I-*ring* if every subring of $Q_{\max}(R)$ containing R is integrally closed in $Q_{\max}(R)$. We note that every commutative FPF ring is an I-ring (see [17, p. 88]) while the converse is not true (e.g. see [34, Example 14]).

Remark 2.4. In [25, Definition 1, p. 210], a ring is called an I-*ring* if every nonnil right ideal contains a nonzero idempotent. It can be checked that a ring R is an I-ring if and only if every nonnil left ideal contains a nonzero idempotent. An I-ring is also called a *Zorn ring* by Kaplansky [27, p. 19]. See also [7, p. 8] for I-rings.

Let R be a commutative principal ideal domain which is not a field. Then R is integrally closed, see [28, Theorem 50, p. 33]. Hence, R is an I-ring in the sense of Eggert. But R is not an I-ring because it is not a field.

Recall that a space X is said to have a *clopen* π*-base* if every open set contains a clopen subset. We note that $C(X)$ is an I-ring if and only if X has a clopen π-base, see [39] for more details. A space X is called *extremally disconnected* if the closure of every open subset is clopen in X. It is known that $C(X)$ is an I-ring in the sense of Eggert [13] if and only if X is an extremally disconnected space, see [37]. It is clear that every extremally disconnected space has a clopen π-base but the converse is not true (e.g. see [20, 4N, p. 64]). Hence, there is an I-ring that it is not an I-ring in the sense of Eggert.

The above examples show that I-rings are different from I-rings in the sense of Eggert [13].

In the following, we determine when a commutative ring is Π-semihereditary.

Theorem 2.5. *Let R be a commutative ring. The following statements are equivalent*:

(1) *R is a* Π-*semihereditary ring.*
(2) *R is semihereditary and* Π-*coherent.*
(3) *R is semihereditary and $Q_{\rm cl}(R)$ is self-injective.*
(4) *R is a reduced* FPF *ring.*
(5) *R is a* B-*ring.*
(6) *R is a reduced* I-*ring in the sense of Eggert* [13].

Proof. (1) ⇔ (2) It follows from [26, Theorem 2.11] (or see [2, Theorems 4.7 and 4.11]).

(2) ⇔ (3) It follows from [12, Theorem 12].
(3) ⇔ (4) It follows from [17, Corollary 2.12].
(1) ⇔ (5) It follows from Theorem 2.2.

(4) ⇔ (6) It follows from [17, Corollary 5.7]. □

Remark 2.6. In [16], it is shown that the FPF condition is not left–right symmetric (see [16, Example 5.2]), while as was mentioned before the notion of Π-semihereditary rings is left-right symmetric by Jones [26]. In [24], it is constructed a non-commutative reduced FPF ring that is not semihereditary (hence is not Π-semihereditary). Thus, the "commutativity" hypothesis in Theorem 2.5 is not superfluous.

Remark 2.7. In [34, p. 60], Lucas pointed out that "reduced I-rings in the sense of Eggert are the same as semi-hereditary rings". Also, [18, Corollary 9.32, p. 187] states that "a reduced ring R is an I-ring in the sense of Eggert if and only if R is semihereditary". These conclusions are not true. In fact, the "self-injectivity" hypothesis for $Q_{\rm cl}(R)$ is not superfluous in Theorem 2.5((3) ⇔ (6)).

For example, let R be the von Neumann regular (hence semihereditary) ring constructed before Example 2.1. Then $Q_{\max}(R) = \prod_{n=1}^{\infty} F_n$ and $Q_{\rm cl}(R) = R$. Hence $Q_{\max}(R) \neq Q_{\rm cl}(R)$ and so R is not an I-ring in the sense of Eggert by [13, Theorem 9].

Let us consider another example. Recall that a space X is *basically disconnected* if the closure of any cozeroset is open in X. Suppose X is a basically disconnected space which is not extremally disconnected (e.g. see [20, 4N, p. 64]). Then $C(X)$ is a reduced semihereditary ring which is not an I-ring in the sense of Eggert [13], for more details see Corollary 4.2.

Before stating the next result of this section, we need to recall some definitions. A ring R is said to be *semilocal* if $R/J(R)$ is semisimple. A ring R is called *semiperfect* if R is semilocal, and idempotents of $R/J(R)$ can be lifted to R. A ring is said to be *strongly right bounded* if every nonzero right ideal contains a nonzero ideal.

According to [10], a right R-module M is called *cofaithful* if it generates every injective right R-module; equivalently, if there exists a finite subset $\{m_1, m_2, \ldots, m_n\}$ of elements of M such that $\mathrm{Ann}_r(\{m_1, m_2, \ldots, m_n\}) = 0$. So any cofaithful module is faithful.

Lemma 2.8. *If R is a strongly right bounded ring such that every finitely generated cofaithful right R-module is a generator then R is* FPF.

Proof. Let R be a strongly right bounded ring such that every finitely generated cofaithful right R-module is a generator and M be a finitely generated faithful right R-module, say $M = x_1R + x_2R + \cdots + x_nR$. Take $A = \mathrm{Ann}_r(\{m_1, m_2, \ldots, m_n\})$. If $A \neq 0$, there is a nonzero ideal B of R such that $B \subseteq A$. Then, $MB = (x_1R + x_2R + \cdots + x_nR)B \leq x_1B + x_2B + \cdots + x_nB = 0$, a contradiction. Hence, $A = 0$ and so M is cofaithful and then generator of Mod-R. This means that R is right FPF. □

Theorem 2.9. *Let R be a semiperfect ring with a strongly right bounded basic ring with right and left Ore conditions. Then the following conditions are equivalent*:

(1) *R is a reduced ring, every finitely generated right ideal of R containing a regular element* (*i.e. a nonzero-divisor*) *is a generator and $Q^r_{\mathrm{cl}}(R)$ is right self-injective.*
(2) *R is a reduced right* FPF *ring.*

Proof. (1) ⇒ (2) Since R satisfies the left and right Ore conditions, there exist $Q^r_{\mathrm{cl}}(R), Q^l_{\mathrm{cl}}(R)$ and $Q^r_{\mathrm{cl}}(R) = Q^l_{\mathrm{cl}}(R)$. As $Q^r_{\mathrm{cl}}(R)$ is right self-injective, $Q^r_{\mathrm{cl}}(R)$ is also right R-injective and hence $Q^r_{\mathrm{cl}}(R) = Q^l_{\mathrm{cl}}(R) = E(R_R)$.

Since R is a reduced ring, $Q^r_{\max}(R)$ is a right self-injective, von Neumann regular ring and $Q^r_{\max}(R) = E(R_R)$ by [32, 13.36, p. 376]. Now we have $Q^l_{\mathrm{cl}}(R) = Q^r_{\mathrm{cl}}(R) = Q^r_{\max}(R) = E(R_R)$.

Now we are in position to prove that R is FPF. Let M be a finitely generated cofaithful right R-module. First we will prove that $M \otimes_R E(R_R)$ is also a cofaithful $E(R_R)$-module. Set $N = M \otimes_R Re_0$, where e_0 is the basic idempotent of R. Then N is a right e_0Re_0-module. By [1, Exercise 19.4], $M \otimes_R Re_0 \cong Me_0$, hence if $Me_0re_0 = 0$, then $e_0re_0 = 0$, proving that N is a faithful right e_0Re_0-module, and since e_0Re_0 is strongly right bounded, N is a cofaithful e_0Re_0-module. Now by [43, Proposition 3.2, Chap. X], e_0Qe_0 is the maximal right ring of quotient of e_0Re_0. Set $Q = Q^r_{\mathrm{cl}}(R)$. Then Q is left flat over R and so $IQ = I \otimes_R Q$ for all right ideals I of R. Now for a right ideal H of e_0Re_0, H is of the form $H = e_0Ie_0$ for a right ideal I of R and a similar proof in [16, Theorem 2.23] yields

$$H \otimes_{e_0Re_0} e_0Qe_0 \cong e_0IQe_0$$

so that e_0Qe_0 is left flat over e_0Re_0.

Since N is a cofaithful e_0Re_0-module, there exists an exact sequence

$$0 \to e_0Re_0 \to N^m$$

for some $m \in \mathbb{N}$.

Now we can tensor this sequence with e_0Qe_0 over e_0Re_0 to obtain the following exact sequence:

$$0 \to e_0Re_0 \otimes_{e_0Re_0} e_0Qe_0 \to N^m \otimes_{e_0Re_0} e_0Qe_0$$

and

$$0 \to e_0Qe_0 \to (M \otimes Re_0)^m \otimes_{e_0Re_0} e_0Qe_0 \cong (M \otimes_R Re_0 \otimes_{e_0Re_0} e_0Qe_0)^m$$
$$\cong (Me_0 \otimes_{e_0Re_0} e_0Qe_0)^m,$$

so $Me_0 \otimes_{e_0Re_0} e_0Qe_0 \cong M \otimes Re_0 \otimes_{e_0Re_0} e_0Qe_0$ is a cofaithful e_0Qe_0-module. Since Q is similar to e_0Qe_0, it follows that $M \otimes_R Q$ is a cofaithful Q-module. By the condition $Q^l_{\mathrm{cl}}(R) = Q^r_{\mathrm{cl}}(R) = Q^r_{\max}(R) = E(R_R) = Q$, we obtain that every finitely generated cofaithful Q-module generates Q, especially, $M \otimes_R Q$ generates Q. So there are finitely many homomorphisms $f_i : M \otimes_R Q \to Q$ such that

$$\sum_{i,j} f_i(m_j \otimes q_{ij}) = 1.$$

Note that the image of M in $M \otimes Q$ generates $M \otimes_R Q$, and we can take $\{m_j\}$ to be a spanning set for M.

Let $f_i(m_j \otimes 1) = q'_{ij} = b_{ij}^{-1}a_{ij}$ and $q_{ij} = c_{ij}d_{ij}^{-1}$. We can find regular elements b and d so that $bf_i(m_j \otimes 1) \in R$ for all i, j and $q_{ij}d \in R$ for all i, j.

We can restrict the homomorphism f_i by $bf_i : M \to R$ defined by $m_j \mapsto bf_i(m_j \otimes 1)$. Then

$$\sum_{i,j} bf_i(m_j \otimes q_{ij}d) = b\left(\sum_{i,j} f_i(m_j \otimes q_{ij})\right)d = bd,$$

so the regular element bd is contained in the trace of M. By assumption, $t(M)$ generates R, moreover M generates $t(M)$. Thus M generates R, proving that on R, every finitely generated cofaithful is a generator, same as the basic ring of R. But the basic ring R is strongly right bounded so it is right FPF. Again, R is similar to the basic ring, so R is right FPF.

(2) $\Rightarrow$ (1) Since R is a reduced ring, $Q^r_{\max}(R)$ is a right self-injective, von Neumann regular ring and $Q^r_{\max}(R) = E(R_R)$ by [32, 13.36, p. 376]. By [32, 13.12, p. 367], there exists an unique embedding from $Q^r_{\mathrm{cl}}(R)$ to $Q^r_{\max}(R)$ extending the

identity map on R. So we have $Q^r_{\mathrm{cl}}(R)$ is right self-injective. Now, if I is a finitely generated right ideal of R containing a regular element i, then $\mathrm{Ann}_r(i) = 0$. Thus, $\mathrm{Ann}_r(I) = 0$ and I is a finitely generated faithful right R-module over a right FPF ring, so I is a generator. □

Theorem 2.10. *Let R be a semiperfect ring with strongly right bounded basic ring. Then the following conditions are equivalent*:

(1) *R is a reduced right* FPF *ring in which every regular element is a unit in R.*
(2) *R is a reduced right self-injective ring.*

In this case, R is semisimple.

Proof. (1) ⇒ (2) If R is a reduced ring then R is right (and left) nonsingular by [43, Lemma 5.1, Chap. XII]. By [16, 3.3], R is semiprime. By [16, 2.1A], R has finite right Goldie dimension. By [32, 11.13, p. 324], R is right Goldie. So $Q^r_{\mathrm{cl}}(R)$ is semisimple. By assumption, every regular element is a unit in R, so $R = Q^r_{\mathrm{cl}}(R) = E(R_R)$. So R is right self-injective.

(2) ⇒ (1) By [16, 2.2A], R is a reduced right FPF ring. Then $Q^r_{\max}(R)$ is a right self-injective, von Neumann regular ring and $Q^r_{\max}(R) = Q^r_{\mathrm{cl}}(R) = E(R_R)$ by [32, 13.36, p. 376], and hence $R = Q^r_{\mathrm{cl}}(R)$. So every regular element is a unit in R. □

3. Π-Coherent Rings versus Baer Rings

In this section, we will focus on Baer rings and Π-coherent rings. Note that the Π-coherent property does not imply the Baer property. For instance, the ring $\mathbb{Z}_4$, ring of integers modulo 4, is a Π-coherent ring that is not Baer. On the other hand, let X be a P-space without isolated points (e.g. [20, 13 P, p. 193]) and let R be the integral closure of $C(X)$ in $Q_{\max}(X)$. Then R is a Baer ring that is not Π-coherent. As another example, let R be the ring defined in [44, Example 4.4]. Then R is a Baer ring that is not Π-coherent, see [12].

Recall that a ring R is called *right cononsingular* if for any right ideal I of R with $\mathrm{Ann}_l(I) = 0$ is essential as a right R-submodule of R_R. A left cononsingular ring is defined similarly. Note that any commutative semiprime ring is right (also left) cononsingular. The following is due to Chatters and Khuri [9].

Theorem 3.1 ([9, Theorem 2.1]). *Let R be a ring. Then R is right* CS *and right nonsingular if and only if R is Baer and right cononsingular.*

From Theorem 3.1, we get the next result.

Proposition 3.2. *Let R be a ring. Then the following are equivalent*:

(1) *R is a von Neumann regular right cononsingular Baer ring.*
(2) *R is a right nonsingular right continuous ring.*

Proof. (1) $\Rightarrow$ (2) Assume that R is a von Neumann regular right cononsingular Baer ring. By Theorem 3.1, R is a right CS and right nonsingular ring. To show that R is right continuous, let N_R be a right R-submodule of R_R such that N_R is R-isomorphic to a direct summand of R_R. Then there exists $e^2 = e \in R$ such that $N_R \cong eR_R$. So $N = aR$ for some $a \in R$. Since R is von Neumann regular, $N = aR = fR$ for some $f^2 = f \in R$. Hence N_R is a direct summand of R_R. Therefore R is right continuous.

(2) $\Rightarrow$ (1) Assume that R is right nonsingular and right continuous. Since R is right continuous, R is right CS. By Theorem 3.1, R is a right cononsingular Baer ring. To show that R is von Neumann regular, take $a \in R$. As R is a Baer ring, R is a right p.p. ring. Hence aR_R is projective. Therefore aR_R is isomorphic to a direct summand of R_R. Because R is right continuous, aR_R is a direct summand of R_R. So R is von Neumann regular. □

As was mentioned before, we note that any commutative semiprime ring is (right) cononsingular. Further, note that for a commutative ring R, R is semiprime if and only if R is (right) nonsingular if and only if R is reduced. Hence, we get the following immediately from Proposition 3.2.

Corollary 3.3. *Let R be a commutative ring. The following statements are equivalent*:

(1) *R is a von Neumann regular Baer ring.*
(2) *R is a reduced continuous ring.*

Theorem 3.4. *Let R be a commutative ring. The following statements are equivalent*:

(1) *R is a von Neumann regular* Π-*coherent ring.*
(2) *R is a reduced self-injective ring.*
(3) *R is a reduced* FPF *ring in which every regular* (*nonzero-divisor*) *element is a unit.*

Proof. (1) $\Rightarrow$ (2) As was mentioned in Corollary 2.3, it follows from [30, Theorem 2; 8, Theorem 1].

(2) $\Rightarrow$ (1) It suffices to show that R is a Π-coherent ring. Since R is a self-injective ring, we infer that R is FPF. By Theorem 2.5, R is Π-semihereditary and hence R is Π-coherent.

(2) $\Leftrightarrow$ (3) It follows from [15, Corollary 1.2, p. 73]. □

Remark 3.5. A ring in which every regular element (elements that are neither left nor right zero-divisors) is a unit is called a *classical ring*, and various examples are provided, see [32, pp. 320–322] for more details. We know that any right self-injective ring R is a classical ring, see [31, Exercise 11.8, p. 248]. It is clear the converse is not true, even in the commutative case. In view of [15, Corollary 1.2, p. 73], we deduce that a commutative classical ring is self-injective if and only if it is an FPF ring.

4. Applications to $C(X)$

We now turn our attention to rings of continuous functions. We are interested in a result due to Fine, Gillman and Lambek [19, 3.5] which states $C(X) = Q_{\max}(X)$ if and only if X is an extremally disconnected P-space. It is well-known that $C(X)$ is a Baer ring if and only if X is an extremally disconnected space, see for example [4]. It is known that $C(X)$ is a von Neumann regular ring if and only if X is a P-space, see [20, 4J(8)]. Recall that a space X is said to be a *P-space*, if every zero-set of X is open. Conditions equivalent to a space being a P-space are given in [20, 4J and 14.29]. Hence, we infer that $C(X)$ is a self-injective ring if and only if $C(X)$ is a von Neumann regular Baer ring, for other proofs see [4, 14]. Note that their result does not apply to commutative reduced rings in general, see, e.g. [41, p. 1149, 4.4]. It is natural to ask why this is so? The "only if" part of this result naturally follows from the well-known fact in ring theory which states every commutative (even, non-commutative) reduced self-injective ring is a von Neumann regular Baer ring, see for examples [38, Proposition 1.7]; [29, Theorem 2.2] and Corollary 3.3. (We note that the condition "reduced" in the above fact is not superfluous, for example, $\mathbb{Q}[x]/(x^2)$ is a self-injective ring that is not von Neumann regular.) The "if" part leads us to the discovery of a result that gives some algebraic characterizations for a $C(X)$ to be Baer. Before we proceed further, let us recall some definitions and facts.

It is well known that $C(X)$ is a semihereditary ring if and only if X is a basically disconnected space. It is known that $C(X)$ is a p.p. ring if and only if it is a semihereditary ring, or equivalently, if and only if it is a coherent ring. The reader is referred to [3] for some equivalent conditions for when $C(X)$ is semihereditary.

Now, we give some characterizations for a $C(X)$ to be Baer.

Theorem 4.1. *The following statements are equivalent*:

(1) $C(X)$ *is a Baer ring.*
(2) $C(X)$ *is an* FPF *ring.*
(3) $C(X)$ *is a* Π-*coherent ring.*

Proof. (1) $\Rightarrow$ (2) Assume that $C(X)$ is a Baer ring. This implies that $C(X)$ is a semihereditary ring and $Q_{\mathrm{cl}}(X)$ is a self-injective ring. Hence, $C(X)$ is an FPF ring by Theorem 2.5.

(2) $\Rightarrow$ (1) Suppose that $C(X)$ is an FPF ring. Thus, it is a CS ring, see [15, p. 83]. This implies that $C(X)$ is a Baer ring since $C(X)$ is a reduced ring.

(2) $\Rightarrow$ (3) It follows from Theorem 2.5.

(3) $\Rightarrow$ (2) Assume that $C(X)$ is a Π-coherent ring. Hence, it is a coherent ring and so $C(X)$ is a semihereditary ring. This yields $C(X)$ is a semihereditary Π-coherent ring. This implies that $C(X)$ is an FPF ring by Theorem 2.5. □

The following result is due to Martinez [37, Theorem 2.7]. Just for the record, we retrieve it here as a corollary of Theorems 2.5 and 4.1.

Corollary 4.2. *$C(X)$ is a Baer ring if and only if $C(X)$ is an* I*-ring in the sense of Eggert* [13].

We recall that a topological space X is said to be an *almost P-space*, if every nonempty zero-set of X has a nonempty interior. It is known that X is an almost P-space if and only if $C(X)$ is a classical ring. Although the following result is not new in the context of $C(X)$ (see [4, Theorem 4.2]), we are in a position to show it via an algebraic approach by Corollary 3.3 and Theorems 3.4 and 4.1.

Corollary 4.3. *The following statements are equivalent*:

(1) *X is an extremally disconnected P-space.*
(2) *X is an extremally disconnected almost P-space.*
(3) *$C(X)$ is a self-injective ring.*
(4) *$C(X)$ is a continuous ring.*

According to [23], a space X is called *strongly fraction-dense* if $Q_{\mathrm{cl}}(X) = Q_{\max}(X)$. It is known that every perfectly normal space (e.g. metric space) and extremally disconnected space is strongly fraction-dense.

In view of Theorems 2.5 and 3.4, we make the following easy observation.

Corollary 4.4. *The following statements are equivalent*:

(1) *X is a strongly fraction-dense.*
(2) *$Q_{\mathrm{cl}}(X)$ is a self-injective ring.*
(3) *$Q_{\mathrm{cl}}(X)$ is an* FPF *ring.*

Following [23], a space X is called *fraction-dense* if $Q_{\mathrm{cl}}(X)$ and $Q_{\max}(X)$ have the same idempotents, see [5, 23] for more details. It is easy to see that every strongly fraction-dense space is fraction-dense. It is an open question whether there is a fraction-dense space that is not strongly fraction-dense, see [23, p. 983]. We hope that Corollary 4.6 sheds some light on this question. First, let us recall the following result, which is [22, Theorem 13.13].

Theorem 4.5. *A von Neumann regular ring R is right continuous if and only if R contains all the idempotents of its maximal right quotient ring.*

In view of the above theorem, we make the following result.

Corollary 4.6. *The following statements are equivalent*:

(1) *X is a fraction-dense space.*
(2) *$Q_{\mathrm{cl}}(X)$ is a continuous ring.*

Proof. (1) $\Rightarrow$ (2) If X is fraction-dense, we have $\mathrm{Min}(C(X))$, where $\mathrm{Min}(C(X))$ is the set of minimal prime ideals of $C(X)$ and it is equipped with the hull-kernel topology, is compact by [23, Theorem 1 1]. Thus, we infer that $Q_{\mathrm{cl}}(X)$ is a von Neumann regular ring. Note that $Q_{\mathrm{cl}}(X)$ and $Q_{\max}(X)$ have the same idempotents by [23, Theorem 1 1]. Now, Theorem 4.5 completes the proof.

(2) $\Rightarrow$ (1) Suppose that $Q_{\mathrm{cl}}(X)$ is a continuous ring. Since $Q_{\mathrm{cl}}(X)$ is reduced, $Q_{\mathrm{cl}}(X)$ is a von Neumann regular ring. Using Theorem 4.5, we infer that R and $Q_{\max}(X)$ have the same idempotents. Hence, X is fraction-dense by [23, Theorem 1 1((1) $\Leftrightarrow$ (7))]. ◻

As noted earlier, the class of Baer rings is not closed under matrix rings, even for commutative reduced rings. In view of Theorems 2.5 and 4.1 and, we infer that $C(X)$ is a Baer ring if and only if $\mathrm{M}_n(C(X))$ is a Baer ring for every positive integer n. In the following, we give some characterizations for $C(X)$ to be Baer. First, let us recall some definitions and facts.

Birkenmeier *et al.* [7, Theorem 6.1.3] proved that a ring R is left semihereditary and right Π-coherent if and only if $\mathrm{M}_n(R)$ is a Baer ring for every $n \geq 1$.

Beidar *et al.* [6, Theorem 4.15] proved that for a reduced ring R and every $n \geq 1$, R is semihereditary and $Q_{\mathrm{cl}}(R) = Q_{\max}(R)$ if and only if $\mathrm{M}_n(R)$ is a CS-ring.

According to [35], a ring R is said to be a *right* AFG *ring* if the right annihilator of every nonempty subset of R is a finitely generated right ideal, or equivalently, if every cyclic torsionless right R-module is finitely presented. A left AFG ring is defined similarly. Note that the definition of AFG rings is not left–right symmetric, see [35, Example 2.4]. A ring which is both right and left AFG is called AFG. It is clear that every Baer ring is a left and right AFG. However, the ring $\mathbb{Z}_4$ is an AFG ring that is not a Baer ring. We note that R is right Π-coherent if and only if $\mathrm{M}_n(R)$ is a right AFG ring for every $n \geq 1$, see for examples [8, Theorem 1; 36, Corollary 2.5].

Using the above facts and Theorem 4.1, we make the following result.

Corollary 4.7. *The following statements are equivalent*:

(1) *$C(X)$ is a Baer ring.*
(2) *$\mathrm{M}_n(C(X))$ is a Baer ring for every $n \geq 1$.*
(3) *$\mathrm{M}_n(C(X))$ is a* CS-*ring for every $n \geq 1$.*
(4) *$\mathrm{M}_n(C(X))$ is an* AFG *ring for every $n \geq 1$.*

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