Asian-European Journal of Mathematics
Vol. 17, No. 12 (2024) 2450087 (22 pages)

DOI: 10.1142/S1793557124500876

# On weak Ikeda–Nakayama rings

Le Van Thuyet
*Department of Mathematics*
*College of Education, Hue University, 34 Le Loi, Hue City, Vietnam*
*lvthuyet@hueuni.edu.vn*

Abdoul Djibril Diallo
*Département de Mathématiques et d'Informatique*
*Faculté des Sciences et Techniques*
*Université Cheikh Anta Diop de Dakar, Sénégal*
*dialloabdoulaziz58@yahoo.fr*

Papa Cheikhou Diop
*Département de Mathématiques*
*Université Iba Der Thiam de Thiès, Sénégal*
*cheikpapa@gmail.com*

Truong Cong Quynh*
*Faculty of Mathematics*
*The University of Danang — University of Science and Education*
*Da Nang 550000, Vietnam*
*tcquynh@ued.udn.vn*

*Communicated by A. R. Rajan*
Received June 13, 2024
Accepted July 30, 2024
Published 30 September 2024

A ring $R$ is called a left Ikeda–Nakayama ring (briefly, left IN-ring) if $r(I \cap K) = r(I) + r(K)$, for all left ideals $I$ and all left ideals $K$ of $R$. A ring $R$ is a called a left $WIN$-ring if $r(I \cap K) = r(I) + r(K)$ for all finitely generated semisimple left ideals $I$ and all left ideals $K$ of $R$. It is clear that a left $IN$ ring must be left $WIN$. Right $WIN$-rings can be defined similarly. It is shown that a left $WIN$-ring may not be right $WIN$ and a left $WIN$ ring may not be left $IN$. One of the aims of this paper is to investigate left $WIN$-rings satisfying additional conditions. We show that this weak injectivity property is useful in obtaining semiperfect rings. Moreover, we give several new characterizations of $PF$ rings and $QF$ rings via $WIN$-rings. Finally, left $C_{11}$, $WIN$-rings were considered.

*Keywords*: $IN$-ring; $WIN$-ring; PF-ring; Kasch ring; quasi-Frobenius ring; dual ring.

AMS Subject Classification: 16D50, 16D70, 16D80

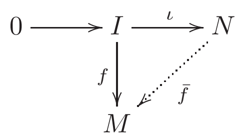
*Corresponding author.

## 1. Introduction

Throughout this paper, all rings $R$ are associative with identity and all modules are unitary right $R$-module. We use $S_l$, $S_r$, $J$, $Z_l$ and $Z_r$ to denote the left socle, the right socle, the Jacobson radical, the left singular ideal and the right singular ideal, respectively. The notation $N \leq_e M$ means that $N$ is an essential submodule. If $X$ is a subset of a ring $R$, the right (left) annihilator in $R$ is denoted by $r(X)\,(l(X))$.

Recall that a ring $R$ is called *right CF* if every cyclic right $R$-module embeds in a free module.

Let $M$ be a right $R$-module. We consider a right $R$-module $N$, $I$ a submodule of $N$ and $f : I \to M$ a homomorphism. Take the following diagram:




- If $M = N = R$ and there exists $\bar{f}$ for every minimal right ideal $I$, then $R$ is called *right mininjective*.
- If $M = N = R$ and there exists $\bar{f}$ for every right ideal $I$ of $R$ with $f(I)$ simple, then $R$ is called *right simple-injective*.
- If there exists $\bar{f}$ with $I = \mathrm{Soc}(N)$ simple, then $M$ is called *soc-$N$-injective*. $M$ is called *soc-injective* if $M$ is soc-$R$-injective. $M$ is called *strongly soc-injective* if $M$ is soc-$N$-injective for all $R$-modules $N$. A ring $R$ is called *strongly left* (*right*, *respectively*) *soc-injective* if ${}_RR$ ($R_R$), respectively, is strongly soc-injective.

A ring $R$ is called *right* (*left*, *respectively*) *quasi-dual* if every essential right (left, respectively) ideal of $R$ is a right (left, respectively) annihilator [15]. $R$ is called *quasi-dual* if it is two-sided quasi-dual. A ring $R$ is called *right GP-injective* (respectively, *right AGP-injective*) if for each $0 \neq a \in R$, there exists $n \in \mathbb{N}$ such that $a^n \neq 0$ and $lr(a^n) = Ra^n$ (respectively, $Ra^n$ is a direct summand of $lr(a^n)$) ([27]).

Recall that a module $M$ is said to be a *$C_{11}$-module* if every submodule of $M$ has a complement which is a direct summand [20]. A ring $R$ is called a *right $C_{11}$-ring* if $R_R$ is a $C_{11}$-module. Clearly, every $CS$-module satisfy the $C_{11}$-condition. However, the converse is not true in general (see [20, p. 1814]).

There are several results in the literature that are important sources of semiperfectness: For example, in [7], it was shown that every left self-injective right Kasch ring is semiperfect. Also, it was proved later in [26] that if $R$ is left $CS$ and the dual of every simple right $R$-module is simple, then $R$ is semiperfect with $S_r = S_l \leq_e {}_RR$. The latter result was extended in [9] to left min-$CS$ ring. Motivated by these results, we introduce the notion of left $WIN$-rings (i.e. $r(I \cap K) = r(I) + r(K)$ for all finitely generated semisimple left ideals $I$ and all left ideals $K$ of $R$) as a generalization of left $IN$-rings (i.e. $r(I \cap K) = r(I) + r(K)$ for all left ideals $I$ and $K$ of $R$) and hence of left self-injective rings. In this paper, we show that this weak injectivity property is useful in obtaining semiperfect rings. We also investigate left $WIN$-rings with $ACC$ on right annihilators. Furthermore, we use left $WIN$-rings to characterize Pseudo-Frobenius rings and quasi-Frobenius rings.

In Sec. 2, we give the properties of some classes of $WIN$-rings. Among other things, we prove that if $R$ is left $WIN$-ring, then $R$ is right Kasch with $S_r \subseteq S_l$ if and only if $R$ is semiperfect with $S_r = S_l$ and $\mathrm{Soc}(Re)$ is simple and essential for every local idempotent $e$ of $R$. As a corollary of this result, we prove that if $R$ is left $WIN$, then the dual of every simple right $R$-module is simple if and only if $R$ is semiperfect with $S_r = S_l$ and $\mathrm{Soc}(Re)$ is simple and essential for every local idempotent $e$ of $R$.

In Sec. 3, some results on Kasch rings and Pseudo-Frobenius rings are obtained via $WIN$-rings. It is shown that a right Kasch, right $SF$-injective and $WIN$-ring is two-sided $GPF$, two-sided finitely cogenerated and right continuous. It is also proved that a ring $R$ is left $PF$ if and only if it is left automorphism-invariant, left $WIN$ and the dual of every simple right $R$-module is simple. Moreover, it is shown that every strongly right soc-injective left Kasch ring with $S_l \subseteq S_r$ is right $PF$. The two latter results extend the work in [2, Theorem 5.6(5), 26].

In Sec. 4, we provide new characterizations of quasi-Frobenius via left $WIN$-rings. Among other results, we show that a ring $R$ is quasi-Frobenius if and only if it is left $WIN$ right $CF$ with $S_r \subseteq S_l$ if and only if it is left $WIN$ left $GP$-injective with $ACC$ on left annihilators. Recall that a module $M$ is called *uniserial* if its submodules are linearly ordered by inclusion. A ring $R$ is called *right* (*left*) *uniserial* if $R_R$ (${}_RR$) is uniserial. It was shown in [12, Theorem 2], that a left uniserial right perfect ring is left aratinian whose factor rings are right $P$-injective. Using this result and our work, we prove that every left uniserial right perfect ring is quasi-Frobenius.

Let $P$ be a property of rings. A ring $R$ is said to be *completely* $P$ if each factor ring of $R$ has the property $P$. At the end of this section, we characterize completely quasi-Frobenius rings in terms of completely $WIN$-rings by showing for example that a ring $R$ is completely quasi-Frobenius if and only if it is completely $WIN$ completely quasi-dual.

In Sec. 5, it is shown that every right cogenerator left $C_{11}$-ring is right $PF$. We also prove that if $R$ is a left $C_{11}$ right $CF$ ring, then $R$ is quasi-Frobenius if and only if $\mathrm{Soc}(Re) \neq 0$ for every local idempotent $e$ of $R$. We also prove that a ring $R$ is QF if and only if it is a right $C$-continuous (i.e. right $C_2$ and right $C_{11}$) left $WIN$-ring with $ACC$ on right annihilators if and only if it is a right $C$-continuous left WIN-ring and $R/S_r$ is right Goldie.

## 2. On Certain Classes of Left WIN-Rings

We have a very interesting property of a left self-injective ring as follows: If $R$ is a left self-injective ring then

$$r(I \cap K) = r(I) + r(K),$$

for all left ideals $I$ and all left ideals $K$ of $R$. A ring satisfies this condition is called a *left Ikeda–Nakayama ring* (briefly, left $IN$-ring). Of course, left self-injective ring $\Rightarrow$ left $IN$-ring. We will consider a weak class of $IN$-rings as follows.

**Definition 2.1.** A ring $R$ is called a left weakly Ikeda–Nakayama ring (briefly, left $WIN$-ring) if $r(I \cap K) = r(I) + r(K)$ for all finitely generated semisimple left ideals $I$ and all left ideals $K$ of $R$. Right $WIN$-rings can be defined similarly. And a ring $R$ is called a $WIN$-ring if it is a left and right $WIN$-ring.

We obtain immediately the following implication:

$$\text{a left IN-ring} \Rightarrow \text{a left WIN-ring}.$$

Firstly, we give some basic properties of left $WIN$-rings.

**Proposition 2.1.** *Let $R$ be a left $WIN$-ring. Then*:

(1) *If $T$ is a finitely generated semisimple left ideal of $R$, then $T \leq_e lr(T)$.*
(2) *If $lr(S_l) = S_l$, then $lr(T) = T$ for all finitely generated semisimple left ideals $T$ of $R$.*
(3) *If $R$ is right Kasch and $lr(S_l) = S_l$, then $kR$ is simple whenever $Rk$ is simple. In particular, $S_l \subseteq S_r$.*
(4) *If $T$ is a finitely generated semisimple left ideal of $R$ and $r(T) \subseteq J$, then $T \leq_e {}_RR$.*

**Proof.** (1) Let $T$ be a finitely generated semisimple left ideal of $R$. Assume that there exists $c \in lr(T)$ such that $T \cap Rc = 0$. Then by hypothesis, $r(T \cap Rc) = r(T) + r(c) = R$. As $c \in lr(T)$, then $r(T) \subseteq r(c)$, from which it follows that $R = r(c)$. Thus, $c = 0$, and so $T \leq_e lr(T)$.

(2) Assume that $lr(S_l) = S_l$ and $T$ is a finitely generated semisimple left ideal of $R$. Then by (1), $T \leq_e lr(T)$. Since $lr(T) \subseteq lr(S_l) = S_l$, it follows that $T = lr(T)$.

(3) Let $Rk$ be simple left ideal of $R$ and let $T$ be a maximal right ideal of $R$ such that $r(k) \subseteq T$. By (2), $l(T) \subseteq Rk$. Since $l(T) \neq 0$, $l(T) = Rk$, and so $T = r(k)$. Therefore, $kR$ is simple and consequently, $S_l \subseteq S_r$.

(4) Suppose that there exists $c \in R$ such that $T \cap Rc = 0$. Then $R = r(T) + r(c)$, and so by hypothesis, $R = J + r(c)$. This implies that $R = r(c)$, and so $c = 0$. □

A ring $R$ is called *right minsymmetric* if for any minimal right ideal $kR$ of $R$, $Rk$ is a minimal left ideal of $R$. Our next result characterizes the right mininjective rings among the left $WIN$-rings.

**Proposition 2.2.** *Let $R$ be a left $WIN$-ring. Then the following conditions are equivalent*:

(1) *$R$ is right mininjective*;
(2) *$R$ is right minsymmetric*;

(3) $S_r \subseteq S_l$.

*In particular, a commutative $WIN$-ring is mininjective.*

**Proof.** (1) $\Rightarrow$ (2) $\Rightarrow$ (3) follow from [13, Theorem 2.21].

(3) $\Rightarrow$ (1) Suppose $S_r \subseteq S_l$. Let $kR$ be a simple right ideal of $R$. According to [13, Lemma 2.1], we need to show that $lr(k) = Rk$. Now, let $0 \neq x \in lr(k)$. Then, $r(k) \subseteq r(x)$. Since, $r(k)$ is a maximal right ideal of $R$, $r(k) = r(x)$. Consequently, $xR$ is a right simple ideal of $R$, and so $xR \subseteq S_l$. It follows that $Rk \subseteq lr(k) \subseteq S_l$. So $lr(k)$ is a semisimple left $R$-module containing $Rk$. Therefore, $lr(k) = Rk$ by Proposition 2.1(1), as desired. □

Examples of left $WIN$-rings include left simple-injective rings (see [4, Lemma 2.2]) and strongly left soc-injective rings (see [2, Proposition 5.2]).

**Corollary 2.1.** *Let $R$ be a left simple-injective or strongly left soc-injective ring. Then the following conditions are equivalent*:

(1) *$R$ is right mininjective*;
(2) *$R$ is right minsymmetric*;
(3) $S_r \subseteq S_l$.

It is clear that left $IN$-rings and hence left uniserial rings are left $WIN$. But neither of the converses is true, in general as illustrated in the following examples. The second example shows also that a left $WIN$-ring need not be right $WIN$.

**Example 2.1 ([2, Example 5.9]).** Let $K$ be a field and let $R$ be the ring of all lower triangular, countably infinite square matrices over $K$ with only finitely many off-diagonal entries. Let $S$ be the $K$-subalgebra of $R$ generated by 1 and $J$. By [2, Proposition 5.2], $S$ is left perfect left $WIN$ which is not left finite dimensional. Therefore, $S$ is not left $IN$ by [13, Theorem 6.32].

**Example 2.2 ([13, Example 6.41]).** Consider the Bjørk Example in [13, Example 2.5]. Let $F$ be a field and assume that

$$F \to \bar{F} \subseteq F$$

$$a \mapsto \bar{a}$$

is an isomorphism, where the subfield $\bar{F} \neq F$. Let $R$ denote the left vector space on basis $\{1, t\}$, and make $R$ into an $F$-algebra by defining $t^2 = 0$ and $ta = \bar{a}t$ for all $a \in F$. Then $R$ is local, $R/J \cong F, J^2 = 0$ and $J = Rt = Ft$ is the only proper left ideal of $R$. Moreover, $lr(L) = L$ for all left ideals $L$ of $R$. From this, $R$ is left $WIN$ and $S_l \subseteq S_r$. However, $R$ need not be right $WIN$. In addition, $R$ is not left mininjective. If $R$ were right $WIN$, then it would be left mininjective by Proposition 2.2, a contradiction. Moreover, as $R$ is not left mininjective, then $R$ is neither left simple-injective nor strongly left soc-injective.

A ring $R$ is said to be *left minannihilator* (respectively, *left min-CS*) if every minimal left ideal $I$ of $R$ is an annihilator (respectively, $I$ is essential in a direct summand).

**Theorem 2.1.** *Let $R$ be a right Kasch left WIN-ring in which $S_r \subseteq S_l$. The following statements hold*:

(1) *$R$ is semiperfect*;
(2) *$S_r = S_l$ is finitely generated and essential as a left ideal*;
(3) *$R$ is left minannihilator*;
(4) *$R$ is left min-CS*;
(5) $\mathrm{Soc}(Re)$ *is simple and essential for every local idempotent $e$ of $R$*;
(6) *For every $x \in R$, $Rx$ is a simple left ideal if and only if $xR$ is simple right ideal.*

*Conversely, if $R$ is semiperfect with $S_r = S_l$ and $\mathrm{Soc}(Re)$ is simple and essential for every local idempotent $e$ of $R$, then $R$ is right Kasch.*

**Proof.** (1) By Proposition 2.2, $R$ is right mininjective. Now, let $T$ be a maximal right ideal of $R$. Since $R$ is right Kasch, $l(T) \neq 0$. There exists $0 \neq a \in R$ such that $aT = 0$. Thus, $T = r(a)$, and so $R/r(a) \cong aR$ is a simple right ideal. As $R$ is right mininjective, then $Ra$ is a left simple ideal by [13, Theorem 2.21]. Let $L$ be a left ideal maximal with respect to $Ra \cap L = 0$. By hypothesis, $r(Ra \cap L) = r(a) + r(L) = r(0) = R$. On the other hand, we have $T \cap r(L) = r(a) \cap r(L) = r(Ra + L)$ and $Ra + L = Ra \oplus L \leq_e {}_RR$. Consequently, $T \cap r(L) \subseteq Z_l$. As $R$ is right Kasch, then $Z_l \subseteq J$ by [13, Proposition 1.46]. Thus, $R$ is semilocal by [18, Corollary 2.2], and so $l(J) = S_r$ is a finitely generated semisimple left ideal by [13, Theorem 5.52]. Hence, by hypothesis $r(l(J) \cap I) = rl(J) + r(I) = J + r(I)$ for every left ideal $I$ of $R$. Using [10, Theorem 3.8], we deduce that idempotents can be lifted over $J$. Therefore, $R$ is semiperfect.

(2) By the proof of (1), $S_r$ is a finitely generated semisimple left ideal of $R$. But $R$ is right Kasch. Then $r(S_r) = J$ and we conclude by Proposition 2.1(4) that $S_r \leq_e {}_RR$. Therefore, it follows from (1), Proposition 2.2 and [13, Proposition 5.54] that $S_r = S_l$ is finitely generated and essential as a left ideal.
(3) Since $R$ is right Kasch, $r(S_r) = J$. Thus, by (1) and (2), $lr(S_l) = S_l$. Using Proposition 2.1(2), we deduce that $R$ is left minannihilator.
(4) By (1) and (2), $R$ is semiperfect and $S_r \leq_e {}_RR$. Thus, by [13, Lemma 4.2], $lr(T)$ is essential in a direct summand of ${}_RR$ for every left ideal $T$ of $R$. Therefore, by (3), $R$ is left min-$CS$.
(5) and (6) follow from (4), Proposition 2.2 and [13, Theorem 4.8].

Conversely, assume that $R$ is semiperfect with $S_r = S_l$ and $\mathrm{Soc}(Re)$ is simple and essential for every local idempotent $e$ of $R$. Then, $S_l \leq_e {}_RR$. Therefore, being semiperfect, $R$ is right Kasch by [13, Lemma 4.2]. □

Following [13, Theorem 2.31], a ring $R$ is right Kasch right minininjective if and only if the dual of every simple right $R$-module is simple.

**Corollary 2.2.** *Let $R$ be a left $WIN$-ring. Then, the dual of every simple right $R$-module is simple if and only if $R$ is semiperfect with $S_r = S_l$ and* $\mathrm{Soc}(Re)$ *is simple and essential for every local idempotent $e$ of $R$.*

**Proof.** This follows from Theorem 2.1 and [13, Theorem 2.31]. □

Now, we will close this section by investigating left $WIN$-rings satisfying ACC on right annihilators.

**Proposition 2.3.** *If $R$ is a left WIN-ring and satisfying ACC on right annihilators, then* $\mathrm{Soc}({}_RR)$ *is finitely generated.*

**Proof.** If $\mathrm{Soc}({}_RR) = 0$, then we are done. Otherwise, assume that $\mathrm{Soc}({}_RR)$ is not finitely generated, then it contains $\bigoplus_{i=1}^{\infty} Ra_i$ with $Ra_i$ simple. Call $I_n = r(a_n, a_{n+1}, \ldots)$ for all $n \geq 1$. Then, we have

$$I_1 \leq I_2 \leq \cdots \leq I_n \leq \cdots.$$

Since $R$ has ACC on right annihilators, there exists $m \geq 1$ such that $I_m = I_k$ for all $k \geq m$. It follows that $r(a_{m+1}, a_{m+1}, \ldots) \leq r(a_m)$. As $\bigoplus_{i=m+1}^{\infty} Ra_i \cap Ra_m = 0$ and $R$ is a left WIN-ring, $R = r(\bigoplus_{i=m+1}^{\infty} Ra_i \cap Ra_m) = r(\bigoplus_{i=m+1}^{\infty} Ra_i) + r(Ra_m)$. Then, we have $R = r(a_{m+1}, a_{m+2}, \ldots) + r(a_m) = r(a_m)$ and so $a_m = 0$, a contradiction. We deduce that $\mathrm{Soc}({}_RR)$ is finitely generated. □

**Corollary 2.3.** *Assume that $R$ is a left WIN-ring and satisfies ACC on right annihilators. If $r(S_l) \leq J(R)$ then $R$ is left finitely cogenerated.*

**Corollary 2.4.** *Let $R$ be a right perfect left WIN-ring. If $R$ is left pseudo-coherent, then $R$ is left finitely cogenerated.*

**Proof.** Since $R$ right perfect, $R$ has $DCC$ on finitely generated left ideal. Hence, if $X \subseteq R$, then $l(X) = l(X_0)$ for some finite subset $X_0$ of $X$. It follows that $R$ satisfies $DCC$ on left annihilators and so $R$ has $ACC$ on right annihilators. But $R$ is left WIN-ring. Then $R$ is left finitely cogenerated by Proposition 2.3. □

**Lemma 2.1 ([25, Lemma 4.3]).** *If $R$ has ACC on right annihilators and $S_l \leq_e R_R$, then $R$ is semiprimary.*

**Lemma 2.2.** *Let $R$ be a semiperfect ring in which $S_l \leq_e R_R$. Then*:

(1) *$R$ is left Kasch and $rl(T)$ is essential in a direct summand of $R_R$ for every right ideal $T$ of $R$.*
(2) *If $R$ is left $WIN$, then $R$ is right mininjective.*

**Proof.** (1) follows from [13, Lemma 4.2].

(2) By (1), $R$ is left Kasch, and so $l(S_l) = J$. On the other hand, since $S_l \leq_e R_R$, we have $l(S_l) \subseteq Z_r$. Therefore, $J \subseteq Z_r$. Now, let $y \in S_r$. Then, $Z_r y = 0$, and so $y \in r(Z_r) \subseteq r(J)$. Since $R$ is semiperfect, $r(J) = S_l$. Thus, $S_r \subseteq S_l$ and we deduce from Proposition 2.2 that $R$ is right mininjective. □

**Theorem 2.2.** *Let $R$ be a left $WIN$-ring with $ACC$ on right annihilators in which $S_l \leq_e R_R$. Then $R$ is right mininjective, left Artinian and satisfies the conditions* (1) *through* (6) *of Theorem* 2.1.

**Proof.** Since $R$ has $ACC$ on right annihilators with $S_l \leq_e R_R$, it follows from Lemma 2.1 that $R$ is semiprimary. Thus, being left $WIN$, $R$ is right mininjective by Lemma 2.2. In [13, Theorem 3.12], it was proved that a right mininjective semiprimary ring is right Kasch. Thus, since $R$ is left $WIN$, we infer from Theorem 2.1 that $S_r = S_l$ is finitely generated and essential as a left ideal. Therefore, according to [13, Lemma 3.30], $R$ is left Artinian. The last part follows from Theorem 2.1(2) because $R$ is right Kasch. □

In general, a right $AGP$-ring with $ACC$ on right annihilators need not be left Artinian as showed in [27, P. 339]. The following corollary shows that the condition "$R$ is left $WIN$-ring" forces "a right $AGP$-injective ring with $ACC$ on right annihilators to be left Artinian".

**Corollary 2.5.** *Let $R$ be a left $WIN$, right $AGP$-injective ring with $ACC$ on right annihilators. Then $R$ is right mininjective, left Artinian and satisfies the conditions* (1) *through* (6) *of Theorem* 2.1.

**Proof.** Since $R$ is right $AGP$-injective with $ACC$ on right annihilators, we infer from [27, Lemma 1.3 and Corollary 1.6] that $R$ is semiprimary and $J = Z_r$. Thus, $S_r \subseteq S_l$ and so $S_l \leq_e R_R$. Therefore, the claim follows from Theorem 2.2. □

**Proposition 2.4.** *Let $R$ be a left $WIN$-ring with $ACC$ on right annihilators such that the dual of every simple right $R$-module is simple. Then $R$ is left Artinian.*

**Proof.** We firstly prove that $J$ is nilpotent. Since the dual of every simple right $R$-module is simple, $R$ is right Kasch right mininjective by [13, Theorem 2.31]. Thus, $R$ is semilocal and $S_r = S_l$ by Theorem 2.1. Hence, [13, Lemma 3.36] implies that $l(J^n) = r(J^n)$ for all $n \geq 1$. Since $R$ has $ACC$ on right annihilators, there exists an integer $m$ such that $l(J^m) = r(J^m) = r(J^{2m}) = l(J^{2m})$. Then the following proof is owing to [17, Theorem 18]. Suppose that $J$ is not nilpotent. Then $J^m \neq 0$, and so $M_R = R/l(J^m)$ is a nonzero $R$-module. Hence, by [17, Lemma 17], the non-empty set $\{r_R(a) : 0 \neq a \in M\}$ has a maximal element, say $r_R(a_1)$. Write $a_1 = x + l(J^m)$ where $x \in R$. Then, $xJ^m \neq 0$. Since $l(J^m) = l(J^{2m})$, $xJ^m \nsubseteq l(J^m)$. So, there exists $b \in J^m$ such that $xb \notin l(J^m)$. Since $R$ is semilocal, it follows from

Theorem 2.1(2) that $l(J) \leq_e {}_RR$ and hence $l(J^m) \leq_e {}_RR$. So, $Rxb \cap l(J^m) \neq 0$. Thus, there exists $y \in R$ such that $0 \neq yxb \notin l(J^m)$. Let $a_2 = yx + l(J^m) \in M$. Then, $a_2 \neq 0$ and $b \in r_R(a_2)$. But $b \notin r_R(a_1)$. So the inclusion $r_R(a_1) \subset r_R(a_2)$ is proper. This contradicts the choice of $a_1$. Therefore, $J$ is nilpotent. Hence, $R$ being semilocal, $R$ is semiprimary. Note that $S_r = S_l$. Therefore, the claim follows from Theorem 2.2. □

## 3. On Kasch Rings and Pseudo-Frobenius Rings via WIN-Rings

Following [24], a ring $R$ is called right simple-$FJ$-injective if every right $R$-homomorphism from a small finitely right ideal to $R$ with a simple image, can be extended to an endomorphism of $R_R$. A ring $R$ is called left $P$-injective ring if every $R$-homomorphism from a principal left ideal to $R$ extends to an endomorphims of ${}_RR$. A ring $R$ is called left $GPF$ if $R$ is a left $P$-injective, semiperfect ring and $S_l \leq_e {}_RR$.

**Proposition 3.1.** *Let $R$ be a right Kasch, right simple-$FJ$-injective and left $WIN$-ring. Then, $R$ is left $GPF$, two-sided finitely cogenerated and right continuous.*

**Proof.** Since $R$ is right simple-$FJ$-injective, $R$ is right mininjective by [24, Lemma 3.3]. So, being right Kasch and left $WIN$, $R$ is semiperfect and $S_r = S_l \leq_e {}_RR$ by Theorem 2.1. Therefore, it follows from [24, Proposition 3.7] that $\mathrm{Soc}(eR)$ is either simple or zero for all local idempotents $e$ of $R$. Hence, $R$ is left mininjective by [13, Proposition 3.5]. Now, since $R$ is semiperfect and $S_r = S_l \leq_e {}_RR$, $\mathrm{Soc}(eR) \neq 0$ by [13, Theorem 3.12]. Using [24, Proposition 3.8], we deduce that $R$ is left $GPF$ and two-sided finitely cogenerated. Hence, $R$ is left Kasch. Note that $rl(I) = I$ for every finitely generated right ideal $I$ of $R$ by [24, Proposition 3.8]. Thus, every finitely generated right ideal of $R$ is essential in a direct summand of $R_R$ by [13, Lemma 4.2]. Therefore, being left Kasch, $R$ is right continuous by [6, Corollary 7.8]. □

**Corollary 3.1.** *Let $R$ be a right Kasch and left simple-injective ring with $S_r \subseteq S_l$. Then, $R$ is right $GPF$, two-sided finitely cogenerated and left continuous.*

**Proof.** Since $R$ is left simple-injective, it is left $WIN$ by [4, Lemma 2.2]. Using Theorem 2.1, we deduce that $R$ is semiperfect with essential left socle. Thus, $R$ is left Kasch by [13, Theorem 6.16]. Therefore, the claim follows from the left version of Proposition 3.1. □

Following [24], a ring $R$ is called *right $SF$-injective* if every homomorphism from a small finitely generated right ideal to $R_R$ can be extended to an endomorphism of $R_R$.

**Proposition 3.2.** *Let $R$ be a right Kasch, right $SF$-injective and left $WIN$-ring. Then, $R$ is two-sided $GPF$, two-sided finitely cogenerated and right continuous.*

**Proof.** Since $R$ is right $SF$-injective, $R$ is right mininjective by [24, Proposition 2.6]. So, being right Kasch and left $WIN$, $R$ is semiperfect and $S_r = S_l \leq_e {}_RR$ by Theorem 2.1. Therefore, it follows from [24, Proposition 2.7 and Theorem 2.10] that $R$ is two-sided $P$-injective. Hence, $R$ is two-sided $GPF$ and two-sided finitely cogenerated by [13, Theorem 5.31]. Note that $rl(I) = I$ for every small finitely generated right ideal $I$ of $R$ by [24, Proposition 2.7]. Now, let $I$ be a finitely generated right ideal of $R$. Since $R$ is semiperfect, there exists a decomposition $R_R = e_1R \oplus e_2R$ such that $e_1R \subseteq I$ and $e_2R \cap I$ are a small right ideal of $R$. It follows that $I = e_1R \oplus (I \cap e_2R)$, and hence $l(I) = R(1-e_1) \cap l(I \cap e_2R)$. Thus, $rl(I) = r[R(1-e_1) \cap l(I \cap e_2R)] = e_1R + I \cap e_2R = I$ by [14, Lemma 2.1]. Using [13, Lemma 4.2], we deduce that every finitely generated right ideal of $R$ is essential in a direct summand of $R_R$. Therefore, being left Kasch, $R$ is right continuous by [6, Corollary 7.8]. □

A ring $R$ is *right* (left, respectively) *dual* if every right (left, respectively) ideal of $R$ is a right (left, respectively) annihilator. A ring is called a *dual ring* if it is left and right dual. It was proved in [13, Theorem 6.18] that a ring is dual if and only if it is two-sided Kasch two-sided simple-injective. In the next proposition, we show that the condition "two-sided Kasch" can be weakened to "one-sided Kasch" by using the $WIN$-rings.

**Proposition 3.3.** *The following conditions are equivalent*:

(1) *$R$ is a dual ring*;
(2) *$R$ is a one-sided Kasch two-sided simple-injective ring.*

**Proof.** (1) ⇒ (2) follows from [13, Theorem 6.18].

(2) ⇒ (1) Assume that $R$ is a right Kasch two-sided simple-injective ring. Since $R$ is left simple-injective, $R$ is left $WIN$ by [4, Lemma 2.2]. So, $R$ is left $GPF$ by Proposition 3.1. Therefore, $R$ is left Kasch by [13, Theorem 5.31]. Using [13, Theorem 6.18], we deduce that $R$ is a dual ring. Similarly, if we assume that $R$ is left Kasch two-sided simple-injective, then we can show that $R$ is a dual ring. □

The following result extends the work in [26, Theorem 2].

**Theorem 3.1.** *Then following conditions are equivalent for a ring $R$*:

(1) *$R$ is a left $PF$-ring*;
(2) *$R$ is a left automorphism-invariant left $WIN$-ring such that the dual of every simple right $R$-module is simple.*

**Proof.** By Theorem 2.1, $R$ is left min-$CS$ and $S_l$ is a finitely generated and essential left ideal of $R$. Then, $S_l = \oplus_{i=1}^{n} S_i$, where each $S_i$ is a simple left ideal, $1 \leq i \leq n$. Since $R$ is left min-$CS$, there exists an idempotent $e_i$ of $R$ such that $S_i \leq_e Re_i$, $1 \leq i \leq n$. As $\{S_i\}_{1\leq i\leq n}$ is an independent family, then so is $\{Re_i\}_{1\leq i\leq n}$ by

[8, Proposition 1.1($d$)]. On the other hand, it is well known that a left automorphism-invariant ring is left $C_3$. Hence, $\oplus_{i=1}^{n} Re_i$ is a direct summand of $R$. Since $S_l \subseteq \oplus_{i=1}^{n} Re_i$ and $S_l \leq_e {}_RR$, $\oplus_{i=1}^{n} Re_i \leq_e {}_RR$. Consequently, $R = \oplus_{i=1}^{n} Re_i$. Let $A$ be a nonzero submodule of $Re_i$. Since $S_i \leq_e Re_i$, $A \cap S_i \neq 0$. But $S_i$ is simple. Then, $A \cap S_i = S_i$, i.e. $S_i \subseteq A$. Similarly, for any nonzero submodule $B$ of $Re_i$, we have $S_i \subseteq B$. Thus, $0 \neq S_i \subseteq A \cap B$, and hence each $Re_i$ is uniform, $1 \leq i \leq n$. Therefore, $R$ is left self-injective by [1, Lemma 3.5]. Note that $S_l$ is a finitely generated and essential left ideal of $R$. Then, $R$ is left $PF$. □

It was shown in [2] that if $R$ strongly left soc-injective and the dual of every simple right $R$-module is simple, then $R$ is left $PF$. We extend this result by using the $WIN$-rings in the following theorem, improving in passing [26, Theorem 2].

**Theorem 3.2.** *Then following conditions are equivalent for a ring $R$:*

(1) *$R$ is a right $PF$-ring;*
(2) *$R$ is a strongly right soc-injective left Kasch ring with $S_l \subseteq S_r$.*

**Proof.** (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) By [2, Proposition 5.2], $R$ is right $WIN$. Thus, $R$ is semiperfect with essential right socle by the right version of Theorem 2.1. Using [2, Corollary 3.2], we deduce that $R$ is right self-injective. Therefore, $R$ is right $PF$. □

A module $M$ is said to be $ef$-extending if every closed essentially finite submodule of $M$ is essential in a direct summand of $M$. A ring $R$ is called *right $ef$*-extending if $R_R$ is $ef$-extending.

**Corollary 3.2.** *Then the following conditions are equivalent for a ring $R$:*

(1) *$R$ is a right $PF$-ring;*
(2) *$R$ is left Kasch with $S_l \subseteq S_r$ and $R \oplus R$ is $ef$-extending as a right $R$-module;*
(3) *$R$ is a right self-injective left Kasch ring with $J \subseteq Z_l$;*
(4) *$R$ is a right self-injective left Kasch ring with $S_l \subseteq S_r$.*

**Proof.** (1) $\Rightarrow$ (2), (3), (4) are clear.

(2) $\Rightarrow$ (1) Being right $ef$-extending and left Kasch, $R$ is right self-injective by the proof of [16, Theorem 2.7]. Therefore, we conclude by Theorem 3.2 that $R$ is right $PF$.

(3) $\Rightarrow$ (4) Being right self-injective left Kasch, $R$ is semiperfect. Let $x \in S_l$. Then $xZ_l = 0$, and hence $x \in l(Z_l) \subseteq l(J) = S_r$ (for $R$ is semiperfect).

(4) $\Rightarrow$ (1) follows from Theorem 3.2. □

**Remark 3.1.** There exists a left Kasch ring $R$ with $S_l \subseteq S_r$ such that the dual of a simple left $R$-module need not be simple. In fact, the ring $R$ in [13, Example 2.5] is left continuous left Artinian, and hence $R$ is left Kasch with $S_l \subseteq S_r$. However,

$R$ is not left mininjective. Therefore, the dual of a simple left $R$-module can't be simple by [13, Theorem 2.31].

Following [23], a ring is called *right FSG* if every finitely generated cofaithful $R$-module is a generator.

**Theorem 3.3.** *Then following conditions are equivalent for a ring $R$:*

(1) *$R$ is a left $PF$-ring*;
(2) *$R$ is a right Kasch, left $WIN$, left $FSG$ ring and $S_r \subseteq S_l$*;
(3) *$R$ is a left $WIN$, left $FSG$ ring and the dual of every simple right $R$-module is simple.*

**Proof.** (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3) By Theorem 2.1, $R$ is right mininjective. Hence, by [13, Theorem 2.31], the dual of every simple right $R$-module is simple.

(3) $\Rightarrow$ (1) By [13, Theorem 2.31] and Theorem 2.1, $R$ is semiperfect with essential left socle. Therefore, being left $FSG$, $R$ is left $PF$ by [23, Theorem 3.8]. □

**Proposition 3.4.** *Then following conditions are equivalent for a left $WIN$-ring $R$:*

(1) *$R$ is a right $PF$-ring*;
(2) *$S_r \subseteq S_l$ and every 2-generated right $R$-module is torsionless*;
(3) *$R$ is right Kasch right small injective.*

**Proof.** (1) $\Rightarrow$ (2), (3) are clear.

(2) $\Rightarrow$ (1) Clearly, $R$ is right Kasch. Thus, in view of Theorem 2.1, $R$ is semiperfect with essential left socle. Therefore, $R$ is right finitely cogenerated by [13, Theorem 5.31]. So, it remains to show that $R$ is right self-injective. Since $J = Z_r$ by [13, Theorem 5.31], this can be proved by arguing as in [14, Theorem 2.8].

(3) $\Rightarrow$ (1) Being right small injective, $R$ is right mininjective. Thus, $R$ is semilocal by Theorem 2.1. Therefore, the claim follows from [19, Theorem 3.16]. □

## 4. On Quasi-Frobenius Rings via WIN-Rings

There exist commutative noetherian $WIN$-rings that are not Artinian (for example $\mathbb{Z}$). However, we do have the following result.

**Proposition 4.1.** *Then following conditions are equivalent for a $WIN$-ring $R$:*

(1) *$R$ is quasi-Frobenius*;
(2) *$R$ is two-sided Kasch with $ACC$ on right or left annihilators and $S_r \subseteq S_l$.*

**Proof.** (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) By Theorem 2.1, $R$ is semiperfect, two-sided mininjective and $S_r = S_l$ is essential as a left and right ideal of $R$. Thus, according to [13, Theorem 3.31], $R$ is quasi-Frobenius. □

**Theorem 4.1.** *Let $R$ be a left $WIN$-ring with $ACC$ on right annihilators such that $S_l \leq_e R_R$. If $xR \leq_e rl(x)$ for every simple right ideal $xR$, then $R$ is quasi-Frobenius.*

**Proof.** Since $R$ has $ACC$ on right annihilators with $S_l \leq_e R_R$, it follows from Lemma 2.1 that $R$ is semiprimary. Then, by Lemma 2.2, $R$ is right mininjective and $rl(T)$ is essential in a direct summand of $R_R$ for every right ideal $T$ of $R$. So, from the hypothesis, we deduce that $R$ is right min-$CS$. On the other hand, being right mininjective, $R$ is right Kasch by [13, Theorem 3.12]. Then, in view of [13, Lemma 4.5], $\mathrm{Soc}(eR)$ is simple for every local idempotent $e \in R$. Note that $S_l = S_r$ by Theorem 2.2. Thus, being semiperfect, $R$ is left mininjective by [13, Proposition 3.5]. As $R$ is right mininjective with $S_r \leq_e R_R$, then according to [13, Theorem 3.31], $R$ is quasi-Frobenius. □

**Corollary 4.1.** *Then following conditions are equivalent for a ring $R$:*

(1) *$R$ is quasi-Frobenius*;
(2) *$R$ is $WIN$ and left quasi-dual with $ACC$ on right annihilators*;
(3) *$R$ is $WIN$ and right $AGP$-injective with $ACC$ on right annihilators*;
(4) *$R$ is $WIN$ with $ACC$ on right annihilators and $S_l \leq_e R_R$.*

**Proof.** (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3) follows from [15, Lemma 2.6].

(3) $\Rightarrow$ (4) By [27, Lemma 3.1 an Corollary 1.6], $R$ is semiprimary and $J = Z_r$. It follows that $S_r \subseteq S_l$. Since $S_r \leq_e R_R$, $S_l \leq_e R_R$.

(4) $\Rightarrow$ (1) Since $R$ is right $WIN$, it follows from the right version of Proposition 2.1 that $xR \leq_e rl(x)$ for every simple right ideal $xR$. Now, we conclude by Theorem 4.1 that $R$ is quasi-Frobenius. □

**Theorem 4.2.** *Then following conditions are equivalent for a ring $R$:*

(1) *$R$ is quasi-Frobenius*;
(2) *$R$ is left $WIN$ and left $GP$-injective with $ACC$ on left annihilators*;
(3) *$R$ is a left $WIN$ right $GP$-injective ring with $ACC$ on right annihilators.*

**Proof.** (1) $\Rightarrow$ (2), (3) are clear.

(2) $\Rightarrow$ (1) By [27, Corollary 1.9] and its proof, $R$ is right minannihilator and right Artinian. Thus, since $R$ is left mininjective, we infer from [13, Corollary 3.13] that $S_r = S_l$. Hence, by Proposition 2.2, $R$ is right mininjective. Therefore, the claim follows from [13, Theorem 3.31].

(3) $\Rightarrow$ (1) Since right $GP$-injective with $ACC$ on right annihilators, we infer from [27, Corollary 1.9] that $R$ is left Artinian. Thus, $R$ is right Kasch by [23, Proposition 2.2]. Note that $R$ is left $WIN$. Then for every $x \in R$, $xR$ is simple, whenever, $Rx$ is simple by Theorem 2.1. Therefore, in view of [23, Theorem 2.4], $R$ is quasi-Frobenius. □

**Theorem 4.3.** *Then the following conditions are equivalent for a ring $R$:*

(1) *$R$ is quasi-Frobenius*;
(2) *$R$ is right $CF$ and left $WIN$ with $S_r \subseteq S_l$*;
(3) *$R$ is right Johns and left $WIN$ with $S_r \subseteq S_l$.*

**Proof.** (1) $\Rightarrow$ (2), (3) are clear.

(2) $\Rightarrow$ (1) Since $R$ is right $CF$, it is right Kasch. Thus, $R$ is semilocal and right mininjective by Proposition 2.2 and Theorem 2.1, and we conclude by [13, Theorem 8.11] that $R$ is quasi-Frobenius.

(3) $\Rightarrow$ (1) By Proposition 2.2, $R$ is right mininjective. Therefore, we infer from [13, Theorem 8.11] that $R$ is quasi-Frobenius. □

It was shown in [12, Theorem 2], that a left uniserial right perfect ring is left aratinian whose factor rings are right $P$-injective. Using this result and Theorem 4.3, we prove in the next corollary that every left uniserial right perfect ring is quasi-Frobenius.

**Corollary 4.2.** *Let $R$ be a left uniserial right perfect ring. Then $R$ is quasi-Frobenius.*

**Proof.** Being left uniserial right perfect, $R$ is left Artinian right $P$-injective by [12, Theorem 2]. Note that every left uniserial ring is left $WIN$ and every right $P$-injective ring is right $GP$-injective. Therefore, we conclude by Theorem 4.3 that $R$ is quasi-Frobenius. □

Let $P$ be a property of rings. A ring $R$ is said to be *completely $P$* if each factor ring of $R$ has the property $P$.

**Theorem 4.4.** *Suppose $R$ is completely left $WIN$, completely right Kasch and completely right mininjective. Then $R$ is left Artinian.*

**Proof.** Let $I$ be a two-sided ideal of $R$. Since $R$ is completely left $WIN$, completely right Kasch and completely right mininjective, $\overline{R} = R/I$ has a finitely generated and essential left socle by Theorem 2.1. Using [13, Lemma 1.52], we deduce that $R$ is left Artinian. □

**Corollary 4.3.** *Suppose $R$ is left perfect, completely left $WIN$, completely right mininjective. Then $R$ is left Artinian.*

**Proof.** Let $I$ be a two-sided ideal of $R$. Since $R$ is left perfect completely right mininjective, $\overline{R} = R/I$ is right Kasch right mininjective by [13, Theorem 3.12]. Now, being completely left $WIN$, $R$ is left Artinian by Theorem 4.4. □

**Theorem 4.5.** *The following conditions are equivalent for a ring* $R$:

(1) *$R$ is completely quasi-Frobenius;*
(2) *$R$ is completely $WIN$, completely quasi-dual;*
(3) *$R$ is completely $WIN$, completely right Kasch and* $\mathrm{Soc}(\overline{R}_{\overline{R}}) \subseteq \mathrm{Soc}({}_{\overline{R}}\overline{R})$ *for every factor ring $\overline{R}$ of $R$;*
(4) *$R$ is completely $WIN$, completely right Kasch and completely right mininjective.*

**Proof.** (1) ⇒ (2) is clear.

(2) ⇒ (3) Let $I$ be a two-sided ideal of $R$. Since $R$ is completely quasi-dual, $\overline{R} = R/I$ is two-sided Kasch and $\mathrm{Soc}(\overline{R}_{\overline{R}}) = \mathrm{Soc}({}_{\overline{R}}\overline{R})$ by [15, Theorem 2.8].

(3) ⇒ (4) follows from Proposition 2.2.

(4) ⇒ (1) Let $I$ be a two-sided ideal of $R$. Then, $\overline{R} = R/I$ is left Artinian by Theorem 4.4. Moreover, $\mathrm{Soc}(\overline{R}_{\overline{R}}) = \mathrm{Soc}({}_{\overline{R}}\overline{R})$ by Theorem 2.1. Therefore, the claim follows from Corollary 4.1. □

**Corollary 4.4.** *The following conditions are equivalent for a ring* $R$:

(1) *$R$ is completely quasi-Frobenius;*
(2) *$R$ is left perfect, completely $WIN$ and completely right mininjective.*

## 5. On Left $C_{11}$-Rings and WIN-Rings

A direct summand of a $C_{11}$-module need not be a $C_{11}$-module (see [21, Example 4.33]). According to [21, p. 192], we say that a module $M$ satisfies $P^+$ if and only if every direct summand of $M$ satisfies $P$.

**Lemma 5.1 ([20, Theorem 4.3]).** *Let $R$ be a ring and let $M$ be a $C_{11}$ right $R$-module with the $C_3$ condition. Then $M$ is a $C_{11}^+$-module (i.e. every direct summand of $M$ is a $C_{11}$-module).*

**Lemma 5.2 ([20, Proposition 2.3(iii)]).** *Let $R$ be a ring and let $M$ be an indecomposable $C_{11}$ right $R$-module. Then $M$ is uniform.*

The proofs of the following lemmas are motivated by [14, Lemmas 2.2 and 2.3].

**Lemma 5.3.** *Let $R$ be a right Kasch, left $C_{11}$-ring. Then,* $S_r \leq_e {}_RR$.

**Proof.** Since $R$ is left $C_{11}$, there exists $e^2 = e \in R$ such that $S_r \leq_e Re$ by [20, Lemma 2.8]. Hence, $(1-e)R \subseteq r(S_r)$. But $R$ is right Kasch. Then $J = r(S_r)$, and so $1 - e \in J$. It follows that $1 = e$. Therefore, $S_r \leq_e {}_RR$. □

**Lemma 5.4.** *Let $R$ be a right dual, left $C_{11}$-ring. Then $R$ is semiperfect and has finite left uniform dimension.*

**Proof.** Let $T$ be a right ideal of $R$. Since $R$ is left $C_{11}$, there exists an idempotent $e$ of $R$ such that $l(T) \cap Re = 0$ and $l(T) \oplus Re \leq_e {}_RR$ by [20, Proposition 2.3]. Hence, by [20, Lemma 2.2], $Re$ is a complement to $l(T)$ in $R$. Thus, $Re$ is maximal with respect to $l(T) \cap Re = 0$. So, $R = T + r(e)$ by [14, Lemma 2.1]. Suppose now there exists $K \subseteq r(e)$ such that $R = T + K$. Then, $Re \subseteq lr(e) \subseteq l(K)$. As $l(T) \cap l(K) = 0$, then the maximality of $Re$ implies that $Re = l(K)$, from which it follows that $K = rl(K) = r(e)$. Therefore, $R$ is semiperfect by [11, Theorem 11.1.5]. Now, write $R = Re_1 \oplus \cdots \oplus Re_n$ where each $e_i$ is a local idempotent. As ${}_RR$ is left $C_{11}$ module with $C_2$-condition, then each $Re_i$ is a uniform module by Lemmas 5.1 and 5.2. Therefore, $R$ has finite left uniform dimension. □

**Proposition 5.1.** *Let $R$ be a right cogenerator ring.*

(1) *If $R$ is left $C_{11}$, then $R$ is right $PF$.*
(2) *If $R \oplus R$ is $C_{11}$ as a left $R$-module, then $R$ is right $PF$.*

**Proof.** (1) Being right cogenerator, $R$ is right dual. Hence, $R$ is semiperfect by Lemma 5.4. In particular, $R$ has a finite number of isomorphism classes of simple right (and left) $R$-modules. Since $R$ is right cogenerator, $R$ is right self-injective, and hence right $PF$.

(2) Since $R$ is right cogenerator, it is left $P$-injective, and so $J = Z_l$. On the other hand, $R$ is semiperfect by Lemma 5.4. Using [13, Example 7.18], we deduce that $R \oplus R$ satisfies the $C_2$-condition as a left $R$-module. Thus, ${}_RR$ is a $C_{11}$-module by Lemma 5.1 and we conclude by (1) that $R$ is right $PF$. □

The following result extends [14, Theorem 2.8] from left $CS$ rings to left $C_{11}$-rings.

**Proposition 5.2.** *Then following conditions are equivalent for a left $C_{11}$-ring $R$:*

(1) *$R$ is a right $PF$-ring;*
(2) *$J \subseteq Z_r$ and every 2-generated right $R$-module is torsionless.*

**Proof.** (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) Clearly, $R$ is right dual. Thus, in view of Lemmas 5.4 and 5.1, $R$ is semiperfect with $S_r \leq_e {}_RR$. Then, it follows from (2) that $S_r \subseteq S_l$. Thus, $S_l \leq_e {}_RR$. Since $R$ is left $P$-injective, we infer from [13, Theorem 5.31] that $R$ is right finitely cogenerated. So, it remains to show that $R$ is right self-injective and this can be proved by arguing as in [14, Theorem 2.8]. □

The following theorem extends [14, Theorem 2.9] from left $CS$ rings to left $C_{11}$-rings.

**Theorem 5.1.** *The following conditions are equivalent for a left* $C_{11}$ *right* $CF$ *ring* $R$:

(1) $R$ *is quasi-Frobenius*;
(2) $J \subseteq Z_r$;
(3) $S_r \subseteq S_l$;
(4) $\mathrm{Soc}(Re) \neq 0$ *for every local idempotent* $e$ *of* $R$.

**Proof.** (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3) Being right $CF$, $R$ is right dual. Hence, by Lemma 5.4, $R$ is semiperfect. But $J \subseteq Z_r$. Then, $S_r \subseteq S_l$.

(3) $\Rightarrow$ (4) Since $R$ is right dual, it is right Kasch. Hence, $S_r \leq_e {}_RR$ by Lemma 5.3. By hypothesis, $S_r = S_l$. Therefore, $S_l \leq_e {}_RR$ from which it follows that $\mathrm{Soc}(Re) \neq 0$ for every local idempotent $e$ of $R$.

(4) $\Rightarrow$ (1) Being right $CF$, $R$ is right dual. Hence, $R$ is semiperfect by Lemma 5.4. If $1 = e_1 + \cdots + e_n$, where each $e_i$ is a local idempotent, then $S_l = \oplus_{i=1}^{n} \mathrm{Soc}(Re_i)$. Since $R$ is right Kasch, it is left $C_2$. Thus, $Re_i$ is uniform for each $i$ by Lemmas 5.1 and 5.2. Therefore, by hypothesis, $\mathrm{Soc}(Re_i) \leq_e Re_i$ for each $i$. It follows that $S_l \leq_e {}_RR$. Hence, $R$ is right finitely cogenerated by [13, Theorem 5.31]. As $R$ is right $CF$, then $R$ is right Artinian. Now, let $e$ be any local idempotent of $R$. As ${}_RR$ is a $C_{11}$-module satisfying the $C_2$-condition, then by Lemma 5.1, ${}_RR$ is a $C_{11}^{+}$-module. Hence, since $Re$ is indecomposable, it follows from Lemma 5.2 that $Re$ is uniform. Note that $\mathrm{Soc}(Re) \neq 0$. Therefore, $\mathrm{Soc}(Re)$ is a minimal left ideal. So, by [22, Corollary 7], $\mathrm{Soc}(eR)$ is simple for every local idempotent $e$ of $R$. Hence, using [13, Theorems 3.12(4) and 3.7(1)], we deduce that $S_r = S_l$. Being semiperfect, $R$ is right mininjective by [13, Proposition 3.5]. Therefore, according to [13, Theorem 3.31], $R$ is quasi-Frobenius. □

Recall that a ring is called left $QF$-2 if it is a direct sum of uniform left ideals. According to [20, Theorem 2.4], every left $QF$-2 ring is left $C_{11}$. Hence, we can obtain the following corollary.

**Corollary 5.1.** *The following conditions are equivalent for a left* $QF$-2, *right* $CF$ *ring* $R$:

(1) $R$ *is quasi-Frobenius*;
(2) $J \subseteq Z_r$;
(3) $S_r \subseteq S_l$;
(4) $\mathrm{Soc}(Re) \neq 0$ *for every local idempotent* $e$ *of* $R$.

Now, we introduce the following notion.

**Definition 5.1.** We call a ring $R$ right (left) $C$-continuous if $R_R$ (${}_RR$) is a $C_{11}$-module and satisfies the $C_2$-condition. It is clear that a continuous ring is $C$-continuous. But the converse is not true in general as illustrated in the following example.

**Example 5.1 ([21, Example 77]).** Let $F$ be a field which has a proper subfield $K$, set $F_n = F$ and $K_n = K$ for $n = 1, 2, \ldots$, and $Q = \prod F_n$. Let $R = \{x \in Q : x_n \in K_n\}$. By [8, Example 13.8], $R$ is a commutative $C_{11}$-ring and $M_2(R)$ is a von Neumann regular ring which is neither right nor left continuous. On the other hand, since $R$ is $C_{11}$, $M_2(R)$ is both left and right $C_{11}$ by [21, Corollary 4.82]. But $M_2(R)$ is a right and left $C_2$-ring (for, $M_2(R)$ is von Neumann regular). Then $M_2(R)$ is both left and right $C$-continuous. This shows that class of right (left) $C$-continuous rings properly contains the class of right (left) continuous rings.

The following lemmas are needed to prove our next theorem.

**Lemma 5.5.** *Let $R$ be a right $C$-continuous ring. Then, $J = Z_r$ and $R/J$ is a von Neumann regular right $C_2$ right $C_{11}$-ring.*

**Proof.** By [21, Theorem 4.64], $J = Z_r$ and $R/J$ is von Neumann regular. Clearly, $R/J$ is right nonsingular. Thus, by [21, Proposition 4.79], $(R/J)_{R/J}$ has $C_{11}$. But $R/J$ is a right $C_2$-ring. Then $R/J$ is right $C$-continuous. □

**Lemma 5.6.** *Let $R$ be a right $C$-continuous ring with $ACC$ on right annihilators. Then $R$ is semiprimary and $S_r \subseteq S_l$.*

**Proof.** Since $R$ has $ACC$ on right annihilators, it is orthogonally finite. Thus, by Lemma 5.5, $R$ is semilocal and $J = Z_r$. Then, $J$ is nilpotent, and so $R$ is semiprimary. It follows that $S_r \subseteq S_l$. □

It is clear that "right continuous" ⇒ "right $C$-continuous" and the converse is not true, in general. So now, we are able to prove the following result which extends [3, Theorem 1].

**Theorem 5.2.** *The following conditions are equivalent for a ring $R$:*

(1) *$R$ is quasi-Frobenius.*
(2) *$R$ is two-sided $C$-continuous with $ACC$ on right annihilators.*
(3) *$R$ is a left $WIN$, left mininjective ring with $ACC$ on right annihilators in which $S_l \leq_e R_R$.*
(4) *$R$ is a right $C$-continuous, left WIN-ring with $ACC$ on right annihilators.*

**Proof.** (1) ⇒ (2), (3), (4) are clear.

(2) ⇒ (1) By Lemma 5.6, $R$ is semiprimary and $S_r = S_l$. Thus, $\mathrm{Soc}(eR)$ and $\mathrm{Soc}(Re)$ are nonzero for all local idempotent $e$ of $R$. By Lemmas 5.1 and 5.2, $eR$ and $eR$ are uniform. So, both $\mathrm{Soc}(Re)$ and $\mathrm{Soc}(eR)$ are simple. Now, being semiperfect with $S_r = S_l$, $R$ is two-sided mininjective by [13, Proposition 3.5]. Therefore, $R$ is QF by [13, Theorem 3.31].

(3) ⇒ (1) Since $R$ is a left $WIN$-ring with $ACC$ on right annihilators in which $S_l \leq_e R_R$, $R$ is right mininjective, left Artinian and satisfies the conditions (1) through (6) of Theorem 2.1. Now, $R$ is QF by [22, Corollary 13].

(4) ⇒ (1) In this case, by Lemma 5.6, $R$ is semiprimary and $S_r \subseteq S_l$. Then by Proposition 2.2, $R$ is right mininjective. Now, let $e$ be any local idempotent of $R$. As $R_R$ is a $C_{11}$-module satisfying the $C_2$-condition, then by Lemma 5.1, $R_R$ is a $C_{11}^{+}$-module. Hence, it follows from Lemma 5.2 that $eR$ is uniform. Note that $\mathrm{Soc}(eR) \neq 0$. Therefore, $\mathrm{Soc}(eR)$ is a simple right ideal of $R$. As $S_r \subseteq S_l$ and $S_r \leq_e R_R$, then we have $S_l \leq_e R_R$. By Theorem 2.2, $R$ is right mininjective, left Artinian and satisfies the conditions (1) through (6) of Theorem 2.1. So, $S_r = S_l$. Since $R$ is semiperfect and $\mathrm{Soc}(eR)$ is a simple right ideal of $R$ for all idempotents $e$ of $R$, infer from [13, Proposition 3.5] that $R$ is left mininjective. Now, $R$ is QF by [22, Corollary 13]. □

Recall that a ring $R$ is *right CEP* if every cyclic right $R$-module is essentially embedded in a projective module. A module $M$ is said to be $GC2$ if every submodule of $M$ isomorphic to $M$ is a direct summand of $M$. A ring $R$ is called *right* $GC2$ if $R_R$ is a $GC2$ module.

**Corollary 5.2.** *Then following conditions are equivalent for a ring* $R$:

(1) *$R$ is quasi-Frobenius*;
(2) *$R$ is right Johns, right $GC2$ and left $WIN$*;
(3) *$R$ is right $CEP$ and left $WIN$.*

**Proof.** (1) ⇒ (2) is clear.

(2) ⇒ (3) By [27, Lemma 1.1], $R$ is semilocal. Since $R$ is right Johns, $J$ is nilpotent by [13, Lemma 8.7]. Thus, $R$ is semiprimary. Now, being right noetherian, $R$ is right Artinian. On the other hand, it is clear that $R$ is right dual. Therefore, we deduce from [14, Proposiition 3.3] that $R$ is right $CEP$.

(3) ⇒ (1) By [14, Proposiition 3.3], $R$ is right continuous and right Artinian. Thus, we can apply Theorem 5.2 to show that $R$ is quasi-Frobenius. □

**Proposition 5.3.** *Let $R$ be a left perfect right $C$-continuous left $WIN$-ring. If $J^2 = r(A)$ for a finite subset $A$ of $R$. Then $R$ is quasi-Frobenius.*

**Proof.** Let $J^2 = r(a_1, \ldots, a_n)$. Define $\phi : R/J^2 \to R_R^n$ via $\phi(a + J^2(R)) = r(a_1 a, a_2 a, \ldots, a_n a)$ for $a \in R$. Then $\phi$ is a monomorphism. Hence, we may regard $J^2/J$ as a submodule of $R_R^n$. Also, we have $J/J^2 = \mathrm{Soc}(J/J^2) \subseteq \mathrm{Soc}(R_R^n) = (S_r)^n$. Since $R$ is semiperfect, $R_R$ has a decomposition $R_R = e_1 R \oplus e_2 R \oplus \cdots \oplus e_n R$, where each $e_i$ is a local idempotent. Note that $R_R$ is a $C_3$-module. Then, since $R_R$ is a $C_{11}$-module, it follows from Lemmas 5.1 and 5.2 that each $e_i R$ is uniform. Consequently, $R$ has finite right uniform dimension. Then, $S_r$ is finitely generated and so is $(S_r)^n$. Therefore, as a direct summand of $(S_r)^n$, $J/J^2$ is a finitely generated right $R$-module. Hence, $R$ is right Artinian by [5, Lemma 2.9]. Thus, by Theorem 5.2, $R$ is quasi-Frobenius. □

**Theorem 5.3.** *Let $R$ be a left perfect right $C$-continuous left $WIN$-ring. If $R$ is left (or right) pseudo-coherent, then $R$ is quasi-Frobenius.*

**Proof.** By Lemma 5.5, $J = Z_r$. Since $R$ is semiperfect, it follows that $S_r \subseteq S_l$. Therefore, $R$ is right mininjective by Proposition 2.2. Now, being left perfect, $R$ is right Kasch by [13, Theorem 3.12]. Using Theorem 2.1, we deduce that $S = S_l = S_r$. Since $R$ is left perfect, $\mathrm{Soc}(eR) \neq 0$ for every local idempotent $e$ of $R$. As $eR$ is indecomposable, it follows from Lemmas 5.1 and 5.2 that $eR$ is uniform. Thus, $\mathrm{Soc}(eR)$ is simple. Hence, by [13, Proposition 3.5], $R$ is left mininjective. Therefore, $S = S_l = S_r$ is a finitely generated left and right ideal by [13, Corollary 5.53]. Again by [13, Theorem 3.12], $R$ is left Kasch. As $R$ is right Kasch, then $J = l(S) = r(S)$. By hypothesis, $R$ is left (or right) pseudo-coherent, and so $J$ is left (or right) finitely generated ideal, from which it follows that $J/J^2$ is a finitely generated left (or right) $R$-module. Since $R$ is left perfect, $R$ is left or (right) Artinian by [5, Lemma 2.9]. Now, $R$ is quasi-Frobenius by Theorem 5.2. □

**Remark 5.1.** A left $C$-continuous ring need not be right $C$-continuous. For example, the ring $R$ in [13, Example 2.5] is a left $C$-continuous left $WIN$ two-side Artinian ring that is not right $C$-continuous. Indeed, if $R$ were right $C$-continuous, then being left $WIN$ with $ACC$ on right annihilators, it would be quasi-Frobenius by Theorem 5.2. However, $R$ is not left mininjective, a contradiction.

**Lemma 5.7 ([21, Theorem 4.64]).** *Let $M$ be a module such that $M$ satisfies $C_{11}^{+}$ and $M/\mathrm{Soc}(M)$ has finite uniform dimension. Then $M$ contains a semisimple submodule and a submodule $M_2$ with finite uniform dimension such that $M = M_1 \oplus M_2$.*

By using the technique of proving [13, Lemma 4.21], we can obtain the following result.

**Lemma 5.8.** *Let $R$ be a right $C$-continuous ring. If either $R/S_r$ or $R/S_l$ has $ACC$ on right annihilators, then $R$ is semiprimary.*

**Proof.** We first suppose that $R/S_r$ has $ACC$ on right annihilators. Since $R$ is a right $C_2$ right $C_{11}$-ring, $J = Z_r$ by Lemma 5.5. So $J$ is nilpotent by [13, Lemma 4.20(4)]. Write $\overline{R} = R/S_r$ and $\widetilde{R} = R/J$ and denote by $\overline{J}$ and $\widetilde{S}$ the images of $J$ in $\overline{R}$ and $S_r$ in $\widetilde{R}$, respectively. Then, $\overline{R}/\overline{J} \cong R/(J + S_r) \cong \widetilde{R}/\widetilde{S}$. Note that $\widetilde{R}$ is von Neumann regular by Lemma 5.5. Then $\overline{R}/\overline{J}$ is von Neumann regular. Since $\overline{R}$ is $I$-finite and $\overline{J}$ is nilpotent, $\overline{R}/\overline{J}$ is also $I$-finite. Consequently, $\overline{R}/\overline{J}$ is semisimple Artinian. So, by the previous isomorphism, $\widetilde{R}/\widetilde{S}$ is semisimple Artinian. By Lemma 5.5, $\widetilde{R}$ is a right $C$-continuous ring. Hence, using [20, Theorem 4.3] and Lemma 5.7, we deduce that $\widetilde{S}_{\widetilde{R}}$ is finitely generated. Therefore, $\widetilde{R}$ is semisimple Artinian, and so $R$ is semiprimary.

Now, assume that $R/S_l$ has $ACC$ on right annihilators. By Lemma 5.5, $J = Z_r$, and so $J$ is nilpotent by [13, Lemma 4.20(4)]. Write $\overline{R} = R/S_l$ and $\widetilde{R} = R/J$

and denote by $\overline{J}$ and $\widetilde{S}$ the images of $J$ in $\overline{R}$ and $S_l$ in $\widetilde{R}$, respectively. Then, $\overline{R}/\overline{J} \cong R/(J+S_l) \cong \widetilde{R}/\widetilde{S}$. Note that $\widetilde{R}$ is von Neumann regular by Lemma 5.5. Then $\overline{R}/\overline{J}$ is von Neumann regular too. Since $\overline{R}$ is $I$-finite and $\overline{J}$, $\overline{R}/\overline{J}$ is also $I$-finite. So, by the previous isomorphism, $\widetilde{R}/\widetilde{S}$ is semisimple Artinian. As $\widetilde{R}$ is semiprime, then $\widetilde{S}_{\widetilde{R}} = {}_{\widetilde{R}}\widetilde{S}$. Thus, since $\widetilde{R}$ is a right $C$-continuous ring by Lemma 5.5, we infer from Lemma 5.7 that $\widetilde{S}_{\widetilde{R}}$ is finitely generated. Therefore, $\widetilde{R}$ is semisimple Artinian. This completes the proof. □

**Theorem 5.4.** *Then following conditions are equivalent for a ring $R$:*

(1) *$R$ is quasi-Frobenius*;
(2) *$R$ is a right $C$-continuous, left WIN-ring and $R/S_r$ is right Goldie*;
(3) *$R$ is a right $C$-continuous, left WIN-ring and $R/S_l$ is right Goldie.*

**Proof.** (1) ⇒ (2) is clear.

(2) ⇒ (1) By Lemma 5.8, $R$ is semiprimary. Now, let $e$ be a local idempotent of $R$. Then, by Lemmas 5.1 and 5.2, $eR$ is uniform. Note that $\mathrm{Soc}(eR) \neq 0$. Then, $\mathrm{Soc}(eR)$ is simple. Note by Lemma 5.5 that $J = Z_r$. Now, let $y \in S_r$. Then, $Z_r y = 0$, and so $y \in r(Z_r) \subseteq r(J)$. Since $R$ is semiperfect, $r(J) = S_l$. Thus, $S_r \subseteq S_l$ and we deduce from Proposition 2.2 that $R$ is right mininjective. Hence, using [13, Theorem 3.12(1)] and Theorem 2.1, we deduce that $S_r = S_l$. But $R$ is semiperfect and $\mathrm{Soc}(eR)$ is simple for all local idempotents $e$ of $R$. Then, $R$ is left mininjective by [13, Proposition 3.5] and so $R$ is two-sided minannihilator by [13, Corollary 2.34]. Therefore, according to [13, Corollary 3.25(1) and Theorem 3.38], $R$ is quasi-Frobenius.

Similarly, (1) ⇔ (3). □

## Acknowledgments

The authors are grateful to the referee for her/his valuable comments. Le Van Thuyet and Truong Cong Quynh acknowledge the support/partial support of the Core Research Program of Hue University, Grant No. NCTB.DHH.2024.01.

## ORCID

Le Van Thuyet https://orcid.org/0000-0002-4544-9958
Abdoul Djibril Diallo https://orcid.org/0000-0002-1791-0639
Papa Cheikhou Diop https://orcid.org/0000-0003-0628-0018
Truong Cong Quynh https://orcid.org/0000-0002-0845-0175

## References

1. A. Alahmadi, N. Er and S. K. Jain, Modules which are invariant under monomorphisms of their injective hulls, *J. Austr. Math. Soc.* **79** (2005) 349–369.

2. I. Amin, M. F. Yousif and N. Zeyada, Soc-injective rings and modules, *Comm. Algebra* **33** (2005) 4229–4250.
3. V. P. Camilo and M. F. Yousif, Continuous rings with ACC on annihilators, *Canad. Math. Bull.* **34** (1991) 462–464.
4. J. Chen and N. Ding, On generalizations of injective rings, in *Int. Symp. on Ring Theory*, South Korea, June 28–July 3, 1999.
5. J. Chen, N. Ding and M. F. Yousif, On generalizations of $PF$-rings, *Comm. Algebra* **32**(2) (2004) 521–533.
6. N. V. Dung, D. V. Huynh, P. F. Smith and R. Wisbauer, *Extending Modules*, Pitman Research Notes in Mathematics, Vol. 313 (Longman, Harlow, 1994).
7. J. L. Garcia and J. L. Gómez Padro, Closed submodules of free modules over the endomorphism ring of quasi-injective module, *Comm. Algebra* **16** (1988) 115–137.
8. K. R. Goodearl, *Von Neumann Regular Rings* (Pitman, London, 1979).
9. J. L. Gómez Padro and M. F. Yousif, Semiperfect min-$CS$ rings, *Glasgow Math. J.* **41** (1999) 231–238.
10. C. R. Hajarnavis and N. C. Norton, On dual rings and their modules, *J. Algebra* **93** (1985) 253–266.
11. F. Kasch, *Modules and Rings*, London Mathematical Society Monograph, Vol. 17 (Academic Press, New York, 1982).
12. W. K. Nicholson and M. F. Yousif, On completely principally injective rings, *Bull. Austral. Math. Soc.* **50** (1994) 513–518.
13. W. K. Nicholson and M. F. Yousif, *Quasi-Frobenius Rings* (Cambridge Uinversity Press, Cambridge, 2003).
14. W. K. Nicholson and M. F. Yousif, Annihilators and the $CS$-condition, *Glasgow Math. J.* **40** (1998) 213–222.
15. S. Page and Y. Zhou, Quasi-dual rings, *Comm. Algebra* **28** (2000) 489–504.
16. T. C. Quynh and L. V. Thuyet, Some properties of $ef$-extending rings, *Math. J. Okayama Univ.* **52** (2010) 123–131.
17. L. Shen, A note on $\aleph_0$-injective rings, *J. Algebra Appl.* **10**(3) (2011) 529–536.
18. L. Shen, A note on Faith-Menal conjecture, *Comm. Algebra* **41**(1) (2016) 95–100.
19. L. Shen and J. Chen, New characterizations of quasi-Frobenius rings, *Comm. Algebra* **34**(6) (2006) 2157–2165.
20. P. F. Smith and A. Tercan, Generalizations of $CS$-modules, in *Communications in Algebra*, Vol. 21 (Taylor & Francis, 1993), pp. 1809–1847.
21. A. Tercan and C. C. Yücel, *Module Theory, Extending Modules and Generalizations*, Frontiers in Mathematics (2015).
22. L. D. Thoang and L. V. Thuyet, On semiperfect mininjective rings with essential socles, *Southeast Asian Bull. Math.* **30** (2006) 555–560.
23. L. D. Thoang and L. V. Thuyet, On generalizations of injectivity, *Acta Math. Univ. Comenianae* **2** (2006) 199–208.
24. L. V. Thuyet and T. C. Quynh, On small injective, simple-injective and quasi-Frobenius rings, *Acta Math. Univ. Comenianae* **2** (2009) 161–172.
25. L. V. Thuyet and T. C. Quynh, On general injective rings with chain conditions, *Algebra. Colloq.* **16**(2) (2009) 243–252.
26. M. F. Yousif, CS rings and Nakayama permutations, *Comm. Algebra* **25** (1997) 3787–3795.
27. Y. Zhou, Rings in which certain right ideals are direct summands on annihilators, *J. Austral. Math. Soc. A* **73** (2002) 335–346.