Computational and Applied Mathematics (2025) 44:301
https://doi.org/10.1007/s40314-025-03254-3

Existence and solution methods for strongly pseudomonotone equilibrium problems on Hadamard manifolds

Nguyen Thai An[1] · Luong Van Nguyen[2] · Nguyen Thi Thu[2]

Received: 17 February 2024 / Revised: 24 October 2024 / Accepted: 4 May 2025

Abstract
This paper is concerned with strongly pseudomonotone equilibrium problems on Hadamard manifolds. We first prove the existence and uniqueness of solution for this class of problems. We then establish lower and upper error bounds for strongly pseudomonotone equilibrium problems. Linear convergence and strong convergence of sequences generated by the modified projection method with suitable choices of step sizes are also investigated. Furthermore, we present finite convergence results, which are new even for the case of linear space setting, for the modified projection method under linear conditioning assumption. Some examples are given to support our results.

Keywords Equilibrium problems · Hadamard manifolds · Strongly pseudomonotone · Modified projection method · Linear conditioning · Finite convergence

Mathematics Subject Classification 49J40 · 65K15 · 90C33 · 58D17

1 Introduction

The equilibrium problem (in short, EP), introduced by Blum and Oettli (1994), is a general mathematical model which contains optimization problems, variational inequality problems, saddle point problems, complementarity problems, fixed point problems, Nash equilibrium problems in noncooperative games and others as special cases (see, e.g., Bigi et al. (2019); Blum and Oettli (1994); Konnov (2007) and references therein). Because of its applications in many areas such as economics, transportations, networks, image reconstructions, elasticity,

✉ Nguyen Thai An
nguyenthaian@hueuni.edu.vn

Luong Van Nguyen
nguyenvanluong@hdu.edu.vn

Nguyen Thi Thu
nguyenthithutn@hdu.edu.vn

1 Department of Mathematics, University of Education, Hue University, 34 Le Loi, Hue City, Vietnam

2 Faculty of Natural Sciences, Hong Duc University, Thanh Hoa City, Vietnam

Published online: 27 May 2025

etc., EP has been studied extensively. Two basic and important issues for EP are the existence of solutions and iterative methods for finding solutions. There have been a large number of papers dealing with the solution existence and solution methods for equilibrium problems in the literature (see, Anh and Hai (2017, 2019); Blum and Oettli (1994); Duc et al. (2016); Hai (2020); Hieu (2019); Muu and Quy (2015); Nguyen et al. (2020); Quoc and Muu (2012); Vuong and Strodiot (2020); Yin et al. (2022) and references therein).

In recent years, several concepts and results of nonlinear analysis and optimization theory have been extended from Euclidean spaces to Riemannian manifolds (see, e.g., Azagra et al. (2005); Ledyaev and Zhu (2007); Li et al. (2009)). It is worth noting that these extensions have some significant benefits. For example, by choosing a suitable Riemannian metric, constrained problems can be reduced to unconstrained problems, non-convex optimization problems can be transformed to convex problems on Riemannian manifolds, and non-monotone bifunctions can be transformed into monotone bifunctions. Colao et al. (2012) was first introduced the equilibrium problem on manifolds. In that paper, by extending the well-known KKM lemma in Knaster et al. (1929), the authors proved an existence result for solutions of equilibrium problems on Hadamard manfolds. As consequences, existence results for solutions of variational inequality, fixed point and Nash equilibrium problems were derived. They also proved the convergence of Picard iteration for firmly nonexpansive mappings in the setting of Hadamrd manifolds and used to devise an algorithm to approximate solutions. The results presented in Colao et al. (2012) were then improved and extended to general Riemannian manifolds by Wang et al. (2019) with a new approach. For other existence results for solutions of equilibrium problems, we refer the reader to, e.g., Al-Homidan et al. (2021); Bento et al. (2022); Cruz Neto et al. (2018); Jana (2022); Pang (2018) and references therein. Several solution methods for equilibrium problems in the setting of manifolds have also been developed. Cruz Neto et al. (2016) presented an extragradient algorithm for solving EPs on Hadamard manifolds to the case where the equilibrium bifunction is not necessarily pseudomonotone. Extragradient type algorithms for solving (strongly) pseudomonotone EPs on Hadamard manifolds were considered in Al-Homidan et al. (2021); Chen et al. (2021); Fan et al. (2021); Khammahawong et al. (2020). Other results concerning solution methods for solving EPs on manifolds can be found in Ansari and Islam (2020); Babu et al. (2022); Li et al. (2016) and references therein. Noting that strongly pseudomonotone equilibrium problems in linear spaces have been studied in many papers and they are still attracted to many researchers (see, e.g., Anh and Hai (2017, 2019); Muu and Quy (2015); Duc et al. (2016); Hai (2020); Hieu (2019); Vuong and Strodiot (2020); Yin et al. (2022) and references therein). However, there were few papers dealt with strongly pseudomonotone equilibrium problems on manifolds.

Motivated by the above-mentioned works, the aim of this paper is to establish some new results for strongly pseudomonotone equilibrium problems on Hadamard manifolds. We first prove the existence and uniqueness for the solution of EPs governed by strongly pseudomonotone bifunctions on Hadamard manifolds. We then present and prove error bounds for EPs when the considered equilibrium bifunction is strongly pseudomonotone and satisfies a Lipschitz-type condition. We aslo study, under the same setting, the convergence property for sequences generated by the modified projection method with different step size rules. These results extend the analogous results from linear spaces to the setting of Hadamard manifolds. We also introduce the notion of linear conditioning for equilibrium problems on Hadamard manifolds and use it to study the finite termination of sequences generated by the modified projection method. An upper bound for the number of iterates for which a sequence generated by the modified projection method converges to the solution of the EP is also given. These finite convergences results are new even for the case of Euclidean spaces. Some examples and numerical experiments are also presented to support our results.

The rest of this paper is organized as follows. In Sect. 2, we recall some notions, definitions and basic results of Hadamard manifolds which will be used in the sequel. Section 3 is devoted to the existence, uniqueness and error bounds for the solution of strongly pseudomonotone EPs on Hadamard manifolds. In Sect. 4.2, we establised the linear and finite convergence for the modified projection method for solving EPs. Finally, some conclusions are presented in Sect. 5.

2 Preliminaries

This section consists of some basic definitions, notations and useful results about Riemannian geometry which can be found in, for instances, do Carmo (1992); Lang (1999); Sakai (1996); Udriste (1994).

Let $\mathcal{M}$ be a connected finite-dimensional smooth manifold. We denoted by $\mathcal{T}_x\mathcal{M}$ the tangent space of $\mathcal{M}$ at a point $x \in M$ and by $\mathcal{TM} = \bigcup_{x\in M} \mathcal{T}_x\mathcal{M}$ the tangent bundle of $\mathcal{M}$. Note that $\mathcal{T}_x\mathcal{M}$ is a vector space with the same dimension as $\mathcal{M}$ and $\mathcal{TM}$ is naturally a manifold. We suppose that $\mathcal{M}$ is endowed with a Riemannian metric to become a Riemannian manifold. We denote by $\langle\cdot,\cdot\rangle_x$ the inner product on $\mathcal{T}_x\mathcal{M}$ and by $||\cdot||_x$ the corresponding norm to the inner product $\langle\cdot,\cdot\rangle_x$. If no confusion occurs, the subscript x is omitted.

The length of a piecewise smooth curve $\gamma : [a, b] \to M$ joining x to y in $\mathcal{M}$, i.e., $x = \gamma(a)$ and $y = \gamma(b)$, is defined by

$$L(\gamma) := \int_a^b ||\gamma'(t)||dt,$$

where $\gamma'(t) \in \mathcal{T}_{\gamma(t)}\mathcal{M}$ is a tangent vector. The Riemannian distance $d(x, y)$ between x and y is the minimal length of all such curves connecting x and y. This distance induces the original topology on $\mathcal{M}$. Let $p \in \mathcal{M}$ and $r > 0$. We denote by $B(p,r)$ and $\overline{B}(p,r)$ which are defined respectively as

$$B(p,r) = \{q \in \mathcal{M} : d(p,q) < r\} \quad \text{and} \quad \overline{B}(p,r) = \{q \in \mathcal{M} : d(p,q) \leq r\},$$

the open metric ball and the closed metric ball at p with radius r, respectively. We denote by $\operatorname{int}\mathcal{C}$ the interior of a set $\mathcal{C} \subset \mathcal{M}$ with respect to the topology induced by the distance d. The distance from a point $p \in \mathcal{M}$ to a subset $\mathcal{C}$ of $\mathcal{M}$ is defined as

$$d(p,\mathcal{C}) := \inf\{d(p,q) : q \in \mathcal{C}\}.$$

A mapping $V : \mathcal{M} \to \mathcal{TM}$ is called a vector field if $V(x) \in \mathcal{T}_x\mathcal{M}$ for each $x \in \mathcal{M}$. Let ∇ be the Levi-Civita connection associated with the Riemannian metric. A vector field V is said to be parallel along a smooth curve γ if $\nabla_{\gamma'(t)}V = \mathbf{0}$, where $\mathbf{0}$ is the zero tangent vector. If γ' is parallel along γ, i.e., $\nabla_{\gamma'(t)}\gamma'(t) = \mathbf{0}$, then we say that γ is a geodesic. A geodesic γ joining x to y is said to be a minimal geodesic if its length equals $d(p,q)$ and in this case the geodesic γ is called a minimizing geodesic. By the Hopf-Rinow theorem, a Riemannian manifold $\mathcal{M}$ is complete if and only if any pair of points in $\mathcal{M}$ can be joined by a minimal geodesic. Moreover, if $\mathcal{M}$ is complete, then $(\mathcal{M}, d)$ is a complete metric space and every bounded closed subset is compact. A Hadamard manifold is a complete, simply connected Riemannian manifold of non-positive sectional curvature. From now on, we always assume that $\mathcal{M}$ is an m-dimensional Hadamard manifold.

The exponential map $\exp_x : \mathcal{T}_x\mathcal{M} \to \mathcal{M}$ at a point $x \in \mathcal{M}$ is defined by $\exp_x v := \gamma_v(1, x)$ for each $v \in \mathcal{T}_x\mathcal{M}$, where $\gamma(\cdot) := \gamma_v(\cdot, x)$ is the geodesic starting from x with velocity v, i.e., $\gamma(0) = x$ and $\gamma'(0) = v$. It is known that $\exp_x tv = \gamma_v(t, x)$ for any real number t and $\exp_x \mathbf{0} = \gamma_v(0, x) = x$. Note that for $x \in \mathcal{M}$, the exponential map $\exp_x : \mathcal{T}_x\mathcal{M} \to \mathcal{M}$ is a diffeomorphism. Thus, there exists an inverse exponential map $\exp_x^{-1} : \mathcal{M} \to \mathcal{T}_x\mathcal{M}$. Moreover, we have $d(x, y) = ||\exp_x^{-1} y||$ for any $x, y \in \mathcal{M}$.

A geodesic triangle $\Delta(x_1, x_2, x_3)$ is a set consisting of three points x_1, x_2 and x_3 in $\mathcal{M}$ and three minimal geodesics γ_i joining x_i to x_{i+1}, where $i = 1, 2, 3(\mathrm{mod}\, 3)$.

Proposition 2.1 (Sakai 1996) *(Comparison result for triangles). Let $\Delta(x_1x_2x_3)$ be a geodesic triangle in $\mathcal{M}$. For each $i = 1, 2, 3(mod\, 3)$, let $\gamma_i : [0, \ell_i] \to \mathcal{M}$ denote the geodesic joining x_i to x_{i+1}, and $\ell_i = L(\gamma_i)$ and α_i be the angle between tangent vectors $\gamma_i'(0)$ and $-\gamma_{i-1}'(\ell_{i-1})$. Then*

(i) $\alpha_1 + \alpha_2 + \alpha_3 \le \pi$;
(ii) $\ell_i^2 + \ell_{i+1}^2 - 2\ell_i\ell_{i+1}\cos\alpha_{i+1} \le \ell_{i-1}^2$;

Since

$$\left\langle \exp_{x_{i+1}}^{-1} x_i, \exp_{x_{i+1}}^{-1} x_{i+2} \right\rangle = d(x_i, x_{i+1})d(x_{i+1}, x_{i+2})\cos\alpha_{i+1},$$

the inequality (ii) of Proposition 2.1 can be rewritten in terms of the distance and the exponential map as

$$d^2(x_i, x_{i+1}) + d^2(x_{i+1}, x_{i+2}) - 2\left\langle \exp_{x_{i+1}}^{-1} x_i, \exp_{x_{i+1}}^{-1} x_{i+2} \right\rangle \le d^2(x_i, x_{i+2}). \tag{1}$$

The following useful property was proved in Tam (2022). We present here a simpler proof.

Lemma 2.1 *For any $x, y, z \in \mathcal{M}$, it holds*

$$||\exp_z^{-1} x - \exp_z^{-1} y|| \le d(x, y). \tag{2}$$

Proof Let $x, y, z \in \mathcal{M}$. Using (1), we have

$$\begin{aligned} ||\exp_z^{-1} x - \exp_z^{-1} y||^2 &= ||\exp_z^{-1} x||^2 + ||\exp_z^{-1} y||^2 - 2\langle \exp_z^{-1} x, \exp_z^{-1} y\rangle \\ &= d^2(x, z) + d^2(y, z) - 2\langle \exp_z^{-1} x, \exp_z^{-1} y\rangle \\ &\le d^2(x, y) \end{aligned}$$

which implies that (2) holds. □

Definition 2.1 (Udriste 1994) A subset $\mathcal{K} \subset \mathcal{M}$ is said to be (geodesically) convex if for any two point x and y in $\mathcal{K}$, the geodesic joining x to y is contained in $\mathcal{K}$, that is, if $\gamma : [a, b] \to \mathcal{M}$ is a geodesic such that $\gamma(a) = x$ and $\gamma(b) = y$, then $\gamma(ta + (1-t)b) \in \mathcal{K}$ for all $t \in [0, 1]$.

The projection of a point $x \in \mathcal{M}$ onto a subset $\mathcal{K}$ of $\mathcal{M}$ is defined by

$$P(x, \mathcal{K}) := \{p \in \mathcal{K} : d(x, p) = d(x, \mathcal{K})\}.$$

Proposition 2.2 (Walter 1974) *Let $\mathcal{K}$ be a closed convex subset of a Hadamard manifold $\mathcal{M}$. Then, for any $x \in \mathcal{M}$, $P(x, \mathcal{K})$ is a singleton set. Also, for any $p \in \mathcal{M}$, the following assertions are equivalent:*

(i) $y = P(p, \mathcal{K})$;
(ii) $\langle \exp_y^{-1} p, \exp_y^{-1} q\rangle \le 0$ *for all* $q \in \mathcal{K}$.

Definition 2.2 (Udriste 1994) Let $\mathcal{C}$ be a nonempty convex set of $\mathcal{M}$. A function $f : \mathcal{C} \to \mathbb{R}$ is said to be (geodesically) convex if for any geodesic $\gamma : [a, b] \to \mathcal{C}$ the composition $f \circ \gamma : [a, b] \to \mathbb{R}$ is convex.

For any $x, y \in \mathcal{M}$, there is a unique minimal geodesic γ joining x to y which is defined by $\gamma(t) = \exp_x(t \exp_x^{-1} y)$ for all $t \in [0, 1]$ (see, e.g., Sakai (1996)). Thus, $f : \mathcal{C} \to \mathbb{R}$ is convex if and only if

$$f(\exp_x(t \exp_x^{-1} y)) \leq (1-t)f(x) + tf(y), \quad \forall x, y \in \mathcal{C} \text{ and } t \in [0, 1].$$

Let $\mathcal{C}$ be a nonempty convex set of $\mathcal{M}$ and $f : \mathcal{C} \to \mathbb{R}$ be convex. The subdifferential of f at a point $x \in \mathcal{C}$ is defined by

$$\partial f(x) := \{v \in \mathcal{T}_x\mathcal{M} : f(y) - f(x) \geq \langle v, \exp_x^{-1} y\rangle, \ \forall y \in \mathcal{C}\}.$$

It is known that if $f : \mathcal{C} \to \mathbb{R}$ is convex, then the set $\partial f(x)$ is nonempty convex and compact for each $x \in \mathcal{C}$ (see, (Udriste, 1994, Theorem 4.6)).

The normal cone of $\mathcal{C}$ at a point $x \in \mathcal{C}$ is defined by

$$N_{\mathcal{C}}(x) = \{v \in \mathcal{T}_x\mathcal{M} : \langle v, \exp_x^{-1} y\rangle \leq 0 \text{ for all } y \in \mathcal{C}\}.$$

We have that

$$N_{\mathcal{C}}(x) = \partial \delta_{\mathcal{C}}(x) \quad \forall x \in \mathcal{C},$$

where $\delta_{\mathcal{C}}$ is the indicator function of the set $\mathcal{C}$ defined by $\delta_{\mathcal{C}}(x) = 0$ if $x \in \mathcal{C}$ and $\delta_{\mathcal{C}}(x) = +\infty$ if $x \notin \mathcal{C}$.

Assume that $f : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ is a bifunction and $\mathcal{C}$ is a convex subset of $\mathcal{M}$ such that $y \mapsto f(x, y)$ is convex on $\mathcal{C}$ for each $x \in \mathcal{C}$. For each $x, y \in \mathcal{C}$, we denote by $\partial_2 f(x, y)$ the subdifferential of $f(x, \cdot)$ at y, that is,

$$\partial_2 f(x, y) := \left\{u \in \mathcal{T}_y\mathcal{M} : \left\langle u, \exp_y^{-1} z\right\rangle + f(x, y) \leq f(x, z), \forall z \in \mathcal{C}\right\}.$$

For our analysis, we need the following result.

Proposition 2.3 *Let $f : \mathcal{M} \to \mathbb{R}$ be a lower semicontinuous function and $\mathcal{C} \subset \mathcal{M}$ be a nonempty closed convex set such that f is convex on an open set containing $\mathcal{C}$. Then, x^* is a minimizer of the problem* $\min\{f(x) : x \in \mathcal{C}\}$ *if and only if*

$$\mathbf{0} \in \partial f(x^*) + N_{\mathcal{C}}(x^*).$$

Proof Consider the function $f_{\mathcal{C}} : \mathcal{M} \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f_{\mathcal{C}}(x) = f(x) + \delta_{\mathcal{C}}(x) \text{ for all } x \in \mathcal{M}.$$

Under our assumption, $f_{\mathcal{C}}$ is a proper lower semicontinuous convex function. Moreover, $x^* \in \mathcal{C}$ is a minimizer of the problem $\min\{f(x) : x \in \mathcal{C}\}$ if and only if x^* is a minimizer of $f_{\mathcal{C}}$ on $\mathcal{M}$. This is equivalent to $\mathbf{0} \in \partial f_{\mathcal{C}}(x^*)$ (see, e.g., (Li et al. 2009, Page 675)). Since $x^* \in \mathcal{C} \cap \operatorname{int}(\operatorname{dom} f)$, by (Li et al. 2011, Proposition 4.3), we have

$$\partial f_{\mathcal{C}}(x^*) = \partial f(x^*) + N_{\mathcal{C}}(x^*).$$

This yields the desired conclusion. □

From now on, unless otherwise stated, let $\mathcal{X}$ be a nonempty closed convex subset of $\mathcal{M}$ and $f : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ be a bifunction satisfying $f(x, x) = 0$ for all $x \in \mathcal{X}$. The *equilibrium problem* in the Riemanian context (in short, EP) consists of finding $x^* \in \mathcal{X}$ such that

$$f(x^*, y) \geq 0 \quad \text{for all} \quad y \in \mathcal{X}. \tag{3}$$

In this case, the bifunction f is said to be an *equilibrium bifunction*. We denote by $\mathcal{X}^*$ the solution set of the equilibrium problem (3).

The equilibrium problem in the manifold context was first considered in Colao et al. (2012) where the authors pointed out some important problems, which can be formulated by (3). In particular, if $f(x, y) = \langle V(x), \exp_x^{-1} y\rangle$ for all $x, y \in \mathcal{X}$, where V is a vector field on $\mathcal{M}$, then the problem (3) reduces to the variational inequality problem on Hdamard manifolds which was first introduced by Németh (2003). Concerning the existence of solutions of equilibrium problems on manifolds under different assumptions, we refer the reader to, e.g., Al-Homidan et al. (2021); Bento et al. (2022); Cruz Neto et al. (2018); Jana (2022); Pang (2018); Wang et al. (2019). Some algorithms for solving equilibrium problems on manifolds can be found in, e.g., Ansari and Islam (2020); Al-Homidan et al. (2021); Babu et al. (2022); Chen et al. (2021); Cruz Neto et al. (2016); Fan et al. (2021); Li et al. (2016) and references therein.

We next recall some concepts related to the equilibrium bifunction.

Definition 2.3 (Cruz Neto et al. 2016; Al-Homidan et al. 2021) Let $f : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ be a bifunction and $\mathcal{X}$ be a nonempty closed convex subset of $\mathcal{M}$. The bifunction f is said to be

(i) monotone on $\mathcal{X}$ if for any $x, y \in \mathcal{X}$,

$$f(x, y) + f(y, x) \leq 0;$$

(ii) strongly monotone on $\mathcal{X}$ with modulus μ if there exists a positive constant μ such that for any $x, y \in \mathcal{X}$,

$$f(x, y) + f(y, x) \leq -\mu d^2(x, y);$$

(iii) pseudomonotone on $\mathcal{X}$ if, for any $x, y \in \mathcal{X}$,

$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0;$$

(iv) strongly pseudomonotone on $\mathcal{X}$ with modulus μ if, there exists a positive constant μ such that for any $x, y \in \mathcal{X}$,

$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq -\mu d^2(x, y).$$

From the definition we have the following implications:

$$(ii) \Rightarrow (i) \Rightarrow (iii) \quad \text{and} \quad (ii) \Rightarrow (iv) \Rightarrow (iii).$$

However, the converse implications do not hold even in the linear space setting.

Definition 2.4 Let $f : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ be a bifunction and $\mathcal{X}$ be a nonempty subset of $\mathcal{M}$. The bifunction f is said to satisfy the Lipschitz-type condition with constant L on $\mathcal{X}$ if there is a constant $L > 0$ such that

$$f(x, y) + f(y, z) \geq f(x, z) - Ld(x, y)d(y, z) \quad \forall x, y, z \in \mathcal{X}. \tag{4}$$

Remark 2.1 The Lipschitz-type condition (4) was introduced by Quoc and Muu (2012) in Hilbert spaces. This condition has been used to investigate the convergence of several algorithms for solving equilibrium problems in the linear space setting in, for instance, Quoc and Muu (2012); Vuong and Strodiot (2020). This condition is weaker than the following Lipschitz-type condition introduced by Antipin (1995):

$$|f(x, y) - f(x, z) + f(u, y) - f(u, z)| \leq Ld(x, u)d(y, z) \quad \forall u, x, y, z \in \mathcal{X},$$

and the condition introduced by Anh and Hai (2017):

$$|f(x, y) + f(y, z) - f(x, z)| \leq Ld(x, y)d(y, z) \quad \forall x, y, z \in \mathcal{X}.$$

On the other hand, the Lipschitz-type condition (4) implies the Lipschitz-type condition introduced by Mastroeni (2003):

$$f(x, y) + f(y, z) \geq f(x, z) - L_1 d^2(x, y) - L_2 d^2(y, z) \quad \forall x, y, z \in \mathcal{X}, \tag{5}$$

where L_1, L_2 are two given positive constants.

Remark 2.2 Assume that $\mathcal{X}$ has more than one element. If $f : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ is strongly monotone with modulus β and satisfies the Lipschitz - type condition (4) with constant L on $\mathcal{X}$, then $\beta \leq L$. Indeed, by the condition (4) and the strong monotonicity of f, we have for all $x, y \in \mathcal{X}$ with $x \neq y$ that

$$-Ld(x, y)d(y, x) \leq f(x, y) + f(y, x) \leq -\beta d^2(x, y).$$

This implies that $\beta \leq L$.

To end this section, we propose the following assumptions on the equilibrium bifunction f which will be required in the sequel.

Assumption (A):

(A1) For each $x \in \mathcal{X}$, the mapping $y \mapsto f(x, y)$ is convex on $\mathcal{X}$.
(A2) For each $x \in \mathcal{X}$, the mapping $y \mapsto f(x, y)$ is lower semicontinuous on $\mathcal{X}$.
(A3) For each $y \in \mathcal{X}$, the mapping $x \mapsto f(x, y)$ is upper semicontinuous on $\mathcal{X}$.
(A4) For each $x \in \mathcal{X}$, the function $y \mapsto f(x, y)$ is convex on an open set containing $\mathcal{X}$.

3 Existence of solutions and error bounds for strongly pseudomonotone equilibrium problems

This section is devoted to the study of existence of solutions and error bounds for strongly pseudomonotone equilibrium problems. The results in this section extend some existing results from linear spaces to Hadamard manifolds.

3.1 Existence of solutions

Before stating our existence result, we recall the following result which will be used for proving our result.

Theorem 3.1 *(see (Wang et al., 2019, Corollary 3.1)). Assume that (A1) and (A3) are satisfied. If $\mathcal{X}$ is compact or there exists a compact set $\mathcal{L} \subset \mathcal{M}$ such that: for any $x \in \mathcal{X} \setminus \mathcal{L}$, there exists $y \in \mathcal{X} \cap \mathcal{L}$ satisfying $f(x, y) < 0$, then EP (3) has a solution.*

Our existence result is stated as follows.

Theorem 3.2 *Assume that (A1) – (A3) are satisfied. If f is strongly pseudomonotone with modulus β on $\mathcal{X}$, then EP (3) has a unique solution.*

Proof We first show that EP (3) has a solution. If $\mathcal{X}$ is bounded, then $\mathcal{X}$ is compact. Then, by Theorem 3.1, EP(3) has a solution.

Assume now that $\mathcal{X}$ is unbounded. We fixed a point $\bar{x} \in \mathcal{X}$. We claim that there exists $r > 0$ such that: for all $x \in \mathcal{X} \backslash \overline{B}(\bar{x}, r)$, there exists $y \in \mathcal{X} \cap \overline{B}(\bar{x}, r)$ satisfying $f(x, y) < 0$. Assume to the contrary that the claim is not true. Then, for each $k \in \mathbb{N}$, there exists $x_k \in \mathcal{X} \setminus \overline{B}(\bar{x}, k)$ such that

$$f(x_k, y) \geq 0 \quad \text{for all } y \in \mathcal{X} \cap \overline{B}(\bar{x}, k).$$

Take $y_0 \in \mathcal{X} \cap \overline{B}(\bar{x}, 1)$. Then, $f(x_k, y_0) \geq 0$ for all k. Since f is strongly pseudomonotone with modulus β on $\mathcal{X}$, we have

$$f(y_0, x_k) + \beta d^2(x_k, y_0) \leq 0 \quad \text{for all } k. \tag{6}$$

Since $f(y_0, \cdot)$ is convex and lower semicontinuous, $\partial_2 f(y_0, \bar{x}) \neq \emptyset$ (see Udriste (1994)). Take $v \in \partial_2 f(y_0, \bar{x})$. Then we have for all k that

$$\langle v, \exp_{\bar{x}}^{-1} x_k \rangle \leq f(y_0, x_k) - f(y_0, \bar{x}).$$

This implies that

$$\begin{aligned} f(y_0, x_k) + \beta d^2(y_0, x_k) &\geq f(y_0, \bar{x}) + \langle v, \exp_{\bar{x}}^{-1} x_k \rangle + \beta d^2(y_0, x_k) \\ &\geq f(y_0, \bar{x}) - ||v||.|| \exp_{\bar{x}}^{-1} x_k|| + \beta d^2(y_0, x_k) \\ &= f(y_0, \bar{x}) - ||v||.d(\bar{x}, x_k) + \beta d^2(y_0, x_k) \\ &\geq f(y_0, \bar{x}) - ||v||[d(\bar{x}, y_0) + d(y_0, x_k)] + \beta d^2(y_0, x_k) \\ &= f(y_0, \bar{x}) - ||v||d(\bar{x}, y_0) + d(y_0, x_k)[\beta d(y_0, x_k) - ||v||]. \end{aligned}$$

Since $d(y_0, x_k) \to \infty$ as $k \to \infty$, we have $f(y_0, x_k) + \beta d^2(y_0, x_k) \to \infty$ as $k \to \infty$. This contradicts (6). Thus, our claim is true. Applying Theorem 3.1, we conclude that EP (3) has a solution.

Assume now that EP has two solutions z_1 and z_2. Then, $f(z_1, z_2) \geq 0$ and $f(z_2, z_1) \geq 0$. By the strongly pseudomonotonicity of f, we have

$$0 \leq f(z_1, z_2) \leq -\beta d^2(z_1, z_2).$$

This implies that $d(z_1, z_2) = 0$, i.e., $z_1 = z_2$. Therefore, EP (3) has a unique solution. □

Theorem 3.2 extends to Hadamard manifolds the result presented in Duc et al. (2016). The following example illustrates the validity of Theorem 3.2.

Example 3.1 Let $\mathcal{M} = \mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$ be the Riemannian manifold with the Riemannian metric

$$\langle u, v \rangle := \frac{1}{x^2} uv, \quad \forall u, v \in \mathcal{T}_x\mathcal{M}.$$

Here, the tangent space $\mathcal{T}_x\mathcal{M}$ at a point $x \in \mathcal{M}$ is equal to $\mathbb{R}$. The Riemannian distance $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}_+$ is defined by

$$d(x, y) = \left| \ln \frac{x}{y} \right|, \quad \forall x, y \in \mathcal{M}.$$

It holds that $\mathcal{M}$ is a Hadamard manifold. The unique geodesic γ starting from a point $x = \gamma(0) \in \mathcal{M}$ with velocity $v = \gamma'(0) \in \mathcal{T}_x\mathcal{M}$ is defined by $\gamma(t) = xe^{(v/x)t}$. In terms of the initial point $\gamma(0) = x$ and the terminal point $\gamma(1) = y$, the geodesic $\gamma(t)$ is defined by $\gamma(t) = x^{1-t}y^t$ for all $t \in [0, 1]$.

Let $\mathcal{X} = [1, \infty)$ be a convex set in $\mathcal{M}$ and $f : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ be defined by

$$f(x, y) = \left(1 + \frac{1}{x}\right) \ln x \cdot \ln \frac{y}{x} + y - x, \quad \forall x, y \in \mathcal{M}.$$

It is evident that the bifunction f satisfies assumptions (A2) and (A3). For each $x \in \mathcal{X}$, the function $f(x, \cdot)$ is geodesically convex. Indeed, for any $y_1, y_2 \in \mathcal{M}$, the geodesic joining y_1 to y_2 is $\gamma(t) = y_1^{1-t}y_2^t$ and thus

$$f(x, \gamma(t)) = \left(1 + \frac{1}{x}\right) \ln x \cdot \ln \frac{y_1^{1-t}y_2^t}{x} + y_1^{1-t}y_2^t - x$$

and

$$f"(x, \gamma(t) = y_1^{1-t}y_2^t \ln^2 \frac{y_2}{y_1} \geq 0$$

for all $t \in [0, 1]$. Hence, $f(x, \cdot) \circ \gamma$ is convex in the Euclidean sense. Therefore, $f(x, \cdot)$ is geodesically convex for each $x \in \mathcal{X}$ and f satisfies assumption (A1). Moreover, f is strongly pseudomonotone on $\mathcal{X}$. Indeed, let $x, y \in \mathcal{X}$ be such that $f(x, y) \geq 0$, i.e.,

$$\left(1 + \frac{1}{x}\right) \ln x \cdot \ln \frac{y}{x} + y - x \geq 0.$$

This implies that $y \geq x$. In this case, one has

$$\begin{aligned} f(y, x) &= \left(1 + \frac{1}{y}\right) \ln y \cdot \ln \frac{x}{y} + x - y \\ &\leq \left(1 + \frac{1}{y}\right) \ln y \cdot \ln \frac{x}{y} + x - y + \left[\left(1 + \frac{1}{y}\right) \ln x \ln \frac{y}{x} + y - x\right] \\ &= \left(1 + \frac{1}{y}\right) (\ln y - \ln x) \ln \frac{x}{y} \\ &= -\left(1 + \frac{1}{y}\right) \ln^2 \frac{x}{y} \\ &\leq -d^2(x, y). \end{aligned}$$

Thus, f is strongly pseudomonotone with modulus $\beta = 1$ on $\mathcal{X}$.

Since all assumptions of Theorem 3.2 are satisfied, EP (3) has a unique solution. In fact, $x^* = 1$ is the unique solution of EP (3).

We note that for each $x \in \mathcal{X}$, the function $y \mapsto h(y) = f(x, y)$ is not convex in the Euclidean sense. Indeed, for all $y > 0$, we have

$$h"(y) = -\left(1 + \frac{1}{x}\right) \ln x \cdot \frac{1}{y^2} < 0,$$

i.e., h is not convex in the Euclidean sense.

3.2 Global error bounds

We establish global bound for the distance between an arbitrary point x to the unique solution of the strongly pseudomonotone equilibrium problem in terms of some easily computable quantities merely depending on x and the data of the considered problem. For our purpose, we consider the mapping $s_\lambda : \mathcal{X} \to \mathcal{X}$ defined by

$$s_\lambda(x) = \operatorname{argmin}\left\{\lambda f(x, y) + \frac{1}{2}d^2(x, y) : y \in \mathcal{X}\right\} \text{ for } x \in \mathcal{X},$$

where λ is a positive real number.

From now on, we always suppose that **Assumption (A)** is satisfied. Under this assumption, the function $f_\lambda : \mathcal{M} \to \mathbb{R}$, with $\lambda > 0$, defined by

$$f_\lambda(y) := \lambda f(x, y) + \frac{1}{2}d^2(x, y)$$

is strongly convex on $\mathcal{X}$. Hence, the mapping s_λ is well-defined and it has single values on $\mathcal{X}$ (see, e.g., Udriste (1994)). It is also noted that for any $\lambda > 0$, x is a solution of EP (3) if and only if $x = s_\lambda(x)$ (see, e.g., (Cruz Neto et al., 2016, Remark 5)).

Theorem 3.3 *Assume that $f : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ is strongly pseudomonotone with modulus β and satisfies the Lipschitz -type condition (4) with constant L on $\mathcal{X}$. Let x^* be the unique solution of EP (3) and $\lambda > 0$. Then, for each $x \in \mathcal{X}$, we have*

$$d(x, x^*) \leq \frac{1 + \lambda\beta + \lambda L}{\lambda\beta} d(x, s_\lambda(x)), \tag{7}$$

and

$$d(x, x^*) \geq \frac{1 - \lambda L}{1 + \lambda L} d(x, s_\lambda(x)). \tag{8}$$

Proof For each $x \in \mathcal{X}$ and $\lambda > 0$, we denote by $z := s_\lambda(x) \in \mathcal{X}$ the unique solution of the strongly convex problem

$$\min\left\{\lambda f(x, y) + \frac{1}{2}d^2(x, y) : y \in \mathcal{X}\right\}.$$

It is evident that (7) and (8) hold if $z = x^*$. Assume now that $z \neq x^*$. By Proposition 2.3, we have

$$\mathbf{0} \in \partial\left[\lambda f(x, \cdot) + \frac{1}{2}d^2(x, \cdot)\right](z) + N_{\mathcal{X}}(z),$$

or

$$\mathbf{0} \in \lambda\partial_2 f(x, z) - \exp_z^{-1} x + N_{\mathcal{X}}(z).$$

Thus, there exists $w \in \partial_2 f(x, z)$ such that

$$-\lambda w + \exp_z^{-1} x \in N_{\mathcal{X}}(z).$$

Using the definition of normal cones, one has

$$\langle \exp_z^{-1} x - \lambda w, \exp_z^{-1} y\rangle \leq 0 \quad \forall y \in \mathcal{X}.$$

This implies that

$$\langle \exp_z^{-1} x, \exp_z^{-1} y\rangle \leq \lambda\langle w, \exp_z^{-1} y\rangle \quad \forall y \in \mathcal{X}.$$

On the other hand, since $w \in \partial_2 f(x, z)$, we have

$$\langle w, \exp_z^{-1} y\rangle \leq f(x, y) - f(x, z) \quad \forall y \in \mathcal{X}.$$

It follows from the last two inequalities that

$$\langle \exp_z^{-1} x, \exp_z^{-1} y\rangle \leq \lambda[f(x, y) - f(x, z)] \quad \forall y \in \mathcal{X}.$$

Replacing $y := x^*$ in the latter inequality, we get

$$\langle \exp_z^{-1} x, \exp_z^{-1} x^*\rangle \leq \lambda[f(x, x^*) - f(x, z)]. \tag{9}$$

Since x^* is the solution of EP (3), we have $f(x^*, z) \geq 0$. By the strong pseudomonotonicity of f,

$$f(z, x^*) \leq -\beta d^2(z, x^*). \tag{10}$$

We have from (9) and (10) that

$$\langle \exp_z^{-1} x, \exp_z^{-1} x^*\rangle + \lambda\beta d^2(z, x^*) \leq \lambda[f(x, x^*) - f(x, z) - f(z, x^*)]. \tag{11}$$

By the Lipschitz-type continuity of f, one has

$$f(x, z) + f(z, x^*) - f(x, x^*) \geq -Ld(x, z).d(z, x^*).$$

This implies that

$$\lambda[f(x, x^*) - f(x, z) - f(z, x^*)] \leq \lambda Ld(x, z).d(z, x^*). \tag{12}$$

It follows from (11) and (12) that

$$\langle \exp_z^{-1} x, \exp_z^{-1} x^*\rangle \leq -\lambda\beta d^2(z, x^*) + \lambda Ld(x, z).d(z, x^*). \tag{13}$$

Then, by the Cauchy-Schwarz inequality, one has

$$-|| \exp_z^{-1} x||.|| \exp_z^{-1} x^*|| \leq -\lambda\beta d^2(z, x^*) + \lambda Ld(x, z).d(z, x^*),$$

or equivalently

$$-d(x, z) \leq -\lambda\beta d(z, x^*) + \lambda Ld(z, x).$$

Thus,

$$d(x, x^*) \leq d(x, z) + d(z, x^*) \leq \frac{1 + \lambda\beta + \lambda L}{\lambda\beta} d(z, x).$$

The upper error bound is proved.

We now prove the lower error bound. Using the Cauchy-Schwarz inequality and Lemma 2.1, we have

$$\begin{aligned}\langle \exp_z^{-1} x, \exp_z^{-1} x^*\rangle &= \langle \exp_z^{-1} x, \exp_z^{-1} x\rangle + \langle \exp_z^{-1} x, \exp_z^{-1} x^* - \exp_z^{-1} x\rangle \\ &\geq || \exp_z^{-1} x||^2 - || \exp_z^{-1} x||.|| \exp_z^{-1} x^* - \exp_z^{-1} x|| \\ &\geq d^2(x, z) - d(x, z)d(x, x^*).\end{aligned}$$

Hence, by (13), one has

$$d^2(x, z) - d(x, z)d(x, x^*) \leq -\lambda\beta d^2(z, x^*) + \lambda Ld(x, z).d(z, x^*) \leq \lambda Ld(x, z).d(z, x^*).$$

This implies that

$$d(x, z) - d(x, x^*) \leq \lambda Ld(z, x^*) \leq \lambda L[d(z, x) + d(x, x^*)].$$

Thus,

$$\frac{1-\lambda L}{1+\lambda L} d(x, z) \leq d(x, x^*).$$

The proof is complete. □

When the equilibrium bifunction f is strongly monotone, we have sharper estimates.

Theorem 3.4 *Suppose that X has more than one element and that f is strongly monotone with modulus β and satisfies the Lipschitz - type condition (4) with constant L on $\mathcal{X}$. Let x^* be the unique solution of EP (3) and $\lambda > 0$. Then, for every $x \in \mathcal{X}$, we have*

$$d(x, x^*) \leq \left(\frac{\lambda L + 1}{2\lambda\beta} + \sqrt{\left(\frac{\lambda L + 1}{2\lambda\beta}\right)^2 - \frac{1}{\lambda\beta}} \right) d(x, s_\lambda(x)). \tag{14}$$

and

$$d(x, x^*) \geq \left(\frac{\lambda L + 1}{2\lambda\beta} - \sqrt{\left(\frac{\lambda L + 1}{2\lambda\beta}\right)^2 - \frac{1}{\lambda\beta}} \right) d(x, s_\lambda(x)), \tag{15}$$

Proof Let $x \in \mathcal{X}, \lambda > 0$. Set $z := s_\lambda(x)$. As in the proof of Theorem 3.3 (see Inequality (9)), one has

$$\langle \exp_z^{-1} x, \exp_z^{-1} x^* \rangle \leq \lambda [f(x, x^*) - f(x, z)].$$

Since x^* is the solution of EP (3), we have $f(x^*, z) \geq 0$. Hence,

$$\langle \exp_z^{-1} x, \exp_z^{-1} x^* \rangle \leq \lambda [f(x, x^*) - f(x, z) + f(x^*, z)].$$

Using Lemma 2.1, the strong monotonicity of f and the Lipschitz-type condition, we have

$$\begin{aligned}
0 &\leq -\langle \exp_z^{-1} x, \exp_z^{-1} x^* \rangle + \lambda [f(x, x^*) - f(x, z) + f(x^*, z)] \\
&= -\langle \exp_z^{-1} x, \exp_z^{-1} x \rangle - \langle \exp_z^{-1} x, \exp_z^{-1} x^* - \exp_z^{-1} x \rangle \\
&\quad + \lambda [f(x^*, x) + f(x, x^*)] + \lambda [f(x^*, z) - f(x, z) - f(x, x^*)] \\
&\leq -d^2(x, z) + || \exp_z^{-1} x || || \exp_z^{-1} x^* - \exp_z^{-1} x || \\
&\quad - \lambda\beta d^2(x, x^*) + \lambda L d(x, x^*) d(x, z) \\
&\leq -d^2(x, z) + d(x, z).d(x, x^*) - \lambda\beta d^2(x, x^*) + \lambda L d(x, x^*) d(x, z) \\
&= -\lambda\beta \left[d(x, x^*) - \frac{\lambda L + 1}{2\lambda\beta} d(x, z) \right]^2 + \left[\lambda\beta \left(\frac{\lambda L + 1}{2\lambda\beta} \right)^2 - 1 \right] d^2(x, z).
\end{aligned}$$

This implies that

$$\lambda\beta \left[d(x, x^*) - \frac{\lambda L + 1}{2\lambda\beta} d(x, z) \right]^2 \leq \left[\lambda\beta \left(\frac{\lambda L + 1}{2\lambda\beta} \right)^2 - 1 \right] d^2(x, z). \tag{16}$$

Since $\beta \leq L$, we have

$$\left(\frac{\lambda L + 1}{2\lambda\beta} \right)^2 - \frac{1}{\lambda\beta} \geq \left(\frac{\lambda\beta + 1}{2\lambda\beta} \right)^2 - \frac{1}{\lambda\beta} = \left(\frac{\lambda\beta - 1}{2\lambda\beta} \right)^2 \geq 0.$$

From this fact and (16), we obtain the desired inequalities (14) and (15). □

Remark 3.1 Since

$$\left(\frac{\lambda L+1}{2\lambda\beta}+\sqrt{\left(\frac{\lambda L+1}{2\lambda\beta}\right)^2-\frac{1}{\lambda\beta}}\right)\leq\frac{1+\lambda\beta+\lambda L}{\lambda\beta},$$

the estimate (14) is sharper than the estimate (7).

4 Modified projection method: linear and finite convergence

This section is devoted to the study of the modified projection method for solving the equilibrium problem (3).

Algorithm 4.1 (Modified Projection Method)

Initialization: Choose an initial point $x_0 \in \mathcal{X}$. Let $\{\lambda_k\} \subset (0, +\infty)$ be a sequence of real numbers and set $k = 0$.

Iterative step: At stage k, given $x_k \in \mathcal{X}$, compute x_{k+1} as

$$x_{k+1} = \operatorname{argmin}\left\{\lambda_k f(x_k, y) + \frac{1}{2}d^2(x_k, y) : y \in \mathcal{X}\right\}. \tag{17}$$

Algorithm 4.1 has been considered by several authors for solving equilibrium problems in linear spaces (see, e.g., Anh and Hai (2017); Duc et al. (2016)).

4.1 Linear convergence of the modified projection method

Our first convergence result extends the analogous result (Anh and Hai, 2017, Corollary 1) in the linear space setting to the manifold context.

Theorem 4.1 *Let $f : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ be strongly pseudomontone with modulus β and satisfy the Lipschitz-type condition (4) with constant L on $\mathcal{X}$. Let $\{x_k\}$ be the sequence generated by Algorithm 4.1 with*

$$0 < a \leq \lambda_k \leq b \leq \frac{2\beta}{L^2} \quad \forall k \in \mathbb{N}, \tag{18}$$

where a and b are some positive constants. Then, $\{x_k\}$ converges linearly to the unique solution x^* of EP (3). Moreover, for all $k \in \mathbb{N}$, it holds that

$$d(x_{k+1}, x^*) \leq \frac{\alpha^{k+1}}{1-\alpha}d(x_1, x_0)$$

and

$$d(x_{k+1}, x^*) \leq \frac{1}{1-\alpha}d(x_k, x_{k+1}),$$

where

$$\alpha = \frac{1}{\sqrt{1 + a(2\beta - bL^2)}} \in (0, 1). \tag{19}$$

Proof As in the proof of Theorem 3.3, letting $x := x_k$, $\lambda := \lambda_k$ and $z := x_{k+1}$ in (13), we have

$$\langle \exp^{-1}_{x_{k+1}} x_k, \exp^{-1}_{x_{k+1}} x^*\rangle \leq -\lambda_k\beta d^2(x_{k+1}, x^*) + \lambda_k L d(x_k, x_{k+1})d(x_{k+1}, x^*). \tag{20}$$

By the Cauchy - Schwartz inequality, one has

$$\lambda_k L d(x_k, x_{k+1}) d(x_{k+1}, x^*) \leq \frac{1}{2}\left[d^2(x_k, x_{k+1}) + \lambda_k^2 L^2 d^2(x_{k+1}, x^*)\right]. \tag{21}$$

On the other hand, by (1), it holds that

$$\langle \exp^{-1}_{x_{k+1}} x_k, \exp^{-1}_{x_{k+1}} x^* \rangle \geq \frac{1}{2}\left[d^2(x_k, x_{k+1}) + d^2(x_{k+1}, x^*) - d^2(x_k, x^*)\right]. \tag{22}$$

From (20) - (22), we have

$$\begin{aligned}\frac{1}{2}\left[d^2(x_k, x_{k+1}) + d^2(x_{k+1}, x^*) - d^2(x_k, x^*)\right] &\leq -\lambda_k \beta d^2(x_{k+1}, x^*) \\ &+\frac{1}{2}\left[d^2(x_k, x_{k+1}) + \lambda_k^2 L^2 d^2(x_{k+1}, x^*)\right]\end{aligned}$$

which implies that

$$\left[1 + \lambda_k(2\beta - \lambda_k L^2)\right] d^2(x_{k+1}, x^*) \leq d^2(x_k, x^*). \tag{23}$$

Under assumption (18), we have $1 < 1 + a(2\beta - bL^2) \leq 1 + 2\lambda_k\beta - \lambda_k^2 L^2$ for all $k \in \mathbb{N}$. Hence, α defined by (19) belongs to $(0, 1)$.

It follows from (23) that

$$d(x_{k+1}, x^*) \leq \frac{1}{\sqrt{1 + 2\lambda_k\beta - \lambda_k^2 L^2}} d(x_k, x^*) \leq \alpha d(x_k, x^*).$$

Thus, the sequence $\{x_k\}$ converges linearly to the solution x^* of EP (3). Moreover, from the latter inequality, one has

$$d(x_{k+1}, x^*) \leq \alpha d(x_k, x^*) \leq \cdots \leq \alpha^{k+1} d(x_0, x^*).$$

Since

$$d(x_k, x^*) \leq d(x_k, x_{k+1}) + d(x_{k+1}, x^*) \leq d(x_k, x_{k+1}) + \alpha d(x_k, x^*),$$

we get

$$d(x_k, x^*) \leq \frac{1}{1-\alpha} d(x_k, x_{k+1}).$$

It follows that

$$d(x_{k+1}, x^*) \leq \alpha^{k+1} d(x_0, x^*) \leq \frac{\alpha^{k+1}}{1-\alpha} d(x_0, x_1).$$

The proof is complete. □

We next consider the convergence of Algorithm 4.1 with diminishing step size rules.

Theorem 4.2 *Let $f : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ be strongly pseudomontone with modulus β and satisfy the Lipschitz-type condition (4) with constant L on $\mathcal{X}$. Let $\{x_k\}$ be the sequence generated by Algorithm 4.1 with*

$$\sum_{k=0}^{\infty} \lambda_k = \infty, \quad \textit{and} \quad \lim_{k\to\infty} \lambda_k = 0. \tag{24}$$

Then, $\{x_k\}$ *converges to the unique solution* x^* *of EP (3). Moreover, there exists* $k_0 \in \mathbb{N}$ *such that* $\lambda_k(2\beta - \lambda_k L^2) > 0$ *and*

$$d(x_{k+1}, x^*) \leq \frac{1}{\sqrt{\Pi_{i=k_0}^{k}[1 + \lambda_i(2\beta - \lambda_i L^2)]}} d(x_{k_0}, x^*) \tag{25}$$

for all $k \geq k_0$.

Proof Since $\lim_{k\to\infty} \lambda_k = 0$, there is some $k_0 \in \mathbb{N}$ such that $\lambda_k L^2 \leq \beta$. Thus, $\lambda_k(2\beta - \lambda_k L^2) > \lambda_k \beta > 0$ for all $k \geq k_0$. As in the proof of Theorem 4.2, we have

$$[1 + \lambda_k(2\beta - \lambda_k L^2)]d^2(x_{k+1}, x^*) \leq d^2(x_k, x^*)$$

which implies that

$$d(x_{k+1}, x^*) \leq \frac{1}{\sqrt{1 + \lambda_k(2\beta - \lambda_k L^2)}} d(x_k, x^*)$$

for all $k \geq k_0$. Hence, we have for all $k \geq k_0$ that

$$\begin{aligned} d(x_{k+1}, x^*) &\leq \frac{1}{\sqrt{1 + \lambda_k(2\beta - \lambda_k L^2)}} d(x_k, x^*) \\ &\leq \frac{1}{\sqrt{1 + \lambda_k(2\beta - \lambda_k L^2)}} \cdot \frac{1}{\sqrt{1 + \lambda_{k-1}(2\beta - \lambda_{k-1} L^2)}} d(x_{k-1}, x^*) \\ &\leq \cdots \\ &\leq \frac{1}{\sqrt{\prod_{i=k_0}^{k}\left[1 + \lambda_i\left(2\mu - \lambda_i L^2\right)\right]}} d\left(x_{k_0}, x^*\right). \end{aligned}$$

This proves (25).

Now, for each k, set $a_k := \lambda_k(2\beta - \lambda_k L^2)$. Since $a_k > \lambda_k \beta$ for all $k \geq k_0$, it follows from (24) that

$$\sum_{k=k_0}^{\infty} a_k = \infty.$$

Hence,

$$\frac{1}{\Pi_{i=k_0}^{k}(1 + a_i)} \leq \frac{1}{1 + \sum_{i=k_0}^{k} a_i} \to 0 \quad \text{as} \quad k \to \infty.$$

Therefore,

$$\begin{aligned} d\left(x_{k+1}, x^*\right) &\leq \frac{1}{\sqrt{\prod_{i=k_0}^{k}\left[1 + \lambda_i\left(2\mu - \lambda_i L^2\right)\right]}} d\left(x_{k_0}, x^*\right) \\ &= \frac{1}{\sqrt{\prod_{i=k_0}^{k}(1 + a_i)}} d\left(x_{k_0}, x^*\right) \to 0 \end{aligned}$$

as $k \to \infty$. This means that the sequence $\{x_k\}$ converges to the unique solution x^* of EP (3). This ends the proof. □

4.2 Finite convergence of the modified projection method

We study the finite convergence of the modified projection method under θ-conditioning assumption. Recall that the concept of θ-conditioning for the bifunction f in connection with the solution set of EP was introduced by Moudafi (2007) to obtain the finite and strong convergence for proximal point method for solving EP in Euclidean spaces. For more results about concerning θ-conditioning and its applications in the linear space setting, we refer the reader to Nguyen et al. (2020).

Definition 4.1 (Moudafi 2007) The equilibrium bifunction f is said to be θ-conditioned with modulus γ if and only if there exist two positive constants γ and θ such that

$$-f\left(x, P_{\mathcal{X}^*}(x)\right) \geq \gamma\left[d\left(x, \mathcal{X}^*\right)\right]^{\theta} \quad \text{for all } x \in \mathcal{X}. \tag{26}$$

We say that f is linearly conditioned if it is 1-conditioned.

Remark 4.1 If (26) holds, we also say that the solution set $\mathcal{X}^*$ is θ-conditioned with modulus γ. In the case when $\theta = 1$, $\mathcal{X}^*$ is said to be linearly conditioned with modulus γ.

When EP has a unique solution x^*, i.e., $\mathcal{X}^* = \{x^*\}$, then $P_{\mathcal{X}^*}(x) = x^*$ for all $x \in \mathcal{X}$. In this case (26) can be rewritten as

$$-f\left(x, x^*\right) \geq \gamma\left[d\left(x, x^*\right)\right]^{\theta} \quad \text{for all } x \in \mathcal{X}. \tag{27}$$

Example 4.1 Let $\mathcal{M}$, $\mathcal{X}$ and f be as in Example 3.1. In this case, $x^* = 1$ is the unique solution of EP(3). Since

$$\left(1+\frac{1}{x}\right)\ln^2 x \geq 0 \quad \text{and} \quad x-1 \geq \ln x, \qquad \forall x \geq 1,$$

we have for all $x \in \mathcal{X}$ that

$$\begin{aligned}
-f(x, x^*) &= -\left[\left(1+\frac{1}{x^*}\right)\ln x . \ln\frac{x^*}{x} + x^* - x\right] \\
&= \left(1+\frac{1}{x}\right)\ln^2 x + x - 1 \\
&\geq \ln x = d(x, x^*).
\end{aligned}$$

Thus, f is linearly conditioned with modulus $\gamma = 1$.

Example 4.2 Let $\mathcal{P}^n$ be the set of all real symmetric matrices of order n, and $\mathcal{P}^n_{++}$ be the cone of all real symmetric positive definite matrices of order n. Then, $\mathcal{M} = (\mathcal{P}^n_{++}, \langle\cdot,\cdot\rangle)$ is a Riemannian manifold with the Riemmanian metric defined by

$$\langle U, V\rangle = \operatorname{tr}(X^{-1}UX^{-1}V), \quad X \in \mathcal{M},\ U, V \in \mathcal{T}_X\mathcal{M},$$

where $\operatorname{tr}(U)$ denotes the trace of matrix $U \in \mathcal{P}^n$ and $\mathcal{T}_X\mathcal{M} \simeq \mathcal{P}^n$ for each $X \in \mathcal{M}$ (see, e.g., Rothaus (1960)). Moreover, $\mathcal{M}$ is a Hadamard manifold (see, e,g, (Lang, 1999, Theorem 1.2, p. 325)). The unique geodesic connecting two points $X, Y \in \mathcal{M}$ is defined by

$$\gamma(t) = X^{1/2}(X^{-1/2}YX^{-1/2})^t X^{1/2}, \quad t \in [0, 1].$$

For any $X \in \mathcal{M}$, the exponential map $\exp_X : \mathcal{T}_X\mathcal{M} \to \mathcal{M}$ and its inverse $\exp^{-1}_X : \mathcal{M} \to \mathcal{T}_X\mathcal{M}$ are defined respectively by: for any $V \in \mathcal{T}_X\mathcal{M}$ and $Y \in \mathcal{M}$,

$$\exp_X V = X^{1/2}e^{(X^{-1/2}VX^{-1/2})}X^{1/2}, \quad \exp^{-1}_X Y = X^{1/2}\ln(X^{-1/2}YX^{-1/2})X^{1/2},$$

where $\ln U$ is the logarithm of matrix U. For $X, Y \in \mathcal{M}$, the Riemannian distance between X and Y is defined as

$$d(X,Y) = \left[\mathrm{tr}\left(\ln^2\left(X^{-1/2}YX^{-1/2}\right)\right)\right]^{1/2} = \left[\sum_{i=1}^{n}\ln^2\left(\lambda_i\left(X^{-1/2}YX^{-1/2}\right)\right)\right]^{1/2},$$

where $\lambda_i\left(X^{-1/2}YX^{-1/2}\right)$ denotes the ith eigenvalue of the matrix $X^{-1/2}YB^{-1/2}$.

Let $\mathcal{X} = \{X \in \mathcal{P}^n_{++} : \det X \geq 1\}$. We consider the bifunction $f : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ defined by

$$f(X,Y) = \mathrm{tr}\ln Y - \mathrm{tr}\ln X + \sqrt{\mathrm{tr}\ln^2 Y} - \sqrt{\mathrm{tr}\ln^2 X}, \quad \forall X, Y \in \mathcal{M}.$$

We claim that $X^* = I_n$, the identity matrix of order n, is the unique solution of EP (3). Indeed, using the property

$$\mathrm{tr}\ln X = \ln\det X, \quad \forall X \in \mathcal{P}^n_{++},$$

we can rewrite f as

$$f(X,Y) = \ln\det Y - \ln\det X + d(Y, I_n) - d(X, I_n).$$

Thus, $f(I_n, Y) \geq 0$ for all $Y \in \mathcal{X}$ and $X^* = I_n$ is a solution of EP. Assume that $Z \neq I_n$ is another solution of EP. Then, $f(Z, I_n) = -\ln\det Z - d(Z, I_n) \geq 0$. That is a contradiction. Therefore, $X^* = I_n$ is the unique solution of EP (3). Now, for every $X \in \mathcal{X}$, we have

$$\begin{aligned} -f(X, X^*) &= -\ln\det X^* + \ln\det X - d(X^*, I_n) + d(X, I_n) \\ &= \ln\det X + d(X, I_n) \geq d(X, I_n). \end{aligned}$$

Hence, f is linearly conditioned with modulus $\gamma = 1$.

Our finite convergence result is state as follows.

Theorem 4.3 *Let $f : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ be strongly pseudomonotone with modulus β and satisfy the Lipschitz-type condition (4) with constant L on $\mathcal{X}$. Let $\{x_k\}$ be a sequence generated by Algorithm 4.1 with*

$$0 < a \leq \lambda_k \leq b < \frac{2\beta}{L^2}, \quad \forall k \in \mathbb{N}$$

where a and b are some positive constants. If f is θ-conditioned with modulus γ for some $\theta \in (0, 1]$ and $\gamma > 0$, then $x_k \in \mathcal{X}^*$ for all k sufficiently large.

Proof Let x^* be the unique solution of EP(3). Since f is θ-conditioned with modulus γ, by definition we have

$$\gamma[d(x, x^*)]^\theta \leq -f(x, x^*) \quad \forall x \in \mathcal{X}.$$

It follows that

$$\gamma[d(x_{k+1}, x^*)]^\theta \leq -f(x_{k+1}, x^*) \quad \forall k \in \mathbb{N}. \tag{28}$$

In (9), letting $x := x_k$, $\lambda := \lambda_k$ and $z := x_{k+1}$, we have

$$\langle \exp^{-1}_{x_{k+1}} x_k, \exp^{-1}_{x_{k+1}} x^* \rangle \leq \lambda_k[f(x_k, x^*) - f(x_k, x_{k+1})]$$

which implies that

$$\frac{1}{\lambda_k}\langle \exp^{-1}_{x_{k+1}} x_k, \exp^{-1}_{x_{k+1}} x^* \rangle \leq f(x_k, x^*) - f(x_k, x_{k+1}). \tag{29}$$

From (28) and (29), we have

$$\gamma[d(x_{k+1}, x^*)]^\theta + \frac{1}{\lambda_k}\langle \exp^{-1}_{x_{k+1}} x_k, \exp^{-1}_{x_{k+1}} x^*\rangle \le -f(x_{k+1}, x^*) + f(x_k, x^*) - f(x_k, x_{k+1}).$$

Then, by the Lipschitz type condition and the Cauchy - Schwarz inequality, one has

$$\begin{aligned}\gamma[d(x_{k+1}, x^*)]^\theta &\le -\frac{1}{\lambda_k}\langle \exp^{-1}_{x_{k+1}} x_k, \exp^{-1}_{x_{k+1}} x^*\rangle + f(x_k, x^*) - f(x_k, x_{k+1}) - f(x_{k+1}, x^*)\\ &\le \frac{1}{\lambda_k}\|\exp^{-1}_{x_{k+1}} x_k\|.\|\exp^{-1}_{x_{k+1}} x^*\| + Ld(x_k, x_{k+1}).d(x_{k+1}, x^*)\\ &= \left(\frac{1}{\lambda_k} + L\right) d(x_k, x_{k+1}).d(x_{k+1}, x^*). \end{aligned} \tag{30}$$

Assume to the contrary that the conclusion of the theorem is not true. Then, there exists a subsequence of $\{x_k\}$ which, without loss of generality, is still denoted by $\{x_k\}$ such that $x_k \neq x^*$ for all k. Thus, by (30), one has

$$\begin{aligned}\gamma &\le \left(\frac{1}{\lambda_k} + L\right) d(x_k, x_{k+1}).[d(x_{k+1}, x^*)]^{1-\theta}\\ &\le \left(\frac{1}{a} + L\right)[d(x_k, x^*) + d(x_{k+1}, x^*).[d(x_{k+1}, x^*)]^{1-\theta}\end{aligned}$$

for all k. Letting $k \to \infty$ in the latter inequality and using the fact that $\lim_{k\to\infty} x_k = x^*$, we get $\gamma \le 0$. This is a contradiction. Therefore, $x_k \in \mathcal{X}^*$ for all k sufficiently large. □

We next give an upper bound for the number of iterations for which sequences generated by Algorithm 4.1 terminate.

Theorem 4.4 *Asume that $f : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ is strongly pseudomonotone with modulus β and satisfy the Lipschitz-type condition (4) with constant L on $\mathcal{X}$. Let $\{x_k\}$ be a sequence generated by Algorithm 4.1 with*

$$0 < a \le \lambda_k \le b < \frac{2\beta}{L^2}, \quad \forall k \in \mathbb{N}$$

where a and b are some positive constants. If f is linearly conditioned with modulus γ, then the sequence $\{x_k\}$ converges to the unique solution x^* of EP in at most $\ell + 1$ iterations with

$$\ell \le \frac{2\beta(1 + aL)^2 d^2(x_0, x^*)}{(2\beta - bL^2)a^2\gamma^2}.$$

Proof As in the proof of Theorem 4.1, from (20) and (22) we have

$$\begin{aligned}&\frac{1}{2}\left[d^2(x_k, x_{k+1}) + d^2(x_{k+1}, x^*) - d^2(x_k, x^*)\right]\\ &\le -\lambda_k \beta d^2(x_{k+1}, x^*) + \lambda_k L d(x_k, x_{k+1}) d(x_{k+1}, x^*)\\ &\le -\lambda_k \beta d^2(x_{k+1}, x^*) + \lambda_k\left[\frac{L^2}{4\beta} d^2(x_k, x_{k+1}) + \beta d(x_{k+1}, x^*)\right]\end{aligned}$$

which implies that

$$\left(1 - \frac{\lambda_k L^2}{2\beta}\right) d^2(x_k, x_{k+1}) \le d^2(x_k, x^*) - d^2(x_{k+1}, x^*).$$

Since $0 < a \le \lambda_k \le b < 2\beta/L^2$ for all k, it follows from the latter inequality that

$$\left(1-\frac{bL^2}{2\beta}\right)d^2(x_k,x_{k+1}) \le d^2(x_k,x^*) - d^2(x_{k+1},x^*) \tag{31}$$

For $0 < N \in \mathbb{N}$, we have from (31) that

$$\begin{aligned}\left(1-\frac{bL^2}{2\beta}\right)\sum_{i=0}^{N} d^2(x_i,x_{i+1}) &\le \sum_{i=0}^{N}\left(d^2(x_i,x^*) - d^2(x_{i+1},x^*)\right)\\ &= d^2(x_0,x^*) - d^2(x_{N+1},x^*)\\ &\le d^2(x_0,x^*).\end{aligned} \tag{32}$$

Since $\lim_{k\to\infty} x_k = x^*$, we have that $\lim_{k\to\infty} d(x_k,x_{k+1}) = 0$. Let ℓ be the smallest integer such that

$$d(x_\ell,x_{\ell+1}) < \frac{a\gamma}{1+aL}. \tag{33}$$

We claim that $x_{\ell+1} = x^*$. If not, by (30), we have

$$\begin{aligned}\gamma d(x_{\ell+1},x^*) &\le \left(\frac{1}{\lambda_k}+L\right)d(x_\ell,x_{\ell+1})d(x_{\ell+1},x^*)\\ &\le \left(\frac{1}{a}+L\right)d(x_\ell,x_{\ell+1})d(x_{\ell+1},x^*).\end{aligned}$$

Since $d(x_{\ell+1},x^*) > 0$, using (33), one has

$$\gamma \le \left(\frac{1}{a}+L\right)d(x_\ell,x_{\ell+1}) < \frac{1+aL}{a}.\frac{a\gamma}{1+aL} = \gamma,$$

which is a contradiction. Thus, $x_{\ell+1} = x^*$. By (32),

$$d^2(x_0,x^*) \ge \left(1-\frac{bL^2}{2\beta}\right)\sum_{i=0}^{\ell-1} d(x_i,x_{i+1}) \ge \frac{(2\beta-bL^2)\ell a^2\gamma^2}{2\beta(1+aL)^2}.$$

This implies that

$$\ell \le \frac{2\beta(1+aL)^2 d^2(x_0,x^*)}{(2\beta-bL^2)a^2\gamma^2}.$$

The proof is complete. □

The results presented in Theorem 4.3 and Theorem 4.4 are new even in the setting of linear spaces. To illustrate our finding, we present an example in Euclidean spaces in which the equilibrium bifunction is linearly conditioned and therefore sequences generated by the modified projection method for solving corresponding problem terminate after a finite number of iterations.

Example 4.3 Let $\mathcal{M} = \mathbb{R}^n$ with the usual inner product $\langle\cdot,\cdot\rangle$ and corresponding norm $||\cdot||$ on $\mathbb{R}^n$. Let $\mathcal{X} = \{(x_1,x_2,\cdots,x_n)^\top \in \mathbb{R}^n : a_i \le x_i \le b_i\}$ be a closed convex subset of $\mathbb{R}^n$, where $a_i, b_i, i = 1,2\cdots n$, are real numbers. Let us consider the bifunction $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(x,y) = g(x)\langle Ax + By + p, y - x\rangle, \quad \forall x,y \in \mathbb{R}^n,$$

where $g : \mathbb{R}^n \to \mathbb{R}$, $A = (a_{ij})$, $B = (b_{ij})$ in $\mathbb{M}^{n\times n}(\mathbb{R})$ and $p = (p_1,\cdots,p_n)^\top$ in $\mathbb{R}^n$ satisfy the following conditions:

i) g is Lipschitz continuous on $\mathcal{X}$ with a constant ℓ, i.e.,

$$|g(x) - g(y)| \leq \ell||x - y|| \quad \forall x, y \in \mathcal{X},$$

and satisfies $0 < \delta_1 \leq g(x) \leq \delta_2$ for all $x \in \mathcal{X}$ and for some constants $\delta_1, \delta_2 \in \mathbb{R}_+$;

ii) $A - B$ is symmetric positive definite and for all $i = 1, 2, \cdots, n$:

$$\min\left\{\sum_{j=1}^{n} a_{ij}x_j + \sum_{j=1}^{n} b_{ij}y_j + p_i : a_j \leq x_j, y_j \leq b_j,\ j = 1, 2, \cdots, n\right\} > 0$$

We show that f satisfies the Lipschitz-type condition (4) on $\mathcal{X}$. Indeed, for $x, y, z \in \mathbb{R}^n$, if we set $h(x, y) = \langle Ax + By + p, y - x\rangle$, then

$$h(x, y) + h(y, z) - h(x, z) \geq -||A - B|| \cdot ||y - x|| \cdot ||z - y||,$$

see Quoc and Muu (2012) for details. Thus, for every $x, y, z \in \mathcal{X}$, we have

$$\begin{aligned}
&f(x, y) + f(y, z) - f(x, z)\\
&= g(x)h(x, y) + g(y)h(y, z) - g(x)h(x, z)\\
&= g(x)[h(x, y) + h(y, z) - h(x, z)] + [g(y) - g(x)]h(y, z)\\
&\geq -g(x)||A - B|| \cdot ||y - x|| \cdot ||z - y|| + [g(y) - g(x)]\langle Ay + Bz + p, z - y\rangle\\
&\geq -\delta_2||A - B|| \cdot ||y - x|| \cdot ||z - y|| - \ell||Ay + Bz + p|| \cdot ||y - x|| \cdot ||z - y||\\
&\geq -\delta_2||A - B|| \cdot ||y - x|| \cdot ||z - y|| - M\ell||y - x|| \cdot ||z - y||\\
&\geq -\left(\delta_2||A - B|| + M\ell\right)||y - x|| \cdot ||z - y||\\
&= -\left(\delta_2||A - B|| + M\ell\right)d(x, y)d(y, z),
\end{aligned}$$

where $M = \sup_{x,y\in\mathcal{X}} ||Ax + By + p||$. Thus, f satisfies the Lipschitz-type condition with constant $L = \delta_2||A - B|| + M\ell$ on $\mathcal{X}$.

We now show that f is strongly pseudomonotone on $\mathcal{X}$. Indeed, let $x, y \in \mathcal{X}$ be such that $f(x, y) \geq 0$. Since $g(x) > 0$, we have$\langle Ax + By + p, y - x\rangle \geq 0$. Thus,

$$\begin{aligned}
f(y, x) &= g(y)\langle Ay + Bx + p, x - y\rangle\\
&\leq g(y)\langle Ay + Bx + p, x - y\rangle - g(y)\langle Ax + By + p, y - x\rangle\\
&= -g(y)\langle(A - B)(y - x), y - x\rangle\\
&\leq -\delta_1\lambda_{\min}(A - B)||x - y||^2 = -\delta_1\lambda_{\min}(A - B)d^2(x, y),
\end{aligned}$$

where $\lambda_{\min}(A - B)$ is the smallest eigenvalue of the positive definite matrix $A - B$. Therefore, f is strongly pseudomonotone on $\mathcal{X}$ with modulus $\beta = \delta_1\lambda_{\min}(A - B)$.

One can see that all assumptions of Theorem 3.2 are satisfied and the equilibrium problem (3) has a unique solution. Let $x^* = (x_1^*, x_2^*, \cdots, x_n^*)^\top \in \mathcal{X}$ be the unique solution of EP (3). Then, for all $y = (y_1, y_2, \cdots, y_n)^\top \in \mathcal{X}$, we have $f(x^*, y) \geq 0$. Since $g(x^*) \geq 0$, it follows that $\langle Ax^* + By + p, y - x^*\rangle \geq 0$. Equivalently,

$$\sum_{i=1}^{n}\left[\left(\sum_{j=1}^{n} a_{ij}x_j^* + \sum_{j=1}^{n} b_{ij}y_j + pi\right)(y_i - x_i^*)\right] \geq 0. \tag{34}$$

Since $\sum_{j=1}^{n} a_{ij}x_j^* + \sum_{j=1}^{n} b_{ij}y_j + p_i > 0$ for all $i = 1, 2, \cdots, n$, it follows from (34) that $x_i^* = a_i$ for all $i = 1, 2, \cdots, n$. Hence, $x^* = (a_1, a_2, \cdots, a_n)^\top$ is the unique solution of EP(3).

Table 1 Finite convergence for Algorithm 4.1 with random data

Samples	Number of iterations						
	$n=2$	$n=5$	$n=10$	$n=20$	$n=50$	$n=100$	$n=200$
Sample 1	2	4	6	8	8	11	15
Sample 2	3	2	5	4	9	9	12
Sample 3	1	5	3	5	8	12	13
Sample 4	3	4	3	5	9	10	11
Sample 5	2	3	5	9	10	9	13

Set

$$\Lambda = \min_{1\le i\le n} \min \left\{ \sum_{j=1}^{n} a_{ij}x_j + \sum_{j=1}^{n} b_{ij}a_j + p_i : a_j \le x_j \le b_j, j = 1, 2, \cdots, n \right\}.$$

We now show that f is linearly conditioned, i.e., for some $\gamma > 0$,

$$-f(x, x^*) \ge \gamma d(x, x^*), \quad \forall x \in \mathcal{X}.$$

For $x = (x_1, x_2, \cdots, x_n)^\top \in \mathcal{X}$, we have

$$\begin{aligned} -f(x, x^*) &= -g(x)\langle Ax + Bx^* + p, x^* - x\rangle \\ &= g(x)\langle Ax + Bx^* + p, x - x^*\rangle \\ &= g(x)\sum_{i=1}^{n}\left[\left(\sum_{j=1}^{n} a_{ij}x_j + b_{ij}a_i + p_i\right)(x_i - a_i)\right] \\ &\ge \delta_1\Lambda\sum_{i=1}^{n}(x_i - a_i) = \delta_1\Lambda\sqrt{\left(\sum_{i=1}^{n}(x_i - a_i)\right)^2} \\ &\ge \gamma\sqrt{\sum_{i=1}^{n}(x_i - a_i)^2} = \gamma d(x, x^*). \end{aligned}$$

where $\gamma = \delta_1\Lambda > 0$.

By Theorem 4.3, if $\{x_k\}$ is a sequence generated by Algorithm 4.1 with $\{\lambda_k\}$ satisfying (18), then $\{x_k\}$ converges to the unqiue solution x^* of EP (3) after a finite number of iterations.

We first test for the case when $\mathcal{X} = \{(x_1, x_2, \cdots, x_n)^\top \in \mathbb{R}^n : 1 \le x_i \le 10, i = 1, 2, \cdots, n\}$, A and B are diagonal matrices and $g(x) = 1$ for all $x \in \mathbb{R}^n$. Table 1 presents the finite convergence of sequences $\{x_k\}$ generated by Algorithm 4.1 in different dimensions where the main diagonal elements of A are randomly chosen in the interval [2.5, 5], the main diagonal elements of B are randomly chosen in the interval [0.5, 2], the elements of vector p are randomly chosen in the interval [0, 5], the elements of the initial point x_0 are also randomly chosen in the interval [5, 10] and the step size $\lambda_k = \lambda = 1.5 \times \beta/L^2$ for all k.

Table 2 Finite convergence for Algorithm 4.1 with different step sizes

Iteration k	$d(x_k, x^*)$		
	$\lambda_k = 0.15$	$\lambda_k = (k+1)/9(k+2)$	$\lambda_k = 0.05$
1	4.482567037341691	6.352483212127498	6.634917948743128
2	1.872654123818203	3.430513285886522	4.379463999702835
3	0.624159022806268	1.595655733535987	2.821055326204551
4	0.019783379375767	0.561901882941611	1.751149114875450
5	0	0.010644649652477	1.034819014029203
6		0	0.521523745901228
7			0.163346799307060
8			0

We next consider the case where A, B and p are chosen, as in Quoc and Muu (2012), by

$$A = \begin{bmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 2 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 1 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

and $p = (1, -2, -1, 2, -1)^\top$. Then, $||A - B|| = 2.905$ and $\lambda_{\min}(A - B) = 0.7192$. We also let $g(x) = 1$ for all $x \in \mathbb{R}^5$ and consider $\mathcal{X} = [0, 5] \times [1, 5] \times [1, 5] \times [0, 5] \times [1, 5]$ a closed convex subset of $\mathbb{R}^5$. Thus, f is strongly pseudomonotone with constant $\beta = \lambda_{\min}(A - B) = 0.7192$ and satisfies the Lipschitz - type condition (4) with constant $L = ||A - B|| = 2.905$. The unique solution of EP (3) is $x^* = (0, 1, 1, 0, 1)^\top$. Table 2 presents the finite convergence results for sequence $\{x_k\}$ generated by Algorithm 4.1 with $x_0 = (5, 5, 5, 5, 5)^\top$ and different step sizes.

5 Conclusions

In this paper, we have obtained several new results for strongly pseudomonotone equilibrium problems (in short, SPEP) on Hadamard manifolds. Under mild conditions, we have established the existence and uniqueness of the solution of SPEP. We have also provided a global error bound for SPEP. We have proposed the modified projection method and proved that sequences generated by the method with suitable step size converges to the unique solution of the SPEP. Moreover, we have shown, under linear conditioning assumption, that sequences generated by the modified projection method converge to the unique solution of the SPEP after a finite number of iterations. Some of our results extends the analogous results from linear spaces to Hadamard manifolds, while some results are new even in the Euclidean space setting. We have also provided several examples and numerical experiments to illustrate our new results.

Acknowledgements This research is funded by Hue University under grant number DHH2023-03-184. A part of this paper was done when the second author was visiting the Vietnam Institute for Advanced Study in Mathematics (VIASM). LVN would like to thank VIASM for supplying a fruitful research environment during the visit.

Data Availability Data sharing is not applicable to this article, as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The authors declare that they have no Conflict of interest.

References

Al-Homidan S, Ansari QH, Islam M (2021) Existence results and two step proximal point algorithm for equilibrium problems on Hadamard manifolds. Carpathian J Math 37(3):393–406

Anh PK, Hai TN (2017) Splitting extragradient-like algorithms for strongly pseudomonotone equilibrium problems. Numer Algor 76(1):67–91

Anh PK, Hai TN (2019) Novel self-adaptive algorithms for non-lipschitz equilibrium problems with applications. J Global Optim 73(3):637–657

Ansari QH, Islam M (2020) Explicit iterative algorithms for solving equilibrium problems on Hadamard manifolds. J. Nonlinear Convex Anal 21(2):425–439

Antipin AS (1995) The convergence of proximal methods to fixed points of extremal mappings and estimates of their rate of convergence. Comput Math Math Phys 35(5):539–551

Azagra D, Ferrera J, López-Mesas F (2005) Nonsmooth analysis and Hamilton-Jacobi equations on Riemannian manifolds. J Funct Anal 220(2):304–361

Babu F, Ali A, Alkhaldi AH (2022) An extragradient method for non-monotone equilibrium problems on Hadamard manifolds with applications. Appl Numer Math 180:85–103

Bento G, Cruz Neto J, Melo I (2022) Combinatorial convexity in Hadamard manifolds: existence for equilibrium problems. J Optim Theory Appl 195(3):1087–1105

Bigi G, Castellani M, Pappalardo M, Passacantando M (2019) Nonlinear programming techniques for equilibria. Springer, Berlin

Blum E, Oettli W (1994) From optimization and variational inequalities to equilibrium problems. Math Student 62(1):127–169

Carmo MP (1992) Riemannian geometry, vol 6. Birkhauser, Boston

Chen J, Liu S, Chang X (2021) Extragradient method and golden ratio method for equilibrium problems on Hadamard manifolds. Int J Comput Math 98(8):1699–1712

Colao V, López G, Marino G, Martín-Márquez V (2012) Equilibrium problems in Hadamard manifolds. J Math Anal Appl 388(1):61–71

Cruz Neto J, Santos P, Soares P (2016) An extragradient method for equilibrium problems on Hadamard manifolds. Optim Lett 10(6):1327–1336

Cruz Neto J, Jacinto F, Soares P, Souza J (2018) On maximal monotonicity of bifunctions on Hadamard manifolds. J Global Optim 72(3):591–601

Duc PM, Muu LD, Quy NV (2016) Solution-existence and algorithms with their convergence rate for strongly pseudomonotone equilibrium problems. Pacific J Optim 12(4):833–845

Fan J, Tan B, Li S (2021) An explicit extragradient algorithm for equilibrium problems on Hadamard manifolds. Comp Appl Math 40(2):1–15

Hai TN (2020) Error bounds and stability of the projection method for strongly pseudomonotone equilibrium problems. Int J Comput Math 97(12):2516–2530

Hieu DV (2019) New inertial algorithm for a class of equilibrium problems. Numer. Algor. 80(4):1413–1436

Jana S (2022) Equilibrium problems under relaxed α-monotonicity on Hadamard manifolds. Rend Circ Mat Palermo 71(2):361–372

Khammahawong K, Kumam P, Chaipunya P, Yao J, Wen C, Jirakitpuwapat W (2020) An extragradient algorithm for strongly pseudomonotone equilibrium problems on Hadamard manifolds. Thai J Math 18:350–371

Knaster B, Kuratowski C, Mazurkiewicz S (1929) Einbeweis des fixpunktsatzes für n-dimensionale simplexes. Fund Math 14:132–137

Konnov I (2007) Equilibrium models and variational inequalities. Elsevier, Amsterdam

Lang S (1999) Fundamentals of differential geometry. Springer, New York

Ledyaev Y, Zhu Q (2007) Nonsmooth analysis on smooth manifolds. Trans Amer Math Soc 359(8):3687–3732

Li C, López G, Martín-Márquez V (2009) Monotone vector fields and the proximal point algorithm on Hadamard manifolds. J London Math Soc 79(3):663–683

Li C, Mordukhovich BS, Wang J, Yao J-C (2011) Weak sharp minima on Riemannian manifolds. SIAM J Optim 21(4):1523–1560
Li X-B, Zhou L-W, Huang N-J (2016) Gap functions and descent methods for equilibrium problems on Hadamard manifolds. J. Nonlinear Convex Anal 17(4):807–826
Mastroeni G (2003) On Auxiliary Principle for Equilibrium Problems. In: P. Daniele, et. al.(eds.) Equilibrium Problems and Variational Models. Kluwer Academic Publishers, Dordrecht, pp. 289-298
Moudafi A (2007) On finite and strong convergence of a proximal method for equilibrium problems. Numer Funct Anal Optim 28(11–12):1347–1354
Muu LD, Quy NV (2015) On existence and solution methods for strongly pseudomonotone equilibrium problems. Vietnam J Math 43(2):229–238
Németh S (2003) Variational inequalities on Hadamard manifolds. Nonlinear Anal 52(5):1491–1498
Nguyen LV, Ansari QH, Qin X (2020) Linear conditioning, weak sharpness and finite convergence for equilibrium problems. J Global Optim 77(2):405–424
Pang J-S (2018) Study of equilibrium problem in Hadamard manifolds by extending some concepts of nonlinear analysis. Rev R Acad Cienc Exactas Fís Nat Ser A Mat RACSAM 112:1521–1537
Quoc TD, Muu LD (2012) Iterative methods for solving monotone equilibrium problems via dual gap functions. Comput Optim Appl 51(2):709–728
Rothaus O (1960) Domains of positivity. Abh Math Sem Univ Hamburg 24:189–235
Sakai T (1996) Riemannian geometry. American Mathematical Society, Providence
Tam VM (2022) Dynamical systems for solving variational inclusion and fixed point problems on Hadamard manifolds. Oper Res Lett 50:91–98
Udriste C (1994) Convex functions and optimization methods on Riemannian manifolds. Springer, Dordrecht
Vuong P, Strodiot J (2020) A dynamical system for strongly pseudo-monotone equilibrium problems. J Optim Theory Appl 185(3):767–784
Walter R (1974) On the metric projection onto convex sets in Riemannian spaces. Arch Math 25(1):91–98
Wang X, López G, Li C, Yao J-C (2019) Equilibrium problems on Riemannian manifolds with applications. J Math Anal Appl 473(2):866–891
Yin L, Liu H, Yang J (2022) Modified golden ratio algorithms for pseudomonotone equilibrium problems and variational inequalities. Appl Math 67:273–296