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Mathematics Teachers' Changes in Knowledge for Teaching Conditional Probability Through Lesson Study

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Abstract. In recent years, probability has become an important content strand in high school mathematics curriculum around the world. Consequently, research on the knowledge for teaching probability has attracted significant interest. However, studies addressing the knowledge for teaching conditional probability remain limited. This study investigated the changes in high school mathematics teachers' knowledge for teaching conditional probability in Vietnam through lesson study. Lesson plans, classroom observations, reflection discussions from a convenience sample of three teachers were used as data in the current study. The data was analyzed using qualitative methods. The findings indicated changes in the teachers' knowledge for teaching conditional probability across four domains: Common Conditional Probability Knowledge (CCPK), Specialized Conditional Probability Knowledge (SCPK), Knowledge of Conditional Probability and Students (KCPS), and Knowledge of Conditional Probability and Teaching (KCPT). Specifically, the results revealed significant shifts in teachers' conceptual understanding of the topic, including its nature, meaning, and related formulas. The study further documented significant progress in the teachers' pedagogical knowledge: improving proficiency in visual representations; enhancing capacity for anticipating and monitoring student thinking and misconceptions; and focusing on leveraging real-world contexts for conceptual introduction. Collectively, these advancements illustrated the positive impact of lesson study on developing knowledge required for teaching conditional probability. Suggestions about future research to support teachers with various experiences in learning to teach new mathematical topics are included.

Keywords: conditional probability; lesson study; teacher knowledge

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1. Introduction

Probability has widespread applications across numerous scientific disciplines and in daily life. Consequently, probability occupies a crucial position in the mathematics curriculum from primary to secondary levels in various countries (Batanero, 2022), including Vietnam (Ministry of Education and Training [MOET], 2018). The 2018 mathematics reform program in Vietnam has integrated this content strand from Grade 2 through Grade 12 including new topics, such as conditional probability and Bayes' theorem. Conditional probability, along with the multiplication rule, the law of total probability, Bayes' theorem, and their practical applications have been incorporated into Grade 12. The integration of conditional probability requires teachers to deepen their knowledge to teach it effectively (Groth, 2010).

However, the abstract nature of probability, coupled with student misconceptions (Batanero & Sanchez, 2005), presents teachers with significant difficulties and challenges in their teaching (Díaz et al., 2010). Probability demands a unique mode of thinking that often runs counter to natural intuition and everyday human experience (Batanero & Sanchez, 2005). The combination of intuitive barriers and complex combinatorial skills renders the teaching of probability considerably more difficult than other mathematical strands (Groth, 2010). Concurrently, inadequate professional preparation focusing on teaching probability in teacher education programs has resulted in ineffective instruction (Estrada et al., 2018).

Lesson study is known as a professional development model driven by a community of teachers who collaboratively design lesson plans, teach and observe research lessons, and discuss and reflect upon teaching practices (Nguyen & Tran, 2023). Lesson study has been utilized in various countries to develop mathematical knowledge for teaching for in-service teachers and pre-service teachers in teaching rational numbers (da Ponte et al., 2022), fractions (Huang et al., 2019), geometry (Moss et al., 2015), and derivatives (Verhoef et al., 2014). Lesson study has also been employed to foster mathematical teaching knowledge and beliefs among in-service teachers (Dudley et al., 2019; Nguyen & Tran, 2023; Ní Shúilleabháin, 2016; Ní Shúilleabháin & Clivaz, 2017; Widjaja et al., 2017) and pre-service teachers (Leavy & Hourigan, 2016; Muñoz et al., 2024).

However, scant research has addressed the use of lesson study to develop mathematical knowledge for teaching probability, particularly conditional probability. While students' understanding and misconceptions in probability are well-documented (Batanero & Álvarez-Arroyo, 2024; Batanero & Sanchez, 2005; Böcherer-Linder et al., 2018; Jones et al., 1999), Batanero (2022) highlights a significant research gap regarding teachers' knowledge for teaching probability, specifically conditional probability. Only a few studies have utilized the Mathematical Knowledge for Teaching (MKT) framework (Ball et al., 2008) to measure teachers' knowledge in this domain (Carrillo-Yañez et al., 2018; Danişman & Tanışlı, 2017), particularly in conditional probability (Groth, 2010). Therefore, this study addresses this gap through a case study of three mathematics teachers at an urban high school in Central Vietnam. The research focuses on analyzing the influence of lesson study on high school mathematics teachers' knowledge about teaching conditional probability. It addressed the research question: How does high school mathematics teachers' knowledge for teaching conditional probability change through lesson study?

2. Literature Review

2.1 Mathematical knowledge for teaching

Shulman (1986) classified different types of teacher knowledge, including: content knowledge; general pedagogical knowledge; curricular knowledge; pedagogical content knowledge; knowledge of learners and their characteristics; knowledge of educational contexts; and knowledge of educational aims and values. Two domains of knowledge that have garnered substantial attention are content knowledge and pedagogical content knowledge. These constructs were later expanded upon by Ball et al. (2008). Building upon Shulman's model of teacher knowledge, Ball et al. developed a model for Mathematical Knowledge for Teaching (MKT) specific to mathematics teaching. MKT comprises Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK). In their framework, SMK comprises Common Content Knowledge (CCK), Specialized Content Knowledge (SCK), and Horizon Content Knowledge (HCK). CCK is defined as the mathematical knowledge used in contexts other than teaching.

These researchers suggested that CCK includes the ability to: (a) solve problems, (b) use common mathematical concepts, terms, and notations, (c) utilize basic mathematical procedures, algorithms, and formulas, (d) recognize the validity (correctness or incorrectness) of problem solutions. SCK refers to mathematical knowledge and skills that are unique to teaching. Differing from CCK, this type of knowledge extends beyond the procedural understanding required by professionals, such as engineers or accountants. It includes a deeper understanding of mathematics and the ability to use that understanding to communicate with learners. SCK includes: (a) analyzing student responses, (b) justifying the validity of students' answers, (c) evaluating student-generated strategies, (c) interpreting mathematical representations created by students.

PCK consists of three domains: Knowledge of Content and Students (KCS), Knowledge of Content and Teaching (KCT) and Knowledge of Content and Curriculum (KCC) (Ball et al., 2008). KCS is an integration of understanding mathematics and understanding students' mathematical learning pathways. KCS includes: (a) anticipating students' mathematical thinking or cognitive pathways, (b) predicting students' possible answers to a mathematical problem, (c) anticipating students' difficulties, errors, misconceptions, and obstacles when performing a learning task or solving a mathematical problem.

KCT represents the intersection of mathematical content knowledge and knowledge of pedagogical strategies. This knowledge pertains to teachers' instructional decisions regarding a specific topic, such as designing learning activities to develop students' mathematical understanding, utilizing appropriate teaching methods and techniques, and selecting assessment tools and methods compatible with the content. KCT includes: (a) designing effective instructional activities or sequences for the mathematical topics in the curriculum, (b) selecting teaching methods, instructional techniques, and using appropriate tools and technology to help students explore the mathematical knowledge, (c) designing situations that help students understand the nature and meaning of mathematical concepts and knowledge, (d) selecting appropriate representations to illustrate a mathematical concept or knowledge, and (e) designing suitable assessment tools to evaluate students' understanding of the mathematical content.

2.2 Mathematical knowledge for teaching conditional probability

Drawing on the MKT framework (Ball et al., 2008) and the Mathematics Teacher's Specialized Knowledge Model (MTSK) (Carrillo-Yañez et al., 2018), Seguí and Alsina (2024) adapted the MTSK for teaching probability. The MTSK model comprises six domains: (a) Knowledge of probability topics, (b) Knowledge of probabilistic structure, (c) Knowledge of practices in probability, (d) Knowledge of probability teaching, (e) Knowledge of features of learning probability, and (f) Knowledge of probability learning standards. This framework was used to measure the knowledge for teaching probability of 25 Spanish primary school mathematics teachers.

Using a mixed-methods approach, Seguí and Alsina found that the teachers scored an average of 12.95 out of 38 on probability knowledge, with scores below 45/100 across all knowledge subtypes. The researchers maintained that teachers' insufficient mathematical and pedagogical knowledge of probability necessitates continuous professional development. Similarly, Danişman and Tanışlı (2017) conducted a case study on three secondary school teachers. This study found that these teachers' PCK of probability was insufficient and was partially influenced by their professional experience. Furthermore, beliefs were the most critical factor affecting PCK. In summary, in-service teachers demonstrate significant gaps in knowledge for teaching probability.

Conditional probability and independence have been included in K-12 mathematics curriculum in many countries (e.g., MOET, 2018; NCTM, 2000). While there has been extensive research on probabilistic reasoning (Batanero & Álvarez-Arroyo, 2024; Böcherer-Linder et al., 2018; Jones et al., 1999), research related to the knowledge necessary for teaching conditional probability is still in its nascent stages (Groth, 2010). In a case study with three secondary mathematics teachers, Groth pointed out that the teachers encountered challenges when teaching conditional probability and independence.

These included: (a) imprecisely formulated problems, (b) using imprecise language related to favorable outcomes of an event, (c) confusing between independent and non-independent events, (d) failing to distinguish between theoretical and experimental probability, (e) focusing on procedural knowledge, and (f) a lack of conditional probability problems with realistic contexts. Groth argued for the need to design quality statistics and probability courses to develop content knowledge and pedagogical knowledge for prospective teachers.

In the current study, we adopted the MKT (Ball et al., 2008), the MTSK (Seguí & Alsina, 2024) frameworks, and the aspects of knowledge necessary for teaching conditional probability and independence (Groth, 2010). We specified four domains of knowledge for teaching conditional probability including: (a) Common Conditional Probability Knowledge, (b) Specialized Conditional Probability Knowledge, (c) Knowledge of Conditional Probability and Student, and (d) Knowledge of Conditional Probability and Teaching (Table 1).

Table 1: Knowledge for teaching conditional probability (Adapting from MKT framework of Ball et al. (2008))

- Common Conditional Probability Knowledge (CCPK)
  - Solving problems related to conditional probability: calculating the probability of compound events, conditional events, and determining the independence or dependence of events.
  - Using common concepts, terminologies, and mathematical notations related to conditional probability (e.g., event, compound events, conditional event, probability of an event, probability of compound events, conditional probability, two-way tables, tree diagrams, and Venn diagrams).
  - Calculating conditional probabilities and applying them to solve real-world problems related to conditional probability in statistical contexts.
  - Distinguishing between the probability of an event, compound events, and conditional probability.
  - Recognizing the validity of an answer to a question related to conditional probability in a mathematical or real-world context.
- Specialized Conditional Probability Knowledge (SCPK)
  - Validating the correctness of students' answers related to conditional probability, compound probability, and independence or dependence of events.
  - Using and linking multiple representations to help learners understand the meaning of conditional probability.
  - Grasping the meaning of conditional probability in diverse real-world contexts to help learners make decisions when solving problems.
- Knowledge of Conditional Probability and Student (KCPS)
  - Anticipating students' thinking pathways when approaching problems related to conditional probability.
  - Anticipating students' answers when they solve mathematical problems related to conditional probability.
  - Anticipating students' difficulties, errors, and misconceptions when solving a problem related to conditional probability.
- Knowledge of Conditional Probability and Teaching (KCPT)
  - Selecting appropriate teaching methods and technological tools for teaching conditional probability.
  - Selecting scenarios to help students distinguish between conditional probability, probability of compound events and of independent events.
  - Using appropriate representations to illustrate the conditional probability formula, the law of total probability, and Bayes' theorem.
  - Designing activities to develop students' understanding of concepts and properties related to conditional probability.
  - Designing instructional scenarios aimed at addressing student misconceptions and biases pertaining to conditional probability.

3. Methodology

3.1 Setting and participants

An explanatory case study involving three high school teachers was conducted. The participant group comprised a convenience sample of two early-career teachers and one experienced from three high schools in Hue city, Vietnam, who agreed to participate in the study. The researchers had established a strong rapport with the teachers over several years. This history of building rapports with the teachers was instrumental in alleviating any suspicion or apprehension the participants may have held towards the research team.

The researchers adopted the role of a participant-observer within the team and introduced the lesson study model and the competency-based teaching approach, stipulated in the Vietnamese education reform program, to the teachers. The participants collaborated to create lesson plans focused on conditional probability and subsequently implemented these lessons. During all team meetings and classroom observations, the researchers reflected on research goals as a participant and collected data for research purposes as an observer.

One of the three teachers in the research group, Tan, held a master's degree in education. A second teacher, Quy, was currently pursuing a master's degree in education, and the final member, Khang, was a novice teacher. Regarding professional experience, Tan possessed twelve years of experience teaching mathematics at the high school level. Other two teachers have taught for one to four years at the high school level. All teachers were participating in lesson study activities for the first time.

3.2 Lesson study and data collection and analysis

3.2.1 Goal setting

The team discussed the research goals for 120 minutes. The researchers initially focused on the recent reform of the Vietnamese high school mathematics curriculum. Subsequently, the research group concentrated on new features within the probability, especially the new topic, conditional probability, and the pedagogical intentions of the curriculum developers. The lesson study team then analyzed the design of this topic across the three popular high school textbooks in Vietnam. They then discussed the applications of the problem-solving approach and using mathematical representations in teaching conditional probability. This discussion was aimed at developing students' conceptual understanding and probabilistic reasoning while simultaneously addressing their misconceptions and common biases.

3.2.2 Lesson planning

After agreeing on using the Vietnamese Department of Education pacing guide, the team spent about an hour discussing each lesson plan before assigning the lessons to Tan, who taught all the research lessons. During meetings for lesson planning, the group primarily focused on strategies to help learners construct the concept of conditional probability and its related formulas through a real-world problem-solving approach. Furthermore, the team also deliberated on how to utilize visual representations to guide students in deriving related probability formulas, such as the conditional probability formula, the law of total probability, and Bayes' theorem.

3.2.3 Classroom observations and reflection

During the lesson study cycle, a total of 12 sessions (45 min each) were observed and videotaped. Each of the three lessons was implemented twice, first in class 12A1 and second in class 12A3. These observations specifically focused on the teachers' task implementation and students' knowledge construction and problem-solving processes. The field notes were subsequently used for post-lesson discussions (30 min each). The discussions concentrated on lesson adaptation, lessons learned during lesson study, CCK, SCK, KCS, KCT, and reflections on the research goals. A total of 12 observation sessions and 12 post-classroom discussions took place over a two-week period during the second semester of the 2024–2025 school year.

Discussions, classroom observations, lesson plans, and teachers' reports collectively formed the data corpus (Table 2).

Table 2: Summary of data collection and analysis

Data Collection Source (N)	Data Analysis Focus (Code/Constructs)
Lesson Plans (N = 3)	Learning intentions and tasks (KCC) Provision of multiple solutions for a problem (CCPK) Anticipation of students' solutions, difficulties and misconceptions (KCPS) Example and task selection and sequencing, real-life tasks selection, visual representations, evaluation of teaching approaches (KCPT)
Classroom Observations, Field Notes, and Videotapes (N = 12)	Learning intentions and tasks (KCC) Responding to students (SCPK) Addressing/Remediating students' difficulties and misconceptions (KCPS) Example or task selection and sequencing, conducting real-life tasks, using visual representations (KCPT)
Group Meetings: Planning (N = 3), and Post-Lesson Discussions (N = 12)	CCPK, SCPK, KCPS, KCPT
Individual Reports during Lesson Study (N = 3)	CCPK, SCPK, KCPS, KCPT

3.2.4 Data analysis

We utilized the coding framework detailed in Table 2 to examine the changes in teachers' knowledge for teaching conditional probability. Specifically, for CCPK analysis, we focused on instances where teachers distinguished relevant terminologies and formulas related to conditional probability, as well as their generation of multiple solution strategies. For SCPK, we emphasized moments when the teachers responded to students in the classroom. We examined how they attended to students' understanding of conditional probability, evaluated the validity of student-generated problem-solving strategies, made sense of students' solutions and representations, and how they linked students' understanding of conditional probability to the learning intentions. For KCPS, we concentrated on how teachers incorporated student learning concerning conditional probability, including difficulties, misconceptions, and biases that were predicted during planning and those noticed during lesson implementation. For KCPT, we documented teachers' decisions regarding task selection and sequencing, as well as their evaluations of the advantages and disadvantages of the visual representations throughout the lesson planning, teaching, and discussion phases. We then compared these types of knowledge as they often exhibited before and during lesson study and triangulated them with what teachers discussed about lessons learned during this process.

4. Results

4.1 Changes in teachers' common and specialized conditional probability knowledge through lesson study

4.1.1 Enhancing conceptual and procedural knowledge

The teachers reported having gained a deeper understanding of the conditional probability concept and related formulas, as well as using multiple problem-solving strategies during lesson study. Specifically, Tan shared that he "had limited knowledge, mainly understanding and applying basic formulas, with a focus on calculation before participating in lesson study". The addition of conditional probability in the Vietnamese high school curriculum caused Tan to "simultaneously learn and teach this topic, often feel vague and uncertain at times".

Through collaborative discussions with colleagues, Tan realized that conditional probability was "not just a formula". He stated that, "investigating problems and solving numerous real-world problems helped me gain a more solid grasp of the knowledge''. Tan gained a clearer understanding of the relationships among conditional probability, the multiplication rule of probability, and Bayes' theorem. This understanding supported his recognition of its important application for calculating the inverse probability $P(B|A)$ when $P(A|B)$ is known. He reflected that he understood the condition for two events to be independent through the property $P(A|B) = P(A)$ and learned how to clearly differentiate the concept of independent events from mutually exclusive events.

Like Tan, Quy shared that he "had gained a comprehensive overview of the theoretical concepts and formulas related to conditional probability, including the conditional probability formula, the multiplication rule, the law of total probability, and Bayes' theorem". He further noted that the derivation of the conditional probability formula by restricting the sample space was also a key insight through discussions with colleagues. According to Quy, "this idea of constructing the conditional probability formula helps students grasp the core nature of conditional probability, which is necessary for re-evaluating the probability of an event given that another event has already occurred". Consequently, this approach enables students to distinguish between conditional probability and general probability.

4.1.2 Proficiency in visual representations

The teachers expressed that collaborative discussions assisted them in learning various solution strategies for a problem, often gleaned from colleagues' shared experiences. For instance, the group discussed the famous Monty Hall problem.

> *Monty Hall Problem: Suppose you are a contestant on a game show, and you are asked to choose one of three doors. Behind one door is a car, and behind the other two doors are goats. You are asked to select one of the three doors. The host, who knows exactly where the car is, opens one of the two unchosen doors that he knows has a goat behind it. He then asks you: "Do you want to stick with your original choice, or do you want to switch to the other unopened door?". Is it better to switch your choice?*

Khang shared that:

> *''Although I knew that after the host revealed a door with a goat, the probability of winning the car would be higher if I switched doors because the occurrence of an event narrows the sample space. However, I still could not explain why the probability of winning the car would be higher if I switched. The group discussion helped me realize that I could explain this visually by listing all possible cases or by using a tree diagram.''*

Khang referred to two strategies in his reflection, including listing all cases and using a tree diagram. The first strategy involves listing all cases: two options to switch the door and win the prize and only one option to stay with the original choice and win (Figure 1).

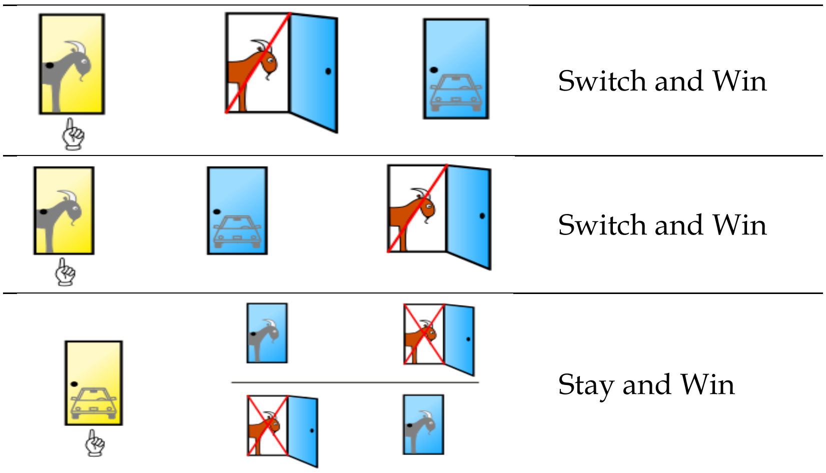


Figure 1: Listing strategy for solving the Monty Hall problem

The second strategy of using a tree diagram to calculate the probability of winning the car. This strategy applies to the scenario where the contestant initially selects a door and then decides to switch doors after the host has revealed a door containing a goat (Figure 2).

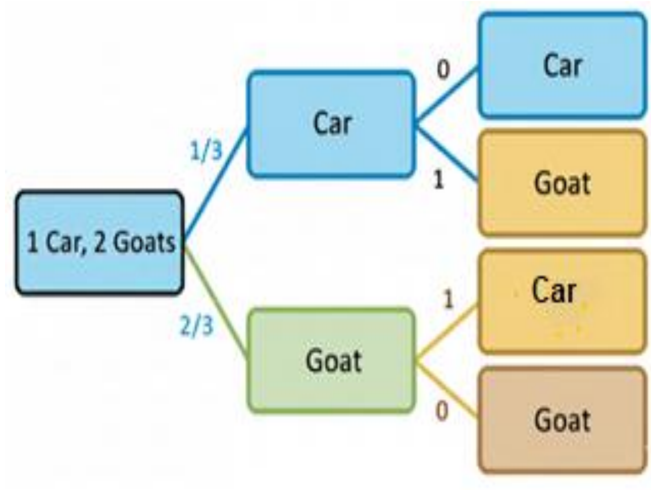


Figure 2: Visual representation strategy using a tree diagram for solving the Monty Hall problem

Furthermore, Tan indicated that he has become proficient in using tree diagrams to solve conditional probability problems. The proficiency was also a shared by Quy as he stated that "I learned how to visually describe problems using tree diagrams, two-way tables, Venn diagrams, and flexibly apply them in problem-solving".

4.1.3 Addressing contextual complexity of a problem and responding to students

The teachers also maintained that their exposure to unfamiliar and realistic problems deepened their awareness of the need for continuous learning to understand the complexity and solutions of real-world problems. Khang shared that he initially misinterpreted the requirement of the Accident problem introduced by Tan in the discussion.

Accident problem: A survey found that in one locality, 2% of drivers use a mobile phone while driving. Among the accidents in that locality, 10% of which involved drivers using a mobile phone while driving. Determine the relative increase in the probability of causing an accident when a driver uses a mobile phone.

Initially, Khang interpreted the problem as calculating P(Accident|Mobile)/P(Accident|No Mobile), whereas the problem asked students to calculate P(Accident|Mobile)/P(Accident). This experience made Khang realize the importance of distinguishing between different types of events, such as the probability of a simple event, conditional events, and compound events.

The teachers noted that they had often ignored the concept of a random experiment when discussing conditional probability in real-world problems. The teachers expressed that they were perplexed by one student's question, Hoa, "What is the random experiment in this situation?" regarding the Concert problem observed during the total probability formula lesson.

Concert problem: The number of spectators attending an outdoor concert is contingent upon weather conditions. Assuming with favorable conditions (no rain), the conditional probability of selling out all tickets is 0.9; the probability of selling out decreases to 0.4 should it rain. The weather forecast predicts the probability of rain during the performance to be 0.75. Determine the overall probability of selling out all tickets.

Immediately, Tan responded to the question by redirecting it to the entire class to elicit ideas while attempting to mobilize his prior knowledge to provide an appropriate answer. However, Tan still provided an incorrect answer about the random experiment. Tan said that "the outcome of this outdoor performance is contingent upon the weather (specifically, the occurrence of rain)". Consequently, in their post-lesson reflection, the teachers noted that they must improve their understanding of experiments and randomness when addressing conditional probability in real-world problems.

4.2 Changes in teachers' knowledge of students and teaching conditional probability through lesson study

4.2.1 Enhancing capacity for anticipating and monitoring student thinking

The ability to access multiple approaches to a problem through collaborative discussions helped the teachers enhance their capacity to anticipate potential student responses. Tan shared that "students could employ various approaches to solving a conditional probability problem, such as using a formula or utilizing tables, tree diagrams, or Venn diagrams". Therefore, instead of only presenting one approach (i.e., using formulas), as he had done previously, Tan was able to predict different answers students might offer while planning.

Furthermore, the teachers' experience in monitoring students' thinking during class observations to capture their mathematical ideas was also accumulated through the lesson observation phase. Khang reflected that he was "particularly impressed by the thinking and problem-solving strategy of the student, Hoa, noting that his approach was unconventional, but highly creative and original". Consequently, Khang argued that teachers need to cultivate their ability to respond immediately to students' problem-solving approaches, to accurately evaluate the correctness, validity, and optimality of those strategies.

4.2.2 Addressing and supporting students overcome their misconceptions

Observing the lessons helped the teachers identify common students' misconceptions, such as the confusion between $P(B|A)$ and $P(B)$, $P(B|A)$ and $P(A|B)$, $P(AB)$ and $P(A|B)$ or the time-order fallacy. Therefore, Khang proposed that students should be allowed to draw their own tree diagrams to represent the situation, and the teacher should subsequently correct the errors made on those diagrams. Specifically, Khang emphasized that it is crucial to explicitly label $P(A)$, $P(B|A)$, and $P(B|\bar{A})$ on the branches of weighted tree diagrams (Figure 3). Khang furthered that "the teacher should encourage students to write the weight on the branch leading to the subsequent event as follows to help them distinguish the probabilities of different types of events."

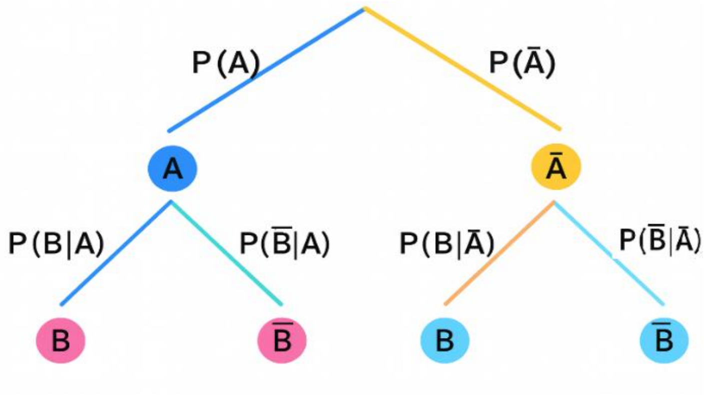
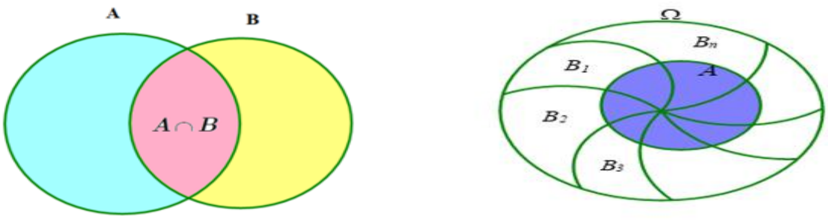


Figure 3: A tree diagram illustrating conditional probability and the multiplication rule

4.2.3. Using visual representations for conceptual and formula establishment

Khang suggested that mathematics teachers should deliberately assign tasks that require using various representations, such as two-way tables and tree diagrams, for problem-solving. Following this, the teacher needed to evaluate the advantages and disadvantages of each method and provided students with guidance on when it was optimal to use a table versus a tree diagram to solve a problem.

For example, regarding the Monty Hall problem, listing-all-case strategies works for all students while using tree-diagram approach is more suitable for more advanced students. Similarly, the use of Venn diagrams to construct the conditional probability formula and the law of total probability was mentioned by both Quy and Tan as a new pedagogical experience for them (Figure 4). This shift occurred because, as shared by Quang, they had previously only encountered these formulas in a ready-established, pre-defined format.



$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad P(A) = \sum_{i=1}^{n} P(B_i).P(A/B_i)$$

Figure 4: Visual representations of formulas related to conditional probability

The adoption of visual representations for the conditional probability formula and the law of total probability led to a shift in the teaching approaches. Previously, teachers only introduced the formula to students directly. Following the lesson study, teachers utilized real-world scenarios and visual representations, such as Venn diagrams, two-way tables, and tree diagrams to elicit and adjust students' initial hypotheses about the probabilities of events. This new approach helped students grasp the conceptual essence of the basic formulas related to conditional probability.

4.2.4 Leveraging real-world contexts for introducing and developing concepts
The teachers learned an effective approach to using relevant real-world situations to initiate a lesson. Specifically, Khang used real data collected in the classroom to construct scenarios that lead students toward new concepts. For instance, Khang cited the situation where the teacher surveyed students' shoe preferences, Adidas vs. Nike (data is presented in Table 3).

Table 3: Student preference for sneaker brands

	Boys	Girls	Total
Addidas	4	4	8
Nike	6	7	13
Total	10	11	21

Using the collected data, the teacher employed a two-way table to display the data and posed questions to guide students in constructing the concept of conditional probability:

> *Tan: What is the probability of selecting a male student?*
> *Student: 10⁄21*
> *Tan: What is the probability of selecting a student who likes Nike shoes, given that the student is male?*
> *Student: 6/10.*

Following the dialogue, the teacher introduced the concept of conditional probability. Although Khang evaluated this approach as effective, he suggested that Tan should have posed more probing questions related to the shoe preference context to allow students to grasp the essence of conditional probability and formulate the definition and formula directly from this situation. Khang argued that this approach would be superior to immediate introduction of another scenario (e.g., drawing marbles) to construct the conditional probability concept and its formula. This reflection demonstrates a key shift in his pedagogical strategies, specifically his ability to leverage real, empirical data present in the classroom to develop instructional ideas. This strategy, in his view, "not only enhances student focus and engagement but also contributes to efficient instructional time".

Quy also shared that using real-world data in the classroom to introduce a new lesson helped him gain a better understanding of inquiry-based and problem-solving methods. Specifically, leveraging the Monty Hall problem to initiate the lesson stimulated students' curiosity about the probability of winning the prize when the teacher had previously revealed a door. He further noted that utilizing real-world situations from classroom data helped him gain experience in designing a sequence of diverse contextual activities or situations, such as those related to preferences, medicine, weather, when teaching conditional probability.

5. Discussion

This study examined how high school teachers' knowledge changed over the course of lesson study, as they engaged in designing and implementing a new topic in the Vietnam's reform curriculum, conditional probability. The analysis revealed a transformation in teachers' conditional probability knowledge after participating in lesson study activities. These changes reflected the positive influence of the lesson study model on teachers' knowledge for teaching conditional probability. The collaborative working environment, focusing on lesson plan design, teaching, classroom observation, and reflection, provided opportunities for the teachers to develop their knowledge for teaching conditional probability. During lesson study activities, the teachers had opportunities to discuss and share problem-solving ideas, utilizing visual representations, such as Venn diagrams, tree diagrams, and two-way tables to support students to explore conditional probability concepts and real-world problem-solving. The effective use of visual representations to help learners grasp the essence of conditional probability (Böcherer-Linder et al., 2018) and develop probabilistic reasoning (Jones et al., 1999) have been advocated in the literature.

Changes in both CCPK and SCPK manifested across several key aspects. The teachers developed a deeper understanding of the conditional probability concept and related formulas as well as using multiple problem-solving strategies. Additionally, they developed a greater awareness of the concept of a random experiment and independence when addressing conditional probability as opposed to misconceptions related probability (Batanero & Sanchez, 2005). This is essential for teachers when teaching a topic of knowledge for the first time. This result addresses the suggestion by Seguí and Alsina (2024) regarding the continuous development of CCPK and SCPK for teachers.

Furthermore, teachers improved their ability to translate problems from real-world language into mathematical language and their capacity to address contextual complexity of a problem and respond to students. These are essential knowledge elements for teaching, particularly for teachers who were not fully prepared during their teacher training, as reflected by Batanero (2022).

The findings also indicated that providing mathematics teachers with a collaborative working environment through lesson study not only helped them deepen their SMK but also their PCK, including both KCPS and KCPT. These changes in KCPS manifested across several key aspects. One area of progress was the teachers' greater attention to anticipating students' solutions and tracking students' thinking during the lesson (Batanero, 2022). Secondly, the teachers changed their KCPS in how they anticipated and addressed students' misconceptions and supported them to overcome their misconceptions (cf. Nguyen & Tran, 2023).

Thirdly, the teachers developed their KCPT in how they used visual representations to help students construct concepts and formulas related to conditional probability (Batanero & Álvarez-Arroyo, 2024; Böcherer-Linder et al., 2018). Finally, teachers gained new teaching experience by using games and tasks in real-world contexts and leveraging the contexts for introducing the conditional probability concept. The effective enhancement of teachers' understanding of gamification and their use of authentic instructional materials in probability lessons are also key factors in helping learners develop both a conceptual understanding of probability and its real-life application as suggested by previous studies (e.g., Batanero & Álvarez-Arroyo, 2024).

6. Limitation

This study used a convenience sample to investigate how the lesson study impacts on teacher knowledge when teaching a new topic introduced in the curriculum. This sampling provided limiting generation of knowledge. However, through rich data collection, it offers empirical evidence and a model for measuring knowledge. Future research could apply this model in different teacher populations, especially diverse teaching experience to examine how and why it works. Additionally, the dual role of researchers as participant-observers might have created potential bias in the study. However, we have carefully collected multiple sources of data for triangulation and built rapports with the teachers so that they could authentically express their knowledge.

7. Conclusion

This study enriches the lesson study literature by addressing teacher knowledge changes in a highly abstract mathematical topic when adopting lesson study outside Japan. The current study contributes to the literature with empirical evidence of the effectiveness of lesson study in teacher knowledge related to a new topic, conditional probability by specific observations of teacher knowledge related to the topics. Additionally, it confirms the positive impact of lesson study has found in the literature (e.g., Nguyen & Tran, 2023). Unique in this study is the combination of teacher experience, instead of only experienced teachers like previous studies (e.g., Nguyen & Tran, 2023).

Interestingly, we found moments when the more experienced teachers struggled with the new topic more than their novice colleagues. We argue that teachers with different experiences might need different learning experiences when teaching new topics and future study could explore this issue. Additionally, this study initially shows the progress in teachers' knowledge following participation in lesson study. This finding suggests a direction for professional development to effectively teach conditional probability in particular, as well as other new mathematics topics introduced into the curriculum, for mathematics teachers through collaborative learning community models such as lesson study.

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