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CASTELNUOVO-MUMFORD REGULARITY OF ELIMINATION IDEALS ASSOCIATED WITH GRAPHS

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ABSTRACT. Let G be a simple connected graph. Anwar and Khalid defined the elimination ideal associated with G, denoted by $I_D(G)$. In this paper, we provide combinatorial bounds for the Castelnuovo-Mumford regularity of these ideals associated with arbitrary simple connected graphs in terms of their degree sequences.

1. Introduction

Let I be a homogeneous ideal in the polynomial ring $S = \mathbf{k}[x_1,\ldots,x_n]$ over an infinite field $\mathbf{k}$. Suppose that the minimal free resolution of I is given by

$$0 \longrightarrow \bigoplus_j S(-j)^{\beta_{l,j}(I)} \longrightarrow \bigoplus_j S(-j)^{\beta_{l-1,j}(I)} \longrightarrow \cdots \longrightarrow \bigoplus_j S(-j)^{\beta_{0,j}(I)} \longrightarrow I \longrightarrow 0$$

where $l \le n$ and $S(-j)$ is the free S-module obtained by shifting the degrees of S by j. The number $\beta_{i,j}(I)$ is called (i,j)*-th graded Betti number of* I, equals the number of minimal generators of degree j in the i-th syzygy module of I. The *Castelnuovo-Mumford regularity of* I, denoted by $\mathrm{reg}(I)$, is defined as the following

$$\mathrm{reg}(I) = \max\{j - i \mid \beta_{i,j}(I) \neq 0\}.$$

A monomial ideal $I \subseteq S$ is called a *Borel type ideal* if $(I : x_t^\infty) = (I : (x_1,\ldots,x_t)^\infty)$ for all $t = 1,\ldots,n$ (see [6]). Herzog et al. [6] showed that I is of Borel type if and only if for any monomial $u \in I$ and for any $1 \le j < i \le n$ and $q > 0$ with $x_i^q | u$, there exists an integer $t > 0$ such that $x_j^t \cdot \dfrac{u}{x_i^q} \in I$. Caviglia and Sbarra [3] also referred to the Borel type ideal as a *weakly stable ideal*.

Let $G = (V,E)$ be a simple connected graph, where $V = \{v_1,\ldots,v_n\}$ is the vertex set and E is the edge set. To study the algebraic properties of G, one can associate certain monomial ideals to it; one such ideal is the edge ideal $I(G)$. This paper focuses on the study of the *elimination ideals* $I_D(G)$ associated with G, which were first introduced by Anwar and Khalid [2]. It is well known that elimination ideals are monomial ideals of Borel type and are not square-free. Anwar and Khalid [2] also introduced a combinatorial invariant associated with G, called the *graphical degree stability* of G, denoted by $\mathrm{Stab}_d(G)$. They proposed a method to compute the graphical degree stability, called the *dominating vertex elimination method* (DVE method). Using the DVE method, Anwar and Khalid computed the graphical degree stability for several families of graphs, including complete graphs, star graphs, path graphs, cycle graphs, fan graphs, wheel graphs, and complete bipartite graphs. As

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a consequence, they also provided upper bounds for the Castelnuovo-Mumford regularity of the elimination ideals associated with these families of graphs.

Motivated by the work of Anwar and Khalid [2], we pose the following question.

Question 1.1. *Let $G=(V,E)$ be a finite simple connected graph with the degree sequence $(d_1,\dots,d_n)$, where $n=|V|$ and let $I_D(G)$ denote the elimination ideal associated with the graph G.*

(i) *Can one establish an exact expression for the graphical degree stability of G in terms of n?*
(ii) *Can one obtain a bound for the Castelnuovo-Mumford regularity of $I_D(G)$ in terms of $d_1,\dots,d_n$?*

In general, an exact formula for the graphical degree stability of a graph is not available. However, we present the following result concerning the second part of Question 1.1 in the next theorem.

Theorem 3.7 *Let $G=(V,E)$ be a simple connected graph with the degree sequence $(d_1,d_2,\dots,d_n)$, where $n=|V|$ and $I_D(G)$ be the elimination ideal associated with the graph G. Then*

$$\operatorname{reg}(I_D(G)) \le d_1+d_2+\dots+d_n-n+1.$$

In particular, we compute the graphical degree stability of full m-ary trees with all leaves of level h as well as of circulant graphs. By applying Theorem 3.7, we obtain combinatorial bounds for the regularity of the elimination ideals associated with these graphs. This leads to the following theorems.

Theorem 4.2 *Let T_h be a full m-ary tree with all leaves of level h, where $m\ge 2$ and $h\ge 1$. Then*

(i) $\operatorname{Stab}_d(T_h)=\begin{cases}0 & \text{if } h=1,2;\\ m-1 & \text{if } h=3.\end{cases}$

(ii) *If $h=3k+1$ for $k=1,2,\dots$, then*

$$\operatorname{Stab}_d(T_h)=m^{3k-1}+m^{3k-4}+\dots+m^2.$$

If $h=3k+2$ for $k=1,2,\dots$, then

$$\operatorname{Stab}_d(T_h)=m^{3k}+m^{3k-3}+\dots+m^3.$$

If $h=3k$ for $k=2,3,\dots$, then

$$\operatorname{Stab}_d(T_h)=m^{3k-2}+m^{3k-5}+\dots+m^4+(m-1).$$

(iii) $\operatorname{reg}(I_D(T_h))\le m^h+m^{h-1}+\dots+m^2+m.$

Theorem 4.5 *Let $n\ge 4$ be an integer and $C_n\left(1,\dots,\widehat{k},\dots,\left\lfloor\frac{n}{2}\right\rfloor\right)$ be a circulant graph for any $1\le k\le \left\lfloor\frac{n}{2}\right\rfloor$. Then*

(i) $\operatorname{Stab}_d\left(C_n\left(1,\dots,\widehat{k},\dots,\left\lfloor\frac{n}{2}\right\rfloor\right)\right)=\begin{cases}n-3 & \text{if } n \text{ is even and } k=\left\lfloor\frac{n}{2}\right\rfloor;\\ n-4 & \text{otherwise.}\end{cases}$

(ii) $\operatorname{reg}\left(I_D\left(C_n\left(1,\dots,\widehat{k},\dots,\left\lfloor\frac{n}{2}\right\rfloor\right)\right)\right)\le\begin{cases}n^2-3n+1 & \text{if } n \text{ is even and } k=\left\lfloor\frac{n}{2}\right\rfloor;\\ n^2-4n+1 & \text{otherwise.}\end{cases}$

On the other hand, we recover several known results on the regularity of certain familiar classes of graphs as presented by Anwar and Khalid [2].

The paper is organized into three sections. Section 2 introduces the notation and terminology used throughout the paper. In Section 3, we establish combinatorial bounds for the regularity of elimination ideals associated with a graph in terms of its degree sequence. Section 4 is devoted to computing the graphical degree stability for certain families of graphs and deriving corresponding bounds for the regularity of their elimination ideals.

2. Preliminary

Let $G=(V,E)$ be a simple connected graph with vertex set $V=\{v_1,\dots,v_n\}$. The *degree* of a vertex v_i, denoted by $\deg(v_i)$, is defined as the number of edges incident to v_i. If $\deg(v_i)=0$, then v_i is called an *isolated vertex*.

A vertex v_i is said to be a *dominating vertex* of G if $\deg(v_i)\geq \deg(v_j)$ for all $j\neq i$. The set of all dominating vertices of G is called the *dominating set* of G, and is denoted by $D(G)$.

A graph G is said to be *scattered* if it contains at least one isolated vertex.

Throughout this paper, if $G=(V,E)$ with $V=\{v_1,\dots,v_n\}$ and $d_i=\deg(v_i)$ for $i=1,\dots,n$, we may assume that the degrees are ordered such that $d_1\geq d_2\geq\cdots\geq d_n$.

We are interested in the following question, originally posed in [2, Question 2.1].

Question 2.1. *What is the maximum number of dominating vertices that can be removed such that the resulting graph remains non-scattered?*

Anwar and Khalid [2] introduced the *dominating vertex elimination method* (DVE method) as follows:

Let $G=G_0$ be a simple connected graph, and let $D(G_0)$ denote a dominating set of G_0. Choose a vertex $v_{i_0}\in D(G_0)$ such that the subgraph $G_1=G_0\setminus\{v_{i_0}\}$ is not scattered. Next, choose a vertex $v_{i_1}\in D(G_1)$ such that the subgraph $G_2=G_1\setminus\{v_{i_1}\}$ is non-scattered. Repeating this process, we obtain a chain of subgraphs of G:

$$G=G_0\supset G_1\supset\cdots\supset G_r.$$

Each graph G_i, for $0\leq i\leq r$, has $n-i$ vertices. The maximum integer r such that all G_i are non-scattered for $0\leq i\leq r$, is called the *graphical degree stability* of G, and is denoted by $\mathrm{Stab}_d(G)$.

Let $S=\mathbf{k}[x_1,\dots,x_n]$ be a polynomial ring over an infinite field $\mathbf{k}$. Let G_i be a graph with vertex set $V(G_i)=\{v_{i_1},\dots,v_{i_k}\}$ and the degree sequence $(d_{i_1},d_{i_2},\dots,d_{i_k})$, where

$$d_{i_1}\geq d_{i_2}\geq\cdots\geq d_{i_k}.$$

Then the ideal

$$Q_{G_i}=(x_1^{d_{i_1}},\dots,x_k^{d_{i_k}})$$

is called the *sequential ideal* of G_i.

Suppose that

$$G=G_0\supset G_1\supset\cdots\supset G_r$$

is a chain of subgraphs of G, where $r=\mathrm{Stab}_d(G)$. Assume that $(d_{i_1},\dots,d_{i_{n-i}})$ is the degree sequence of G_i for $0\leq i\leq r$. Then the sequential ideal of G_i is given by

$$Q_{G_i}=(x_1^{d_{i_1}},\dots,x_{n-i}^{d_{i_{n-i}}}).$$

The *elimination ideal* of G, denoted by $I_D(G)$, is defined as

$$I_D(G)=\bigcap_{i=0}^{r}Q_{G_i}.$$

Let I be a monomial ideal of S. For any monomial $u\in S$, define

$$m(u)=\max\{j:x_j\mid u\}\quad\text{and}\quad m(I)=\max\{m(u):u\in\mathscr{G}(I)\},$$

where $\mathscr{G}(I)$ denotes the minimal set of monomial generators of I.

A monomial ideal I is called *stable* if for every monomial $u \in I$ and every $1 \leq j < m(u)$, it holds that

$$x_j \cdot \frac{u}{x_{m(u)}} \in I.$$

Let $d(I)$ denote the maximum degree of a monomial in $\mathscr{G}(I)$, and let $I_{\geq t}$ be the ideal generated by all monomials in I of degree at least t.

3. Regularity of elimination ideals associated with graphs

In this section, we establish combinatorial bounds for the Castelnuovo-Mumford regularity of elimination ideals associated with simple connected graphs. These results generalize the work of Anwar and Khalid [2] to arbitrary simple connected graphs.

Let $G = (V,E)$ be a simple connected graph, and let $r = \mathrm{Stab}_d(G)$. Assume that

$$G = G_0 \supset G_1 \supset \cdots \supset G_r$$

is the chain of subgraphs of G. Suppose that each G_i has vertex set $V_i = \{v_{i_1}, \ldots, v_{i_{n-i}}\}$, with the degree sequence $(d_{i_1}, \ldots, d_{i_{n-i}})$. Define the sequential ideal of G_i as

$$Q_{G_i} = (x_1^{d_{i_1}}, \ldots, x_{n-i}^{d_{i_{n-i}}}),$$

for each $i = 0, \ldots, r$. Moreover, set

$$\gamma(G_i) = d_{i_1} + \cdots + d_{i_{n-i}} - (n-i) + 1.$$

The following lemma shows that $\gamma(G_i) > \gamma(G_{i+1})$ for all $i = 0, \ldots, r-1$.

Lemma 3.1. *Let $G_0 \supset G_1 \supset \cdots \supset G_r$ be the chain of subgraphs of a simple connected graph G, constructed by the DVE method, and let $\gamma(G_i)$ be as defined above. Then, for all $i = 0, \ldots, r-1$, we have*

$$\gamma(G_i) > \gamma(G_{i+1}).$$

Moreover,

$$\gamma(G_0) = \max\{\gamma(G_i) \mid i = 0, \ldots, r\}.$$

Proof. For each $i = 0, \ldots, r$, suppose that G_i has the degree sequence

$$(d_{i_1}, d_{i_2}, \ldots, d_{i_{n-i}}) \quad \text{with } d_{i_1} \geq d_{i_2} \geq \cdots \geq d_{i_{n-i}},$$

and that G_{i+1} has the degree sequence

$$(d_{(i+1)_1}, d_{(i+1)_2}, \ldots, d_{(i+1)_{n-i-1}}) \quad \text{with } d_{(i+1)_1} \geq d_{(i+1)_2} \geq \cdots \geq d_{(i+1)_{n-i-1}}.$$

Without loss of generality, by applying the DVE method, we may assume that v_{i_1} is the vertex removed from the graph G_i to obtain the graph G_{i+1}. In this case, we have $d_{i_1} \geq 2$. Let the neighborhood of v_{i_1} be defined as

$$N_{G_i}(v_{i_1}) = \{v \in V(G_i) \mid v_{i_1}v \in E(G_i)\}.$$

For $v_k \in N_{G_i}(v_{i_1})$, denote by $d_{G_i}(v_k)$ the degree of v_k in G_i. We have $d_{G_i}(v_k) = d_{i_{j_k}}$ for some $j_k \in \{1,\dots,n-i\}$ and $k \in \{1,\dots,|N_{G_i}(v_{i_1})|\}$. In G_{i+1}, each v_k remains a vertex, and its degree in G_{i+1} is

$$d_{G_{i+1}}(v_k) = d_{i_{j_k}} - 1, \quad \text{for all } k = 1,\dots,|N_{G_i}(v_{i_1})|,$$

while the degrees of other vertices remain unchanged except for the removal of v_{i_1}. By rearranging the degree sequence, we have

$$\begin{aligned}
\gamma(G_i) &= d_{i_1} + d_{i_2} + \cdots + d_{i_{n-i}} - (n-i) + 1 \\
&= d_{i_1} + d_{i_2} + \cdots + \underbrace{(d_{i_{j_1}} - 1) + \cdots + (d_{i_{j_{|N_{G_i}(v_{i_1})|}}} - 1)}_{|N_{G_i}(v_{i_1})|\text{-terms}} + \cdots + d_{i_{n-i}} \\
&\quad - (n - |N_{G_i}(v_{i_1})| - i) + 1 \\
&= d_{i_1} + d_{(i+1)_1} + \cdots + d_{(i+1)_{n-(i+1)}} - (n - |N_{G_i}(v_{i_1})| - i) + 1 \\
&= d_{(i+1)_1} + \cdots + d_{(i+1)_{n-(i+1)}} - (n-i-1) + 1 + |N_{G_i}(v_{i_1})| + d_{i_1} - 1 \\
&> \gamma(G_{i+1}),
\end{aligned}$$

where the last inequality holds since $d_{i_1} \geq 2$. The proof is complete. □

In the next lemma, we show that $Q_{G_{i \geq \gamma(G_i)}}$ is stable for all $i = 0,\dots,r$.

Lemma 3.2. *Let $G = (V,E)$ be a simple connected graph and set $r = \mathrm{Stab}_d(G)$. Assume that*

$$G = G_0 \supset G_1 \supset \cdots \supset G_r$$

is the chain of subgraphs of G. Suppose that G_i has the vertex set $V_i = \{v_{i_1},\dots,v_{i_{n-i}}\}$ with the degree sequence $(d_{i_1},\dots,d_{i_{n-i}})$. Let

$$Q_{G_i} = (x_1^{d_{i_1}},\dots,x_{n-i}^{d_{i_{n-i}}})$$

be the sequential ideal of G_i for $i = 0,\dots,r$. For $i = 0,\dots,r$, set

$$\gamma(G_i) = d_{i_1} + \cdots + d_{i_{n-i}} - (n-i) + 1.$$

Then the ideal $Q_{G_{i \geq \gamma(G_i)}}$ is stable for all $i = 0,\dots,r$.

Proof. Take $u \in Q_{G_{i \geq \gamma(G_i)}}$. Then $u = v x_j^{d_{i_j}}$ for some $j = 1,\dots,n-i$ and $v \in (x_1,\dots,x_{n-i})^{\gamma(G_i) - d_{i_j}}$.

If $m(u) > j$, then

$$\frac{x_k u}{x_{m(u)}} = \frac{x_k v}{x_{m(u)}} \cdot x_j^{d_{i_j}} \in Q_{G_{i \geq \gamma(G_i)}}$$

for all $k < m(u)$. Therefore, $Q_{G_{i \geq \gamma(G_i)}}$ is stable.

If $m(u) = j$ then clearly $u \in (x_1,\dots,x_{n-i})^{\gamma(G_i)}$, which is a stable ideal and we have

$$Q_{G_{i \geq \gamma(G_i)}} \subseteq (x_1,\dots,x_{n-i})^{\gamma(G_i)}.$$

We are going to show that

$$(x_1,\dots,x_{n-i})^{\gamma(G_i)} \subseteq Q_{G_{i \geq \gamma(G_i)}}.$$

Let $w \in (x_1,\dots,x_{n-i})^{\gamma(G_i)}$. Then
$$w = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_{n-i}^{\alpha_{n-i}}$$
with $\alpha_j \geq 0$ for all j, and $\sum_{j=1}^{n-i}\alpha_j \geq \gamma(G_i)$. Assume that $\alpha_j < d_{i_j}$ for all $j = 1,\dots,n-i$. Then
$$\sum_{j=1}^{n-i}\alpha_j \leq d_{i_1} + \cdots + d_{i_{n-i}} - (n-i) = \gamma(G_i) - 1,$$
which is a contradiction. Therefore, $\alpha_j \geq d_{i_j}$ for some $j \in \{1,\dots,n-i\}$. Then we can write $w = x_l^{d_{i_l}}x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_l^{\alpha_l - d_{i_l}}\cdots x_{n-i}^{\alpha_{n-i}}$ for some $l \in \{1,\dots,n-i\}$, it follows that $w \in Q_{G_{i_{\geq\gamma(G_i)}}}$. The proof is complete. $\square$

Remark 3.3. In general, one cannot guarantee $Q_{G_{i_{\geq\gamma(G_i)-1}}}$ is stable. For example, the circulant graph (see Definition 4.4) $C_8(\widehat{1},2,3,4)$. In this case, $Q_{G_4} = (x_1^2, x_2^2, x_3, x_4)$ and $\gamma(G_4) = 3$. Then $Q_{G_{4_{\geq 2}}}$ is not stable.

By [1, Proposition 1.1], the case $k = 2$ holds. The general case follows by a straightforward iteration.

Lemma 3.4. *Let I_i be monomial ideals in S with $m_i \geq d(I_i)$ for all $i = 1,\dots,k$. Assume that $(I_i)_{\geq m_i}$ are stable ideals for all $i = 1,\dots,k$. Then*
$$\left(\bigcap_{i=1}^{k} I_i\right)_{\geq \max\{m_1,\dots,m_k\}}$$
is a stable ideal.

Proof. Set $M = \max\{m_1,\dots,m_k\}$. Let u be a monomial in
$$\left(\bigcap_{i=1}^{k} I_i\right)_{\geq M}.$$
Then $u \in \bigcap_{i=1}^{k} I_i$, that is, $u \in I_i$ for all $i = 1,\dots,k$, and
$$\deg(u) \geq M.$$
Since $M \geq m_i$ for all $i = 1,\dots,k$, it follows that
$$u \in (I_i)_{\geq m_i} \quad \text{for all } i = 1,\dots,k.$$
As each $(I_i)_{\geq m_i}$ is stable, for every $1 \leq j < m(u)$ we obtain
$$\frac{x_j u}{x_{m(u)}} \in (I_i)_{\geq m_i} \quad \text{for all } i = 1,\dots,k.$$
Hence,
$$\frac{x_j u}{x_{m(u)}} \in \bigcap_{i=1}^{k} (I_i)_{\geq m_i},$$
and therefore,
$$\frac{x_j u}{x_{m(u)}} \in \bigcap_{i=1}^{k} I_i.$$

On the other hand,

$$\deg\left(\frac{x_j u}{x_{m(u)}}\right) = \deg(u) \geq M.$$

Thus,

$$\frac{x_j u}{x_{m(u)}} \in \left(\bigcap_{i=1}^{k} I_i\right)_{\geq M},$$

for $1 \leq j < m(u)$. The proof is complete. □

From Lemma 3.1, Lemma 3.2, and Lemma 3.4, we obtain the following result.

Lemma 3.5. *Let $G = (V,E)$ be a simple connected graph and set $r = \mathrm{Stab}_d(G)$. Assume that*

$$G = G_0 \supset G_1 \supset \cdots \supset G_r$$

is the chain of subgraphs of G. Suppose that G_i has the vertex set $V_i = \{v_{i_1}, \ldots, v_{i_{n-i}}\}$ with the degree sequence $(d_{i_1}, \ldots, d_{i_{n-i}})$. Let

$$Q_{G_i} = (x_1^{d_{i_1}}, \ldots, x_{n-i}^{d_{i_{n-i}}})$$

be the sequential ideal of G_i for $i = 0, \ldots, r$. For $i = 0, \ldots, r$, set

$$\gamma(G_i) = d_{i_1} + \cdots + d_{i_{n-i}} - (n-i) + 1.$$

Then

$$\left(\bigcap_{i=0}^{r} Q_{G_i}\right)_{\geq \gamma(G_0)}$$

is a stable ideal. Furthermore,

$$I_D(G)_{\geq \gamma(G_0)}$$

is also a stable ideal.

Proof. By Lemma 3.1, we have $\gamma(G_0) = \max\{\gamma(G_i) \mid i = 0, \ldots, r\}$. It is clear that $\gamma(G_i) \geq d(Q_{G_i})$ and according to Lemma 3.2, the ideal $Q_{G_i \geq \gamma(G_i)}$ is stable for all $i = 0, \ldots, r$. From Lemma 3.4, it follows that

$$\left(\bigcap_{i=0}^{r} Q_{G_i}\right)_{\geq \max\{\gamma(G_i) \mid i=0,\ldots,r\}} = \left(\bigcap_{i=0}^{r} Q_{G_i}\right)_{\geq \gamma(G_0)}$$

is a stable ideal. On the other hand, $I_D(G)_{\geq \gamma(G_0)}$ is a stable ideal. □

We recall the following result from [4, Proposition 1.2].

Lemma 3.6. *Let I be a monomial ideal of S with $d(I) = d$, and let $e \geq d$ be an integer such that $I_{\geq e}$ is stable. Then*

$$\mathrm{reg}(I) \leq e.$$

From Lemma 3.5 and Lemma 3.6, we get the main result of this section in the following theorem.

Theorem 3.7. *Let $G = (V,E)$ be a simple connected graph with the degree sequence $(d_1, d_2, \dots, d_n)$, where $n = |V|$ and let $I_D(G)$ be the elimination ideal associated with the graph G. Then*

$$\operatorname{reg}(I_D(G)) \leq d_1 + d_2 + \cdots + d_n - n + 1.$$

Proof. Since $(d_1, d_2, \dots, d_n)$ is the degree sequence of G, we have

$$\gamma(G_0) = d_1 + d_2 + \cdots + d_n - n + 1.$$

By Lemma 3.5, we know that $I_D(G)_{\geq \gamma(G_0)}$ is a stable ideal. According to Lemma 3.6, it follows that

$$\operatorname{reg}(I_D(G)) \leq \gamma(G_0) = d_1 + d_2 + \cdots + d_n - n + 1.$$

□

Remark 3.8. In the case where $I_D(G) = \bigcap_{i=0}^{r} Q_{G_i}$, we have

$$m(I_D(G)) \geq \max\{m(Q_{G_i}) \mid i = 0, \dots, r\} = n.$$

It follows that $m(I_D(G)) = n$. On the other hand,

$$d(I_D(G)) \geq \max\{d(Q_{G_i}) \mid i = 0, \dots, r\} = d_1.$$

Since $I_D(G)$ is a Borel type ideal, by [1, Corollary 2.4] we have

$$\operatorname{reg}(I_D(G)) \leq n\big(d(I_D(G)) - 1\big) + 1.$$

By Theorem 3.7, we obtain

$$\operatorname{reg}(I_D(G)) \leq d_1 + d_2 + \cdots + d_n - n + 1.$$

Hence, in this case, the bound given in Theorem 3.7 improves the general bound for Borel type ideals. It is also worth mentioning that the upper bound obtained above are combinatorial. On the other hand, our results extend the work of Anwar and Khalid [2] to arbitrary simple connected graphs.

Suppose that $G = (V,E)$ is a simple connected graph with its degree sequence $(d_1, \dots, d_n)$ and $n = |V|$. By the handshaking lemma [7, Proposition 1.3.3], we have

$$d_1 + d_2 + \cdots + d_n = 2|E|.$$

From this and Theorem 3.7, we get the following corollary.

Corollary 3.9. *Let $G = (V,E)$ be a simple connected graph and let $I_D(G)$ be the elimination ideal associated with the graph G. Then*

$$\operatorname{reg}(I_D(G)) \leq 2|E| - |V| + 1.$$

4. Applications to trees and circulant graphs

In this section, we show explicitly how Theorem 3.7 is used in the applications. We consider tree graphs and circulant graphs, and we begin by determining the degree sequence (or, for general trees, the numbers of vertices and edges). We then substitute this data into the bound given by Theorem 3.7 or Corollary 3.9 to obtain concrete estimates for the Castelnuovo-Mumford regularity of the elimination ideal $I_D(G)$. In addition, some known results from Anwar and Khalid [2] are recovered as special cases.

Definition 4.1. A *rooted tree* T is a tree in which one vertex is designated as the root, and every edge is directed away from the root. If u and v are vertices in the rooted tree T, then v is called a *child* of u if there is a directed edge from u to v. A vertex with no children is called a *leaf*, while vertices that have children are called *internal vertices*. The *level* of a vertex v in a rooted tree T is the length of the unique path from the root to v. A rooted tree T is said to be a *full m-ary tree* if every internal vertex has exactly m children.

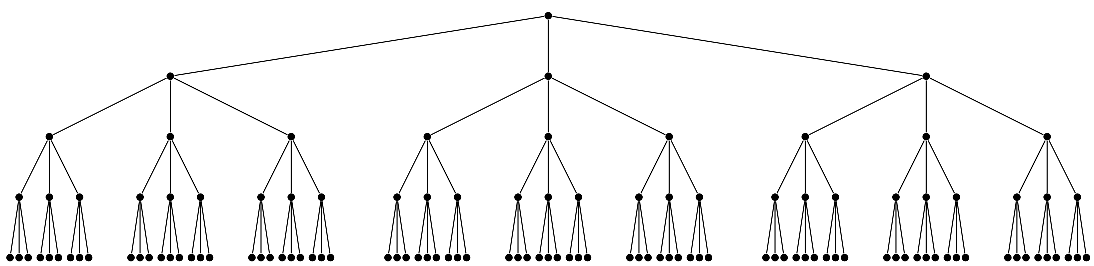

Figure 1: Full 3-ary tree with all leaves of level 4.

The following theorem computes the graphical degree stability of a full m-ary tree with all leaves of level h and provides a bound for the regularity of its elimination ideal.

Theorem 4.2. *Let T_h be a full m-ary tree with all leaves of level h, with $m \geq 2$ and $h \in \mathbb{N}$. Then*

(i) $\mathrm{Stab}_d(T_h) = \begin{cases} 0 & \text{if } h = 1,2; \\ m-1 & \text{if } h = 3. \end{cases}$

(ii) *If $h = 3k+1$ for $k = 1,2,\ldots$, then*

$$\mathrm{Stab}_d(T_h) = m^{3k-1} + m^{3k-4} + \cdots + m^2.$$

If $h = 3k+2$ for $k = 1,2,\ldots$, then

$$\mathrm{Stab}_d(T_h) = m^{3k} + m^{3k-3} + \cdots + m^3.$$

If $h = 3k$ for $k = 2,3,\ldots$, then

$$\mathrm{Stab}_d(T_h) = m^{3k-2} + m^{3k-5} + \cdots + m^4 + (m-1).$$

(iii) $\mathrm{reg}(I_D(T_h)) \leq m^h + m^{h-1} + \cdots + m^2 + m.$

Proof. (i) Let T_h be a full m-ary tree with all leaves of level h.

If $h = 1$, then T_1 is a tree with a root and m leaves. Therefore, the degree of the root is m, and every leaf has degree 1. By the DVE method, we get $\mathrm{Stab}_d(T_1) = 0$.

If $h = 2$, then T_2 has the dominating set consisting of the m vertices of level 1. However, the neighborhood of each dominating vertex contains vertices of degree 1. By the DVE method, we get $\mathrm{Stab}_d(T_2) = 0$.

If $h = 3$, then T_3 has the dominating set consisting of the m vertices of level 1 and the m^2 vertices of level 2. However, the neighborhoods of dominating vertices of level 2 contain vertices of degree 1. Therefore, using the DVE method, we can only remove dominating vertices of level 1. It follows that $\mathrm{Stab}_d(T_3) = m - 1$.

(ii) We will prove the statement by induction on k in the case $h = 3k+1$ for $k = 1, 2, \ldots$ The other cases can be proved similarly.

If $k = 1$, then $h = 4$. We note that T_4 has the dominating set consisting of m vertices of level 1, m^2 vertices of level 2, and m^3 vertices of level 3. However, the vertices of level 3 have neighbors that are leaves, so we can only remove the vertices of level 1 or level 2. Using the DVE method and successively removing the vertices of level 2 from T_4, we obtain $m^3 + 1$ full m-trees with all leaves of level 1. Therefore,

$$\mathrm{Stab}_d(T_4) = m^2 + (m^3 + 1)\mathrm{Stab}_d(T_1).$$

But by (i), $\mathrm{Stab}_d(T_1) = 0$. Thus,

$$\mathrm{Stab}_d(T_4) = m^2.$$

By induction, suppose the statement holds for $k \geq 1$. That is, for all $k \geq 1$ and $h = 3k+1$ we assume that

$$\mathrm{Stab}_d(T_h) = \mathrm{Stab}_d(T_{3k+1}) = m^{3k-1} + m^{3k-4} + \cdots + m^2. \tag{1}$$

We will prove that the statement holds for $k+1$. Note that $T_{3(k+1)+1}$ is a full m-ary tree with all leaves of level $3k+4$ obtained from T_{3k+1} by attaching full m-ary trees T_3 with all leaves of level 3 to the leaves of T_{3k+1}. Suppose we remove the dominating vertices of levels $h-2$, $h-3$, and $h-4$ in the tree T_{3k+4} such that the resulting graphs are non-scattered graphs. Then, the number of dominating vertices at these levels that can be removed is

$$m^{h-2} - (m-1)(a+b),$$

where a is the number of removed vertices of level $h-3$, and b is the number of removed vertices of level $h-4$. As a result, the remaining graphs consist of subtrees with graphical degree stability equal to zero, together with a tree H, which is a subgraph of T_{3k+1} satisfying $D(H) \subseteq D(T_{3k+1})$. Moreover, every removal of a dominating vertex in H corresponds to a removal of a dominating vertex in T_{3k+1}. Therefore,

$$\mathrm{Stab}_d(H) \leq \mathrm{Stab}_d(T_{3k+1}).$$

It follows that the total number of dominating vertices that can be removed from T_{3k+4} , including some of levels $h-2, h-3$ and $h-4$ is

$$m^{h-2} - (m-1)(a+b) + \mathrm{Stab}_d(H).$$

Hence, the maximum number of dominating vertices that can be removed is achieved when $a+b = 0$, i.e., only dominating vertices of level $h-2$ are removed. In this case, we have $H = T_{3k+1}$. Using the DVE method, successively removing the vertices of level $h-2 = 3k+2$ of T_{3k+4}, we obtain a graph consisting of m^{3k+3} full m-ary trees with all leaves of level 1 and a full m-ary tree with all leaves of level $3k+1$. Therefore,

$$\mathrm{Stab}_d(T_{3k+4}) = m^{3k+2} + \mathrm{Stab}_d(T_{3k+1}). \tag{2}$$

From (1) and (2), we obtain

$$\mathrm{Stab}_d(T_{3k+4}) = m^{3k+2} + m^{3k-1} + m^{3k-4} + \cdots + m^2.$$

This completes the proof.

(ii) Let T_h be the full m-ary tree with all leaves of level h. The dominating set of T_h contains m vertices of level 1, m^2 vertices of level 2, $\ldots$, m^{h-1} vertices of level $h-1$. Each dominating vertex of T_h has degree $m+1$. On the other hand, T_h has a root of degree m and m^h leaves of degree 1. Therefore, the degree sequence of T_h is

$$(\underbrace{m+1, m+1, \ldots, m+1}_{(m^{h-1}+m^{h-2}+\cdots+m)\text{-tuple}}, m, \underbrace{1, \ldots, 1}_{m^h\text{-tuple}}).$$

Hence, by Theorem 3.7 we get

$$\begin{aligned}\mathrm{reg}(I_D(T_h)) &\le \underbrace{(m+1)+\cdots+(m+1)}_{(m^{h-1}+m^{h-2}+\cdots+m)\text{-terms}} + m + \underbrace{1+\cdots+1}_{(m^h)\text{-terms}} - (1+m+\cdots+m^h)+1\\ &= m^h + m^{h-1} + \cdots + m^2 + m.\end{aligned}$$

The argument is proved. $\square$

For an arbitrary tree T, we do not know its graphical degree stability and the degree sequence, but we can obtain a combinatorial bound for the regularity of corresponding elimination ideal. This leads to the next corollary.

Corollary 4.3. *Let T be an arbitrary tree with n vertices and $I_D(T)$ be the elimination ideal of T. Then*

$$\mathrm{reg}(I_D(T)) \le n-1.$$

Proof. It is well known that if T is a tree with $|V| = n$ vertices then it has $|E| = n-1$ edges. By Corollary 3.9,

$$\mathrm{reg}(I_D(T)) \le n-1.$$

$\square$

Next, we compute the graphical degree stability of circulant graphs and derive combinatorial bounds for the regularity of their elimination ideals.

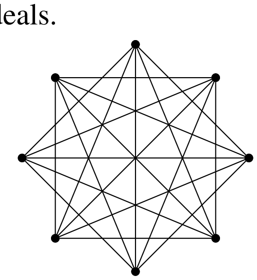

Figure 2: The circulant graph $C_8(2,3,4)$.

Definition 4.4. Let $n \geq 2$ be an integer and $S \subseteq \left\{1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$. The *circulant graph* $C_n(S)$ is the graph on the vertex set $V(C_n(S)) = \{v_1, v_2, \ldots, v_n\}$ such that $v_i v_j \in E(C_n(S))$ if and only if $|i-j| \in S$ or $n - |i-j| \in S$ for all $i, j = 1, \ldots, n$. For convenience, we denote $C_n(\{a_1, \ldots, a_k\})$ by $C_n(a_1, \ldots, a_k)$.

When $S = \left\{1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$, then $C_n(S) \cong K_n$, the *clique* on n vertices. On the other hand, if $S = \{1\}$, then $C_n(1) \cong C_n$, the *cycle* on n vertices. It is well known that the graphical degree stability of K_n, C_n, and the regularity of the elimination ideals associated with the graphs K_n, C_n were studied in [2]. In particular, generalizing the case K_n, we compute $\mathrm{Stab}_d(C_n(S))$ and $\mathrm{reg}(I_D(C_n(S)))$ when

$$S = \left\{1, \ldots, \widehat{k}, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$$

for any $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$, where $\widehat{k}$ means the element k is removed from the set $\{1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\}$.

Theorem 4.5. *Let $n \geq 2$ be an integer and $C_n(1, \ldots, \widehat{k}, \ldots, \left\lfloor \frac{n}{2} \right\rfloor)$ be the circulant graph for any $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$. Then*

(i) $\mathrm{Stab}_d(C_n(1, \ldots, \widehat{k}, \ldots, \left\lfloor \frac{n}{2} \right\rfloor)) = \begin{cases} n-3 & \text{if } n \text{ is even and } k = \left\lfloor \frac{n}{2} \right\rfloor; \\ n-4 & \text{otherwise.} \end{cases}$

(ii) $\mathrm{reg}(I_D(C_n(1, \ldots, \widehat{k}, \ldots, \left\lfloor \frac{n}{2} \right\rfloor))) \leq \begin{cases} n^2-3n+1 & \text{if } n \text{ is even and } k = \left\lfloor \frac{n}{2} \right\rfloor; \\ n^2-4n+1 & \text{otherwise.} \end{cases}$

Proof. (i). Let n is an even integer and $n \geq 4$, and $k = \left\lfloor \frac{n}{2} \right\rfloor$. If $n = 4$ then it is clear that

$$\mathrm{Stab}_d(C_4(1, \widehat{2})) = \mathrm{Stab}_d(C_4)) = 1 = n-3. \tag{3}$$

If $n = 6$ then the circulant graph $C_6(1, 2, \widehat{3})$ has 6 vertices of degree 4 and the dominating vertex $D(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. Without loss of generality, using the DVE method, we remove the vertex v_1 and obtain the graph G_1 with vertex set v_2, v_3, v_4, v_5, v_6 and dominating vertex set $D(G_1) = v_4$. Moreover, the vertices v_2, v_3, v_5, v_6 have degree 3, and the vertex v_4 has degree 4. Every vertex $v_i \in \{v_2, v_3, v_5, v_6\}$ is adjacent to v_4 and two other vertices $v_j \in \{v_2, v_3, v_5, v_6\}$ such that $|i-j| \neq 3$. Therefore, $\{v_2, v_3, v_5, v_6\}$ is the vertex set of the circulant graph $C_4(1, \widehat{2})$. Repeating the DVE method, we remove the vertex v_4 and obtain G_2, which is the circulant graph $C_4(1, \widehat{2})$. It follows that

$$\mathrm{Stab}_d(C_6(1, 2, \widehat{3})) = 2 + \mathrm{Stab}_d(C_4(1, \widehat{2})). \tag{4}$$

From (3) and (4), we have

$$\mathrm{Stab}_d(C_6(1, 2, \widehat{3})) = 3 = n-3.$$

By induction on n, we assume that

$$\mathrm{Stab}_d\left(C_n\left(1, \ldots, \widehat{\left\lfloor \frac{n}{2} \right\rfloor}\right)\right) = n-3, \tag{5}$$

holds for all even integers n. Consider the circulant graph

$$H = C_{n+2}\left(1, \ldots, \widehat{\left\lfloor \frac{n+2}{2} \right\rfloor}\right),$$

which has $n+2$ vertices, each of degree n. Every vertex v_i in H is adjacent to all other vertices except for one vertex v_j such that $|i-j|=k$, where $k=\frac{n+2}{2}$. Without loss of generality, we remove vertex v_1 from H to obtain a graph G_1 on $n+1$ vertices. In G_1, the vertex v_k becomes a dominating vertex of degree n, adjacent to all other vertices. Moreover, for any $v_j \neq v_k$, the only missing edge is to v_{j+k}. Applying the DVE method to G_1, we remove the dominating vertex v_k and the resulting graph G_2 is the circulant graph

$$G_2 = C_n\left(1,2,\dots,\widehat{\left\lfloor \frac{n}{2} \right\rfloor}\right).$$

Hence,

$$\operatorname{Stab}_d\left(C_{n+2}\left(1,2,\dots,\widehat{\left\lfloor \frac{n+2}{2} \right\rfloor}\right)\right) = 2+\operatorname{Stab}_d\left(C_n\left(1,2,\dots,\widehat{\left\lfloor \frac{n}{2} \right\rfloor}\right)\right). \tag{6}$$

From (5) and (6), it follows that

$$\operatorname{Stab}_d\left(C_{n+2}\left(1,2,\dots,\widehat{\left\lfloor \frac{n+2}{2} \right\rfloor}\right)\right) = n-1.$$

(ii). Let n be an odd integer or $k < \lfloor \frac{n}{2} \rfloor$ and $d=\gcd(n,k)$. We consider two cases.

If $d=1$. Assume that $G_0 = C_n(1,\dots,\widehat{k},\dots,\lfloor \frac{n}{2} \rfloor)$ has vertex set $V=\{v_1,\dots,v_n\}$, and note that $\deg(v_j)=n-3$ for all $j\in\{1,\dots,n\}$. Hence, the dominating set of G_0 is the whole vertex set $D(G_0)=V$.

Without loss of generality, by applying the DVE method recursively, we may remove the vertices v_j such that $j \equiv 1+kh \pmod{n}$ for $h=0,\dots,n-5$. The resulting graph is the path graph P_4.

Therefore,

$$\operatorname{Stab}_d\left(C_n\left(1,\dots,\widehat{k},\dots,\lfloor \tfrac{n}{2} \rfloor\right)\right) = (n-4)+\operatorname{Stab}_d(P_4) = n-4.$$

If $d \neq 1$. Assume that $n=ad$ and $k=bd$ for some $a,b\in\mathbb{Z}$. When $n=6$ and $k=2$, it is clear that

$$\operatorname{Stab}_d(C_6(1,\widehat{2},3)) = 4 = n-4.$$

Without loss of generality, by applying the DVE method recursively, we may remove the vertices v_j, where $j\equiv 1+kh \pmod{n}$ for $h=0,\dots,a-1$. At step a, we obtain a graph G_a which is the circulant graph

$$C_{n-a}\left(1,\dots,\widehat{b(d-1)},\dots,\lfloor \tfrac{n-a}{2} \rfloor\right).$$

Therefore,

$$\operatorname{Stab}_d\left(C_n\left(1,\dots,\widehat{k},\dots,\lfloor \tfrac{n}{2} \rfloor\right)\right) = a+\operatorname{Stab}_d\left(C_{n-a}\left(1,\dots,\widehat{b(d-1)},\dots,\lfloor \tfrac{n-a}{2} \rfloor\right)\right). \tag{7}$$

By induction on n and noting that $2b(d-1)=n-a$ if and only if $n=2k$, we have

$$\operatorname{Stab}_d\left(C_{n-a}\left(1,\dots,\widehat{b(d-1)},\dots,\lfloor \tfrac{n-a}{2} \rfloor\right)\right) = n-a-4. \tag{8}$$

From (7) and (8), we get

$$\operatorname{Stab}_d\left(C_n\left(1,\dots,\widehat{k},\dots,\lfloor \tfrac{n}{2} \rfloor\right)\right) = n-4.$$

(iii) If n is even and $k = \lfloor \frac{n}{2} \rfloor$. Consider the graph $G = C_n\left(1, \dots, \widehat{\lfloor \frac{n}{2} \rfloor}\right)$, each vertex v_i is not adjacent to exactly one vertex v_j such that $|i-j| = \frac{n}{2}$. Therefore, every vertex in G has degree $n-2$ and its degree sequence is

$$(d_1, \dots, d_n) = (\underbrace{n-2, \dots, n-2}_{n\text{-tuple}}).$$

By Theorem 3.7, it follows that

$$\operatorname{reg}\left(I_D\left(C_n\left(1, \dots, \widehat{\lfloor \tfrac{n}{2} \rfloor}\right)\right)\right) \le n(n-2) - n + 1 = n^2 - 3n + 1.$$

If n is odd or $k < \lfloor \frac{n}{2} \rfloor$. Consider the graph $G = C_n\left(1, \dots, \widehat{k}, \dots, \lfloor \frac{n}{2} \rfloor\right)$, each vertex v_i is not adjacent to exactly two vertices v_j such that $|i-j| = \frac{n}{2}$ and $n - |i-j| = \frac{n}{2}$. Therefore, every vertex in G has degree $n-3$ and its degree sequence is

$$(d_1, \dots, d_n) = (\underbrace{n-3, \dots, n-3}_{n\text{-tuple}}).$$

By Theorem 3.7, it follows that

$$\operatorname{reg}\left(I_D\left(C_n\left(1, \dots, \widehat{k}, \dots, \left\lfloor \frac{n}{2} \right\rfloor\right)\right)\right) \le n(n-3) - n + 1 = n^2 - 4n + 1.$$

□

Example 4.6. Let $C_8(\widehat{1}, 2, 3, 4)$ be the circulant graph. By applying the DVE method, we obtain

$$\begin{aligned}
Q_{G_0} &= (x_1^5, x_2^5, x_3^5, x_4^5, x_5^5, x_6^5, x_7^5, x_8^5),\\
Q_{G_1} &= (x_1^5, x_2^5, x_3^4, x_4^4, x_5^4, x_6^4, x_7^4),\\
Q_{G_2} &= (x_1^4, x_2^4, x_3^3, x_4^3, x_5^3, x_6^3),\\
Q_{G_3} &= (x_1^3, x_2^3, x_3^2, x_4^2, x_5^2),\\
Q_{G_4} &= (x_1^2, x_2^2, x_3, x_4).
\end{aligned}$$

Therefore, $\operatorname{Stab}_d(C_8(\widehat{1}, 2, 3, 4)) = 4$. Using `Macaulay2` [5], we have

$$\begin{aligned}
I_D(C_8(\widehat{1}, 2, 3, 4)) = (&x_1^5, x_2^5, x_3^5, x_4^5, x_4x_5^5, x_3x_5^5, x_4^2x_6^5, x_3^2x_6^5, x_2^2x_5^5, x_1^2x_5^5, x_4^3x_7^5, x_3^3x_7^5,\\
&x_4x_5^2x_6^5, x_3x_5^2x_6^5, x_2^3x_6^5, x_1^3x_6^5, x_4^4x_8^5, x_3^4x_8^5, x_4x_5^3x_7^5, x_3x_5^3x_7^5, x_2^4x_7^5, x_1^4x_7^5,\\
&x_2^2x_5^2x_6^5, x_1^2x_5^2x_6^5, x_4x_5^4x_8^5, x_3x_5^4x_8^5, x_4^2x_6^3x_7^5, x_3^2x_6^3x_7^5, x_2^2x_5^3x_7^5, x_1^2x_5^3x_7^5,\\
&x_4^2x_6^4x_8^5, x_3^2x_6^4x_8^5, x_2^2x_5^4x_8^5, x_1^2x_5^4x_8^5, x_4x_5^2x_6^3x_7^5, x_3x_5^2x_6^3x_7^5, x_2^3x_6^3x_7^5, x_1^3x_6^3x_7^5,\\
&x_4^3x_7^4x_8^5, x_3^3x_7^4x_8^5, x_4x_5^2x_6^4x_8^5, x_3x_5^2x_6^4x_8^5, x_2^3x_6^4x_8^5, x_1^3x_6^4x_8^5, x_2^2x_5^2x_6^3x_7^5, x_1^2x_5^2x_6^3x_7^5,\\
&x_4x_5^3x_7^4x_8^5, x_3x_5^3x_7^4x_8^5, x_2^4x_7^4x_8^5, x_1^4x_7^4x_8^5, x_2^2x_5^2x_6^4x_8^5, x_1^2x_5^2x_6^4x_8^5, x_4^2x_6^3x_7^4x_8^5,\\
&x_3^2x_6^3x_7^4x_8^5, x_2^2x_5^3x_7^4x_8^5, x_1^2x_5^3x_7^4x_8^5, x_4x_5^2x_6^3x_7^4x_8^5, x_3x_5^2x_6^3x_7^4x_8^5, x_2^3x_6^3x_7^4x_8^5,\\
&x_1^3x_6^3x_7^4x_8^5, x_2^2x_5^2x_6^3x_7^4x_8^5, x_1^2x_5^2x_6^3x_7^4x_8^5).
\end{aligned}$$

and

$$\operatorname{reg}(I_D(C_8(\widehat{1},2,3,4))) = 33.$$

Therefore, the bound given in Theorem 3.7 is a sharp bound.

We also recover some bounds for the regularity of elimination ideals associated with familiar graphs previously studied in [2].

Corollary 4.7 ([2], Theorem 3.11)**.** *Let P_n be a path graph with $n \geq 3$. Then*

$$\operatorname{reg}(I_D(P_n)) \leq n-1.$$

Proof. We have P_n has the degree sequence $(\underbrace{2,\dots,2}_{(n-2)\text{-tuple}},1,1)$ and n vertices. Therefore, by Theorem 3.7, we get

$$\operatorname{reg}(I_D(P_n)) \leq \underbrace{2+\cdots+2}_{(n-2)\text{-terms}}+1+1-n+1 = n-1.$$

$\square$

Corollary 4.8 ([2], Theorem 3.12)**.** *Let C_n be a cycle graph with $n \geq 3$. Then*

$$\operatorname{reg}(I_D(C_n)) \leq n+1.$$

Proof. We have C_n has the degree sequence $\underbrace{(2,\dots,2)}_{n\text{-tuple}}$ and n vertices. By Theorem 3.7, we get

$$\operatorname{reg}(I_D(C_n)) \leq \underbrace{2+\cdots+2}_{n\text{-terms}}-n+1 = n+1.$$

$\square$

Corollary 4.9 ([2], Theorem 3.7)**.** *Let K_n be a clique on n vertices with $n \geq 3$. Then*

$$\operatorname{reg}(I_D(K_n)) \leq (n-1)^2.$$

Proof. We have K_n has the degree sequence $\underbrace{(n-1,\dots,n-1)}_{n\text{-tuple}}$ and n vertices. By Theorem 3.7, we get

$$\operatorname{reg}(I_D(K_n)) \leq n(n-1)-n+1 = (n-1)^2.$$

$\square$

Corollary 4.10. *Let $K_{m_1,m_2,\dots,m_q}$ be a complete q-partite graph, where $m_i \geq 1$ for $i = 1,\dots,q$. Then*

$$\operatorname{reg}(I_D(K_{m_1,m_2,\dots,m_q})) \leq 2\sum_{i=1}^{q} m_i m_{i+1} - \sum_{i=1}^{q} m_i + 1,$$

where $m_{q+1} := m_1$.

Proof. We have

$$|E(K_{m_1,m_2,\dots,m_q})| = \sum_{i=1}^{q} m_i m_{i+1},$$

where $m_{q+1} := m_1$ and

$$|V(K_{m_1,m_2,\dots,m_q})| = \sum_{i=1}^{q} m_i.$$

By Corollary 3.9, we have

$$\operatorname{reg}(I_D(K_{m_1,m_2,\dots,m_q})) \leq 2\sum_{i=1}^{q} m_i m_{i+1} - \sum_{i=1}^{q} m_i + 1,$$

where $m_{q+1} := m_1$. □

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