

Rings Characterized via Pseudo C^* -injectivity

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Abstract—In this paper, we present several significant findings concerning pseudo c^* -injective modules and rings. Furthermore, we introduce additional results, detailed as follows: (i) R is right self-injective iff, for every integer $n \geq 2$, the matrix ring $M_n(R)$ is right pseudo c^* -injective and (ii) R is a regular right self-injective ring if and only if every cyclic right ideal of $M_2(R)$ is a pseudo c^* -injective module. In particular, we show that M is semisimple iff all finitely generated modules in $\sigma[M]$ are pseudo c^* -injective. Moreover, we obtain many characterizations of many kinds of rings, via pseudo c^* -injective modules, that extend known results, for example, with a noetherian ring, it extends Bass–Papp Theorem and Damiano’s result. We show that if M is a pseudo c^* -injective module satisfying certain conditions, then $S = \text{End}(M)$ is a semilocal ring. Moreover, we present a result concerning the symmetry of a Rickart ring.

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1. INTRODUCTION

The concept of injective modules, introduced by Baer in 1940, has played a central role in module and ring theory. Over the decades, various generalizations have been proposed to capture more nuanced extension properties. Among these, the notion of M -injectivity stands out: A module N is considered M -injective provided that any R -homomorphism defined on a submodule $A \subseteq M$ with codomain N admits an extension to the entire module M . In 2005, D.Q. Hai introduced a broader class called *pseudo M -injective* modules (see [10] or [15]). Here the extension condition is relaxed: only monomorphisms from submodules of M to N are required to extend to homomorphisms from M to N . This subtle shift allows for a richer class of modules that still retain meaningful extension behavior. A right R -module A is termed *pseudo injective* if it satisfies the condition of being pseudo A -injective; that is, every homomorphism from a submodule of A into A can be extended to the whole of A . These modules generalize quasi-injective modules and have been the subject of extensive study. Researchers have explored their structural properties, their relationships with quasi-injectivity, and their role in characterizing certain classes of rings, such as semisimple Artinian rings and QF-rings (see [16]). Further developments have introduced the concept of *pseudo M - c^* -injective* modules. A module A is called pseudo B - c^* -injective if, whenever a submodule N of B is isomorphic to a closed submodule of B , any injective map from N into A is extended to B . When $M = N$, we refer to M as a *pseudo c^* -injective* module. These concepts are not only theoretically elegant but also practically useful in understanding the structure of endomorphism rings. For instance, it has been shown that under certain conditions, $\text{End}(M)$ is semilocal if M is pseudo c^* -injective. Moreover, pseudo c^* -injectivity has connections to co-Hopfian modules, CS-modules, and continuous modules, enriching the landscape of module theory.

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In this paper, we aim to contribute further to this growing body of work by presenting new results on pseudo c^* -injective modules, exploring their algebraic behavior, and examining their implications for the symmetry of Rickart rings.

Throughout this work, all rings are assumed to be associative and possess a multiplicative identity. Unless otherwise specified, all modules are taken to be right unital. For any submodule N of a module M , we write $N \leq M$ to indicate that N is a submodule of M , and $N < M$ to denote that N is a proper submodule. The notation $N \leq^\oplus M$ signifies that N is a direct summand of M . We use $N \leq_e M_R$ to mean that N is an essential submodule of M_R . We use the notation $\sigma[M]$ to refer to the full subcategory of $\text{Mod-}R$ consisting of all modules that can be realized as submodules of quotients of direct sums of copies of M ([17]). We denote by $E(M)$ and $E_M(N)$ the injective envelope of M and the M -injective envelope of N , respectively. A module M is called *automorphism-invariant* if, for every automorphism f of its injective envelope, the image $f(M)$ is contained within M ; that is, $f(M) \leq M$. If every finitely generated submodule of M is noetherian, then M is called *locally noetherian*.

Any undefined terms appearing in this article can be found in standard references such as [7, 9, 11, 13, 14], and [17].

We outline the organization of the paper as follows. Section 1 provides the necessary background, including key definitions and concepts that underpin the results to follow. Section 2 concentrates on the analysis of pseudo c^* -injective modules. The property of being right self-injective for a ring R is equivalent to the condition that $\mathbb{M}_n(R)$ is right pseudo c^* -injective for all integers $n \geq 2$. Interestingly, this equivalence already holds when $n = 2$. Additionally, we present novel criteria for identifying regular right self-injective rings through the lens of pseudo c^* -injective modules over the matrix ring $\mathbb{M}_2(R)$. Furthermore, we present several characterizations of important classes of rings—such as noetherian rings, QF-rings, hereditary rings, semisimple artinian rings, artinian rings, and semi-artinian rings—through the lens of pseudo c^* -injectivity. These results extend classical theorems, including the Bass–Papp Theorem and Damiano’s results for noetherian rings (see [8, Corollary 1.7, 1.10]). In Section 3, we turn our attention to pseudo c^* -injective modules and their endomorphism rings. If a module M is pseudo c^* -injective and possesses an essential submodule structured as $\bigoplus_{i=1}^n u_i R$, where each u_i is a uniform closed element of M , then it follows that the endomorphism ring $S = \text{End}(M)$ is semilocal. In addition, we explore the symmetric nature of Rickart rings and show that if a ring R possesses both right pseudo c^* -injectivity and the right Rickart property, then it necessarily exhibits the left Rickart property as well.

2. PSEUDO C^* -INJECTIVE MODULES AND RELATED RINGS

Recall that a module A is pseudo B - c^* -injective provided that, for each submodule N of B isomorphic to a closed submodule of B , every injective map from N into A admits an extension to a homomorphism from B into A . When $A = B$, we refer to A as a *pseudo c^* -injective* module.

Lemma 1 ([16, Theorem 3.3]). *Let $X := P \oplus Q$ be a pseudo c^* -injective module. Then, P is Q -injective.*

Lemma 2. *Let U and V be right R -modules. If $U \oplus V$ is a pseudo c^* -injective right R -module, then every monomorphism from V to U splits.*

Proof. Suppose that $U \oplus V$ forms a pseudo c^* -injective right R -module. According to Lemma 1, this implies that V is injective relative to U . Consequently, any injective homomorphism $f : V \rightarrow U$ admits a splitting. \square

Proposition 1. *Let R be a ring. The following conditions are equivalent:*

- (i) R is semisimple artinian.
- (ii) Every submodule of a projective right R -module is pseudo c^* -injective.
- (iii) Every submodule of $(R \oplus R)_R$ is pseudo c^* -injective.
- (iv) Every submodule of an injective module is pseudo c^* -injective.

Proof. (i) \Rightarrow (ii) \Rightarrow (iii) and (i) \Rightarrow (iv) are obvious.

(iii) \Rightarrow (i) Let P be a right ideal of R . Then, the module $(P \oplus R)_R$ is a submodule of $(R \oplus R)_R$. By assumption (3), we have that $P \oplus R$ is pseudo c^* -injective as a right R -module. Hence, the right ideal P is a direct summand of R by Lemma 2. Therefore, every right ideal of R is a direct summand of R_R , which implies that R is semisimple artinian.

(iv) \Rightarrow (i) Let I be a right ideal of R . Then, $(I \oplus R)_R$ is a submodule of the injective module $E(R_R) \oplus E(R_R)$. Therefore, $(I \oplus R)_R$ is pseudo c^* -injective and so $I \leq^\oplus R_R$ by Lemma 2. \square

Proposition 2. Let Q_R be a module. Then, Q is injective iff $Q \oplus E(Q)$ is pseudo c^* -injective.

Proof. By Lemma 2. \square

Now, we give some characterizations of rings in which rings $\mathbb{M}_n(R)$ are pseudo c^* -injective.

Theorem 1. The following conditions are equivalent for a ring R :

1. R^n is a pseudo c^* -injective right R -module for all $n \geq 2$.
2. R is right self-injective.
3. $\mathbb{M}_n(R)$ is a right pseudo c^* -injective ring for all $n \geq 2$.
4. $\mathbb{M}_2(R)$ is a right pseudo c^* -injective ring.

Proof. (1) \Rightarrow (2), by Lemma 1, and (2) \Rightarrow (1) is obvious.

(2) \Rightarrow (3) \Rightarrow (4) are obvious.

(4) \Rightarrow (2). Assume that $\mathbb{M}_2(R)$ is a right pseudo c^* -injective ring. Write $S = \mathbb{M}_2(R)$, so that $S = e_{11}S \oplus e_{22}S$ with e_{ii} the unit matrix. We have that S is a right pseudo c^* -injective ring and obtain that $e_{11}S$ is $e_{22}S$ -injective by Lemma 1. Note that $e_{ii}S \cong e_{jj}S$ for all $i, j \in \{1, 2\}$. It follows that $e_{11}S$ is $e_{ii}S$ -injective for all $i \in \{1, 2\}$. Therefore, we have that $e_{11}S$ is S -injective. From this, it implies that $e_{11}S_S$ is injective. Similarly, $e_{22}S$ is injective so that S is right self-injective. We may thus infer that R enjoys the property of right self-injectivity. \square

Corollary 1. Let R be a ring. The following conditions are equivalent:

1. $\mathbb{M}_2(R)$ is right pseudo c^* -injective.
2. Endomorphism rings of n -generated projective right R -modules ($n \geq 2$) are right pseudo c^* -injective rings.

Proposition 3. “Right pseudo c^* -injective ring” is not a Morita invariant property.

Proof. Let us assume, contrary to expectation, that the class of right pseudo c^* -injective rings is closed under Morita equivalence. Then, $\mathbb{M}_2(R)$ is a right pseudo c^* -injective ring whenever R is a right pseudo c^* -injective ring, and, by Theorem 1, R is right self-injective. This is a contradiction, since the ring R consisting of all eventually constant sequences of elements from \mathbb{F}_2 is an example of a pseudo c^* -injective ring that R_R is not injective. \square

We now establish novel criteria for identifying regular right self-injective rings, utilizing the framework of pseudo c^* -injective modules over $\mathbb{M}_2(R)$.

Theorem 2. Let R be a ring. The following statements are equivalent:

1. R is regular right self-injective.
2. All cyclic right ideals of $\mathbb{M}_2(R)$ are pseudo c^* -injective.
3. All cyclic right ideals of $\mathbb{M}_2(R)$ generated by a diagonal matrix are pseudo c^* -injective modules.

Proof. (1) \Rightarrow (2). Call $S := \mathbb{M}_2(R)$. We have that regular self-injective is a Morita invariant property and obtain that S is von Neumann regular right self-injective. It follows that all principal right ideals of S are direct summands of S . Thus, all principal right ideals of S are injective modules and so they are pseudo c^* -injective modules.

(2) \Rightarrow (3) is obvious.

(3) \Rightarrow (1). Since $\mathbb{M}_2(R)$ admits generation by a diagonal matrix, it qualifies as a right pseudo c^* -injective ring. Therefore, by Theorem 1, we conclude that R is right self-injective. Next, we show that R

is regular. In fact, let $S := \mathbb{M}_2(R)$, $b \in R$, and let I be a right ideal of S generated by the element $\begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix}$.

One can check that

$$I = \left\{ \begin{bmatrix} bx_1 & bx_2 \\ x_3 & x_4 \end{bmatrix} \mid x_i \in R \right\}. \tag{1}$$

It follows that I is a pseudo c^* -injective right S -module by the equation (1). Take $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and so, S

and R are Morita equivalent via $N \rightarrow Ne$ for all right S -modules N . Since Morita equivalence preserves isomorphisms, direct summands, we obtain that Ie is isomorphic to $bR \oplus R$ as right R -modules. Given that I is pseudo c^* -injective, then $(bR \oplus R)_R$ is pseudo c^* -injective. By Lemma 2, this implies that bR is R -injective. Hence, $bR \leq^\oplus R_R$, and, therefore, R is regular. \square

Corollary 2. *The following statements are equivalent for a ring R :*

1. R is regular right self-injective.
2. All cyclic right ideals of $\mathbb{M}_n(R)$ are pseudo c^* -injective modules for each integer number $n \geq 2$.

Coflat modules are introduced by Damiano [8]. It is well-known that a right R -module M is coflat if for every finitely generated right ideal I of R and every homomorphism $f : I \rightarrow M$, there exists $m \in M$ such that $f(x) = mx$ for all $x \in I$. It follows that every injective right R -module is coflat and a ring R is right noetherian if and only if every coflat right R -module is injective (see [8, Corollary 1.7, 1.10]). We obtain the following result that extends Bass–Papp Theorem and Damiano’s result ([8, Corollary 1.7, 1.10]).

Proposition 4. *Let R be a ring. The following conditions are equivalent:*

- (i) R is right noetherian.
- (ii) If $\{M_i\}_{i \in I}$ is any family of injective right R -modules, then $\bigoplus_{i \in I} M_i$ is a pseudo c^* -injective module.
- (iii) Every countable direct sum of injective right R -modules is a pseudo c^* -injective module.
- (iv) Every coflat right R -module is an injective module.
- (v) Every coflat right R -module is a pseudo c^* -injective module.

Proof. (i) \Rightarrow (ii) \Rightarrow (iii) are obvious.

(iii) \Rightarrow (i). Assume that every countable direct sum of injective right R -modules is an pseudo c^* -injective module. Let $\{E_i\}_{i \in \mathbb{N}}$ be a family of envelopes of simple right R -module $\{K_i\}_{i \in \mathbb{N}}$, and so $[\bigoplus_{i \in \mathbb{N}} E_i] \bigoplus E(R_R)$ is a countable direct sum of injective right R -modules. It follows that $[\bigoplus_{i \in \mathbb{N}} E_i] \bigoplus E(R_R)$ is a pseudo c^* -injective right R -module and so $\bigoplus_{i \in \mathbb{N}} E_i$ is $E(R_R)$ -injective, by Lemma 1, so that it is an injective right R -module. We deduce that R is a right noetherian ring.

(i) \Rightarrow (iv) \Rightarrow (v) are obvious.

(v) \Rightarrow (i). Assume that every coflat right R -module is a pseudo c^* -injective module. Take $\{E_i\}_{i \in \mathbb{N}}$ a family of injective right R -module. Then, $\bigoplus_{i \in \mathbb{N}} E_i \oplus E(R_R)$ is a coflat right R -modules by [8, Corollary 1.9]. Therefore, $\bigoplus_{i \in \mathbb{N}} E_i \oplus E(R_R)$ is a pseudo c^* -injective right R -module and so $\bigoplus_{i \in \mathbb{N}} E_i$ is $E(R_R)$ -injective so that it is an injective right R -module. We deduce that R is a right noetherian ring. \square

For a module M , we also obtain the following similar result.

Proposition 5. *Let M be a right R -module. The following conditions are equivalent:*

1. M is locally noetherian.
2. If $\{M_i\}_{i \in I}$ is any family of M -injective right R -modules in $\sigma[M]$, then $\bigoplus_{i \in I} M_i$ is a pseudo c^* -injective module.

Proof. (1) \Rightarrow (2) \Rightarrow (3) are obvious.

(3) \Rightarrow (1). Assume that every countable direct sum of M -injective right R -modules in $\sigma[M]$ is an pseudo c^* -injective module. Let $\{E_i\}_{i \in \mathbb{N}}$ be a family of M -injective hulls of simple right R -module $\{K_i\}_{i \in \mathbb{N}}$ in $\sigma[M]$. Consider the module $[\bigoplus_{i \in \mathbb{N}} E_i] \oplus E_M(M)$, which is a countable direct sum of M -injective right R -modules contained in $\sigma[M]$. This implies that $[\bigoplus_{i \in \mathbb{N}} E_i] \oplus E_M(M)$ is a pseudo c^* -injective right R -module. Therefore, $\bigoplus_{i \in \mathbb{N}} E_i$ is $E_M(M)$ -injective, by Lemma 1, and, hence, M -injective.

Consequently, we deduce from [17, 27.3] that M is a locally noetherian module. \square

Proposition 6. *Let R be a domain. If every pseudo c^* -injective right R -module is flat, then R is a division.*

Proof. Assume that R is a domain and every pseudo c^* -injective right R -module is flat. Flat modules over a domain are torsion-free, and injective modules, being flat, inherit this property. As a consequence, all R -modules are nonsingular. Consider any essential right ideal I of R . Since R/I is nonsingular, it must be that $I = R$. Therefore, R is semisimple, and consequently a division ring. \square

Proposition 7. Let R be a ring. The following conditions are equivalent:

1. R is a quasi-Frobenius ring.
2. R is a right perfect ring and every pseudo c^* -injective right R -module is flat.

Proof. (1) \Rightarrow (2) is obvious.

(2) \Rightarrow (1). Assume that R is a right perfect ring and every pseudo c^* -injective right R -module is flat. It is well-known that every flat right R -module over a right perfect ring R is projective. It follows that every injective right R -module is projective. Thus, R is quasi-Frobenius. \square

Theorem 3. *Let R be a ring. Then, R is a right hereditary ring iff all factor modules of injective right R -modules are pseudo c^* -injective modules.*

Proof. \Rightarrow if R is a right hereditary ring, then one can check that all factor modules of injective right R -modules are pseudo c^* -injective modules.

\Leftarrow Suppose that N is injective as a right R -module. Let K be a submodule of N . To prove that N/K is injective, consider the module $N/K \oplus E(N/K)$, which is a quotient of $N \oplus E(N/K)$. Since pseudo c^* -injectivity is preserved under factor modules, it follows that $E(N/K) \oplus N/K$ is pseudo c^* -injective. By Lemma 1, this implies that N/K is $E(N/K)$ -injective. Hence, $N/K = E(N/K)$, and thus N/K is injective, as required. \square

Theorem 4. *For a right R -module M , the following statements are equivalent:*

1. M is semisimple.
2. All modules in the category $\sigma[M]$ are pseudo c^* -injective.

3. All finitely generated modules in $\sigma[M]$ are pseudo c^* -injective.
4. All 2-generated modules in $\sigma[M]$ are pseudo c^* -injective.

Proof. One can check that (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4).

(4) \Rightarrow (1). Suppose P is a cyclic module in $\sigma[M]$ and let x be an element of M . Then, the 2-generated module $P \oplus xR$ is pseudo c^* -injective which ensures that P is xR -injective by Lemma 1. From [13, Proposition 1.4], it follows that P is injective relative to M . Consequently, by [9, Corollary 7.14], we deduce that M is semisimple. \square

The corollary below may be viewed as a generalization of Corollary 2.5 in [15]:

Corollary 3. *A ring R is semisimple. if and only if every 2-generated right R -modules are pseudo c^* -injective modules.*

Theorem 5. *Let M be a module with essential socle. The following conditions are equivalent:*

1. M is semisimple.
2. If P and Q are arbitrary pseudo c^* -injective modules in $\sigma[M]$, then $P \oplus Q$ is pseudo c^* -injective.
3. Pseudo c^* -injective right R -modules in $\sigma[M]$ are precisely the injective ones.
4. If $\{T_i\}_{i \in I}$ is any family of pseudo c^* -injective modules in $\sigma[M]$, then $\bigoplus_{i \in I} T_i$ is pseudo c^* -injective.

Proof.(1) \Rightarrow (2) is obvious.

(2) \Rightarrow (3). Suppose that N is a pseudo c^* -injective module. By assumption, the direct sum $N \oplus E_M(N)$ is also pseudo c^* -injective. Invoking Lemma 1, we deduce that N is $E_M(N)$ -injective. It then follows that N is M -injective.

(3) \Rightarrow (1). Let $\{S_i \mid i \in \mathbb{N}\} \subset \sigma[M]$ be a countable family of simple right R -modules. Then, the direct sum $\bigoplus_{i \in \mathbb{N}} S_i$ is semisimple, and, hence, pseudo c^* -injective. By condition (3), it follows that $\bigoplus_{i \in \mathbb{N}} S_i$ is M -injective. Therefore, it is a direct summand of $\bigoplus_{i \in \mathbb{N}} E_M(S_i)$. Since $\bigoplus_{i \in \mathbb{N}} S_i$ is essential in $\bigoplus_{i \in \mathbb{N}} E_M(S_i)$, we conclude that $\bigoplus_{i \in \mathbb{N}} S_i = \bigoplus_{i \in \mathbb{N}} E_i$ is M -injective. Thus, M is locally noetherian.

By [17, 27.4], the injective hull $E_M(M)$ can be decomposed as $E_M(M) = \bigoplus_{i \in I} K_i$, where each K_i is an indecomposable right R -module in $\sigma[M]$. Since each K_i is M -injective, it follows that every K_i is uniform.

For each $i \in I$, the socle $Soc(K_i)$ is M -injective by condition (3), and hence a direct summand of K_i . This implies $Soc(K_i) = K_i$, so each K_i is simple. Therefore, $E_M(M)$ is semisimple, and we conclude that M itself is semisimple.

(1) \Rightarrow (4) \Rightarrow (2). They are obvious. \square

Corollary 4. *Let R be a ring with essential right socle. The following conditions are equivalent:*

1. R is semisimple artinian.
2. If A and B are arbitrary pseudo c^* -injective modules, then $A \oplus B$ is pseudo c^* -injective.
3. Pseudo c^* -injective right R -modules are precisely the injective ones.
4. If each module C_i for $i \in I$ is pseudo c^* -injective, then their direct sum $\bigoplus_{i \in I} C_i$ also possesses the pseudo c^* -injectivity property.

Recall a ring R *right semiartinian*, if all non-zero right R -modules have nonzero socles.

As introduced in [6], a right R -module M is called *socle- N -injective* (abbreviated as *soc- N -injective*) if every R -homomorphism $f : \text{Soc}(N) \rightarrow M$ can be extended to a homomorphism from N to M . Furthermore, M is said to be *strongly soc-injective* if M is soc- N -injective for every right R -module N .

Proposition 8. *The following conditions are equivalent for a ring R :*

1. R is right semiartinian.
2. Strongly soc-injective right R -modules are precisely the injective ones.
3. Strong soc-injectivity implies pseudo c^* -injectivity for right R -modules.

Proof. (1) \Leftrightarrow (2) The result is a direct consequence of [6, Proposition 16] and [1, Proposition 2.22].

(2) \Rightarrow (3) is obvious.

(3) \Rightarrow (2). Assume that P is a strongly soc-injective right R -module. From the strong soc-injectivity of P , we have a decomposition $P = I \oplus H$, where I is injective and $\text{Soc}(H) = 0$ by [6, Proposition 16]. Consider the module $P \oplus E(P)$, which can be expressed as

$$P \oplus E(P) = I \oplus H \oplus E(P) = (I \oplus E(P)) \oplus H. \quad (2)$$

From the equation (2), applying [6, Proposition 16] once again, it follows that $P \oplus E(P)$ is strongly soc-injective. By the hypothesis, this implies that $P \oplus E(P)$ is pseudo c^* -injective. Then, by Proposition 2, we conclude that P is injective. \square

In the sense of [5], a right R -module A is called *simple- B -injective* if, for every submodule $H \leq B$, any homomorphism $q : H \rightarrow A$ with $q(H)$ simple is extended to B . A module A is referred to as *strongly simple-injective* if it satisfies the property of being simple- B -injective for all right R -modules B .

We give some characterizations of right artinian rings via pseudo c^* -injective modules and strongly simple-injective modules.

Proposition 9. *Let R be a ring. The following conditions are equivalent:*

1. R is right artinian.
2. Strongly simple-injective right R -modules are precisely the injective ones.
3. Strong simple-injectivity implies pseudo c^* -injectivity for right R -modules.

Proof. (1) \Rightarrow (2). Let M be a strongly simple-injective right R -module, and let S be a simple submodule of M . We claim that there is an injective direct summand E of M such that S is essential in E . Indeed, let $\iota_1 : S \rightarrow E(S)$ and $\iota_2 : S \rightarrow M$ be the inclusion maps. Since $S \leq_e E(S)$ and M is strongly simple-injective, we have a monomorphism $\mu : E(S) \hookrightarrow M$ such that $\mu\iota_1 = \iota_2$, and so $\mu(x) = x$ for all $x \in S$. Let $E = \mu(E(S))$. Then, $S \leq_e E \leq^\oplus M$, and so S is essential in E .

The socle of M can be decomposed as $\text{Soc}(M) = \bigoplus_{j \in I} S_j$, where the components S_j are simple submodules of M indexed by I . For each $j \in I$, let $S_j \leq_e E_j \leq^\oplus M$, where E_j is an injective module containing S_j essentially. Define $E = \bigoplus_{j \in I} E_j$. We have that R is right noetherian and obtain that $E \leq^\oplus M$, and we can write $M = E \oplus M'$ for some $M' \leq M$ with $\text{Soc}(M') = 0$.

If, in addition, R is right artinian, then every module has finite length, and the condition $\text{Soc}(M') = 0$ implies $M' = 0$. Hence, $M = E$ is injective, as required.

(2) \Rightarrow (3) is obvious.

(3) \Rightarrow (1). By reasoning similar to the proof of (3) \Rightarrow (2) in Proposition 8, we deduce that R is right semiartinian. Since every right noetherian and right semiartinian ring is right artinian, it suffices to verify that R is right noetherian.

Let S_i be a simple right R -modules with $i \in \mathbb{N}$ and consider the module $A = \bigoplus_{i=1}^{\infty} E(S_i)$. We aim to show that A is injective. Note that $A \oplus E(A)$ is strongly simple-injective, and hence pseudo c^* -injective by assumption. Applying Proposition 2, it infers that A is injective. \square

3. ENDOMORPHISM RINGS OF PSEUDO C^* -INJECTIVE MODULES

Let M_R and $S = \text{End}_R(M)$ be the endomorphism ring of M . We define

$$\mathcal{K}_U = \{s \in S \mid U \cap \text{Ker}(s) \neq 0\}, \quad (3)$$

where U is a uniform closed submodule of M . From the equation (3), it is easy to see that \mathcal{K}_U is a left ideal of S . Now, an element u of M is called a *uniform closed element* if uR is a uniform closed submodule of M . In this case, \mathcal{K}_{uR} will be denoted by \mathcal{K}_u and $\mathcal{K}_U = \mathcal{K}_u$ for every nonzero element u of U .

The following lemma is useful for our consideration.

Lemma 3. *Let M be a pseudo c^* -injective module with endomorphism ring $S = \text{End}(M)$, and let $u, v \in M$. Assume that $uR \cong C$ with C a closed submodule of M , and that $r_R(u) = r_R(v)$. Then, the left ideals Su and Sv coincide.*

Proof. Firstly, we prove that $Sv \leq Su$ by considering the mapping $\beta : uR \rightarrow vR$ via $\beta(ur) = vr$ for all $r \in R$. It is easy to check that β is an isomorphism. Given that M is pseudo c^* -injective, one can deduce the existence of an endomorphism $\bar{\beta}$ of M that serves as an extension of β . Hence, $v = \beta(u) = \bar{\beta}(u) \in Su$. It follows that $Sv \leq Su$.

To prove the converse, note that $vR \cong uR$ and so vR is isomorphic to a closed submodule of M . Analogously, since u belongs to Sv , it follows that the left ideals Su and Sv are equal. \square

Proposition 10. *Assume that M_R is a pseudo c^* -injective module with $S = \text{End}(M)$ and u is a uniform closed element of M . Then, \mathcal{K}_u emerges as the only maximal left ideal of S encompassing $l_S(u)$.*

Proof. It is straightforward to verify that $l_S(u) \leq \mathcal{K}_u$. Moreover, since $1 \notin \mathcal{K}_u$, it follows that $\mathcal{K}_u \neq S$. Our next objective is to establish that \mathcal{K}_u constitutes a maximal left ideal in the ring S . Given $s \in S \setminus \mathcal{K}_u$, we observe that $uR \cap \text{Ker}(s) = 0$. As a result, $r_R(u) \subseteq r_R(su)$. Assume $a \in r_R(su)$. Since $sua = 0$, it follows that ua lies in both $\text{Ker}(s)$ and uR , implying $ua = 0$. Therefore, $a \in r_R(u)$, and we deduce that $r_R(u) = r_R(su)$. Using Lemma 3, we infer that Su coincides with Ssu , which implies that S can be expressed as $l_S(u) + Ss$. As a result, The ideal \mathcal{K}_u is recognized as a maximal left ideal in S .

Our next objective is to confirm that \mathcal{K}_u is the sole maximal left ideal containing $l_S(u)$. Assume, contrary to this claim, that there exists a maximal left ideal L of S such that $l_S(u) \subseteq L$ and $L \neq \mathcal{K}_u$. Using analogous reasoning as previously established, we arrive at the conclusion that $S = L$, contradicting the premise that L is a left ideal with $L \neq S$. Therefore, \mathcal{K}_u is the only maximal left ideal of S that contains $l_S(u)$. \square

Naturally, this leads us to a structural characterization of the endomorphism ring $S = \text{End}(M)$, where M is an automorphism-invariant module (see [3, 4] and [12]).

Proposition 11. *Suppose M is an automorphism-invariant module with $S = \text{End}(M)$, and let $u \in M$ be a uniform closed element. Then, among all maximal left ideals of S , only \mathcal{K}_u contains $l_S(u)$.*

Proof. By Proposition 10. \square

We also have the similar result for a pseudo c^* -injective ring.

Corollary 5. *Let R be right pseudo c^* -injective and u be any uniform closed element of R . Then, among all maximal left ideals of R , only \mathcal{K}_u contains $l_R(u)$.*

Now we need the following lemmas to consider some more properties of $\text{End}(M)$ of a pseudo c^* -injective module M .

Lemma 4. *Assume that M is a pseudo c^* -injective right R -module, and denote $S = \text{End}(M)$. Assume that U decomposes as $U = \bigoplus_{i=1}^n u_i R$, with each summand $u_i R$ being a uniform closed submodule within M . If A is a maximal left ideal of S that does not coincide with any ideal of the form \mathcal{K}_u for a uniform closed element $u \in M$, then there exists an element $\theta \in A$ with $\text{Ker}(1 - \theta) \cap U \leq_e U$.*

Proof. Firstly, we consider the uniform closed element u_i . Since $A \neq \mathcal{K}_{u_1}$, we can take the element $k \in A \setminus \mathcal{K}_{u_1}$. This yields $u_1 R \cap \text{Ker}(k) = 0$, implying that $r_R(u_1) = r_R(ku_1)$. Applying Lemma 3, we conclude that $S(ku_1) = Su_1$. As a result, there exists $\alpha_1 \in S$ such that $u_1 = \alpha_1(ku_1)$. Let $\varphi_1 = \alpha_1 k \in SA \subset A$. Then, $(1 - \varphi_1)u_1 = 0$. This shows that $\text{Ker}(1 - \varphi_1) \cap u_1 R = u_1 R \neq 0$. Provided

that $\text{Ker}(1 - \varphi_1) \cap u_i R \neq 0$ holds for each $i \geq 2$, the argument concludes. Under this condition, the module $\bigoplus_{i=1}^n (\text{Ker}(1 - \varphi_1) \cap u_i R) \leq_e U$.

We may, without loss of generality, assume that the intersection $\text{Ker}(1 - \varphi_1) \cap u_2 R$ is trivial. It follows that $(1 - \varphi_1)(u_2 R) \simeq u_2 R$ is uniform. Since $A \neq \mathcal{K}_{(1-\varphi_1)u_2}$, we can take any $h \in A \setminus \mathcal{K}_{(1-\varphi_1)u_2}$. By using the above argument, there exists $\alpha_2 \in S$ such that $(1 - \varphi_1)u_2 = \alpha_2 h(1 - \varphi_1)u_2$. It follows that

$$(1 - (\alpha_2 h + \varphi_1 - \alpha_2 h \varphi_1))u_2 = 0.$$

Let $\varphi_2 = \alpha_2 h + \varphi_1 - \alpha_2 h \varphi_1$. It follows that, for each for $i = 1, 2$ we have $(1 - \varphi_2)u_i = 0$. Proceeding in this fashion, for every $i \in \{1, 2, \dots, n\}$, one can find an element $\theta \in A$ with $\text{Ker}(1 - \theta) \cap u_i R$ nonzero. Consequently, $\text{Ker}(1 - \theta) \cap U$ is an essential submodule of U , fulfilling the required condition. \square

A ring R is referred to as *semilocal* when the quotient $R/J(R)$ forms a semisimple ring.

Lemma 5 ([16, Theorem 3.6]). *Let M be a pseudo c^* -injective right R -module, and let $S = \text{End}(M)$ denote its endomorphism ring. Then, $S/J(S)$ is regular and*

$$J(S) = \{s \in S \mid \text{Ker}(s) \leq_e M\}. \quad (4)$$

Theorem 6. *Assume that M is a pseudo c^* -injective module with $S = \text{End}(M)$ and $W = \bigoplus_{i=1}^n u_i R$ is an essential submodule of M with uniform closed elements u_i of M . Then, the following assertions hold.*

- (1) *For every maximal left ideal in S , there exists a uniform closed element $u \in M$ such that the ideal is given by \mathcal{K}_u .*
- (2) *Moreover, S is a semilocal ring.*

Proof. We begin by observing that M has finite Goldie dimension. As a result, we can find elements u_1, \dots, u_n in M for which the direct sum $W = u_1 R \oplus u_2 R \oplus \dots \oplus u_n R$ constitutes an essential submodule of M , with each u_i being a closed uniform element. Moreover, M is a pseudo c^* -injective module, by Lemma 5 and the equation (4), it follows that $J(S) = \{s \in S \mid \text{Ker}(s) \leq_e M\}$.

(1) Call H a maximal left ideal of S . Assume, contrary to our claim, that H does not take the form \mathcal{K}_u for any uniform closed element $u \in M$. Then, Lemma 4 ensures the existence of an element $\psi \in H$ with $\text{Ker}(1 - \psi) \cap W \leq_e W$. This implies that $1 - \psi \in J(S) \subseteq H$, contradicting the maximality of H . Therefore, there is a uniform closed element $u \in M$ with $H = \mathcal{K}_u$.

(2) Assume that φ belongs to the intersection $\mathcal{K}_{u_1} \cap \mathcal{K}_{u_2} \cap \dots \cap \mathcal{K}_{u_n}$. Then, for each index i , we have $\text{Ker}(\varphi) \cap u_i R \neq 0$, which ensures that $\text{Ker}(\varphi)$ intersects every summand of W non trivially. As a result, $\text{Ker}(\varphi) \leq_e M$, implying that $\varphi \in J(S)$. Therefore,

$$\mathcal{K}_{u_1} \cap \dots \cap \mathcal{K}_{u_n} = J(S).$$

This confirms that the quotient ring $S/J(S)$ is semisimple. \square

Corollary 6. *Assume that R is a right pseudo c^* -injective ring and $W = \bigoplus_{i=1}^n u_i R$ is an essential right ideal of R with uniform closed elements u_i of R_R . Then, the following assertions hold.*

- (1) *If $I \leq R$ is a maximal left ideal, then there exists a uniform closed element $u \in R$ such that $I = \mathcal{K}_u$.*
- (2) *Moreover, R is semilocal.*

A ring R is said to be right *Rickart* if, for every element $a \in R$, the right annihilator $r_R(a)$ can be expressed as eR for some idempotent $e \in R$ ([11]). A left *Rickart* ring is defined similarly. We obtain the following property of a pseudo c^* -injective module which aims at proving the symmetry of Rickart rings.

Theorem 7. *Suppose that M is a pseudo c^* -injective module with endomorphism ring $S = \text{End}(M)$, and let $u, v \in M$ such that the submodule uR is isomorphic to a closed submodule of M . Under these conditions, the following conclusions can be drawn.*

1. If the submodule uR embeds into vR , then the left ideal Su is a homomorphic image of Sv .
2. Furthermore, if $uR \cong vR$, then the left ideals Su and Sv are isomorphic as S -modules.

Proof. (1) Consider a monomorphism $\beta : uR \rightarrow vR$, where $uR \cong C$ with C a closed submodule of M , and suppose $\beta(u) = vt$ for some $t \in R$. By the pseudo c^* -injectivity of M , there exists an endomorphism $\bar{\beta}$ of M that agrees with β on uR . Thus, for every $x \in uR$, we have $\bar{\beta}(x) = \beta(x)$, and in particular, $\bar{\beta}(u) = vt$.

Define a mapping $\gamma : Sv \rightarrow Su$ by setting $\gamma(f(v)) = f(v)t = f\bar{\beta}(u)$ for all $f \in S$. It is straightforward to verify that γ is an S -module homomorphism. To establish that γ is an S -epimorphism, observe that β is injective, which yields $r_R(vt) = r_R(\beta(u))$. Applying Lemma 3, we deduce that $Svt = S\beta(u) = S\bar{\beta}(u)$. Thus, the image of γ coincides with Su , confirming that γ is surjective.

(2) Let $\beta : uR \rightarrow vR$ be an isomorphism. Analogous to the argument in (1), we obtain S -module homomorphisms

$$\begin{aligned} \varphi : Sv &\longrightarrow Su \\ f(v) &\longmapsto f\bar{\beta}(u), \end{aligned}$$

where $\bar{\beta} \in S$ is an extension of β , satisfying $\bar{\beta}(u) = vt = \beta(u)$ for some $t \in R$.

Similarly, there exists a homomorphism

$$\begin{aligned} \varphi' : Su &\longrightarrow Sv \\ f(u) &\longmapsto f\bar{\beta}'(v), \end{aligned}$$

where $\bar{\beta}' \in S$ is an extension of β^{-1} , and $\bar{\beta}'(v) = uq = \beta^{-1}(v)$ for some $q \in R$.

It follows that

$$\bar{\beta}'\bar{\beta}(u) = \bar{\beta}'(\beta(u)) = \bar{\beta}'(vt) = \bar{\beta}'(v)t = \beta^{-1}(v)t = \beta^{-1}(vt) = u.$$

Hence, for any $f \in S$, we have

$$\varphi\varphi'(f(u)) = \varphi(f\bar{\beta}'(v)) = f\bar{\beta}'\bar{\beta}(u) = f(u),$$

which shows that $\varphi \circ \varphi' = 1_{Su}$. Similarly, one can verify that $\varphi' \circ \varphi = 1_{Sv}$. Therefore, φ is an S -isomorphism between Sv and Su . \square

Corollary 7. Suppose that R is a right pseudo c^* -injective ring, and let $u, v \in R$ be such that the right ideal uR is isomorphic to a closed right ideal of R . Then, the following statements hold.

1. If uR embeds into vR , then the left ideal Ru is a homomorphic image of Rv .
2. Furthermore, if $uR \cong vR$, then $Ru \cong Rv$ as left R -modules.

With the groundwork laid, we proceed to prove the following theorem addressing the symmetry property of Rickart rings.

Theorem 8. Let R be a right pseudo c^* -injective ring. Then:

1. Whenever aR is projective as a right ideal for some $a \in R$, the left ideal Ra inherits projectivity as well.
2. \textcircled{a} In addition, the right Rickart property of R ensures that R is necessarily left Rickart as well.

Proof. (1) Suppose aR is a projective right ideal. It follows that there is an isomorphism $aR \rightarrow eR$ for some $e^2 = e \in R$. We deduce that $Ra \cong Re$, and hence Ra is projective as a left ideal by Corollary 7.

(2) Take any element $y \in R$. Given that R is a right Rickart ring, and so $r_R(y) = eR$ for some idempotent $e \in R$. Consequently, we obtain an isomorphism $yR \cong (1 - e)R$, indicating that yR is projective. By the result in (1), the left ideal Ry is also projective. It follows that the left ideal $l_R(y)$ of R is generated by some idempotent of R , thereby establishing that R is a left Rickart ring. \square

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflict of interest.

REFERENCES

1. I. Amin, Y. Ibrahim, and M. F. Yousif, “ C_3 -modules,” *Algebra Colloq.* **22**, 655–670 (2015). doi 10.1142/S1005386715000553
2. A. Alahmadi, N. Er, and S. K. Jain, “Modules which are invariant under monomorphisms of their injective hulls,” *J. Austr. Math. Soc.* **79**, 349–360 (2005). <https://doi.org/10.1017/S1446788700010946>
3. A. N. Abyzov, T. C. Quynh, and D. D. Tai, “Dual automorphism-invariant modules over perfect rings,” *Sib. Math. J.* **58**, 743–751 (2017). <https://doi.org/10.1134/S0037446617050019>
4. A. N. Abyzov, L. V. Thuyet, T. C. Quynh, and A. A. Tuganbaev, “Modules coinvariant under the idempotents endomorphisms of their covers,” *Sib. Math. J.* **60**, 1191–1208 (2019). <https://doi.org/10.1134/S0037446619060016>
5. I. Amin, Y. Fathi, and M. F. Yousif, “Strongly simple-injective rings and modules,” *Algebra Colloq.* **15**, 135–144 (2008). <https://doi.org/10.1142/S100538670800014X>
6. I. Amin, M. F. Yousif, and N. Zeyada, “Soc-injective rings and modules,” *Comm. Algebra* **33**, 4229–4250 (2005). <https://doi.org/10.1080/00927870500279001>
7. F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules* (Springer, Berlin, 1974).
8. R. F. Damiano, “Collat rings and modules,” *Pacif. J. Math.* **81**, 349–369 (1979). <https://doi.org/10.2140/pjm.1979.81.349>
9. N. V. Dung, D. V. Huynh, P. F. Smith, and R. Wisbauer, *Extending Modules*, Vol. 313 of *Pitman Research Notes in Mathematics* (Longman, Harlow, 1994).
10. D. Q. Hai, “A note on pseudo-injective modules,” *Comm. Algebra* **33**, 361–369 (2005). <https://doi.org/10.1081/AGB-200040917>
11. T. Y. Lam, *Lectures on Modules and Rings* (Springer, New York, 1998).
12. T. K. Lee and Y. Zhou, “Modules which are invariant under automorphisms of their injective hulls,” *J. Algebra Appl.* **12**, 1250159 (2013). <https://doi.org/10.1142/S0219498812501599>
13. S. H. Mohamed and B. J. Müller, *Continous and Discrete Modules* (Cambridge Univ. Press, Cambridge, 1990).
14. W. K. Nicholson and M. F. Yousif, *Quasi-Frobenius Rings* (Cambridge Univ. Press, Cambridge, UK, 2003).
15. T. C. Quynh and M. T. Koşan, “On automorphism-invariant modules,” *J. Algebra Appl.* **14**, 1550074 (2015). <https://doi.org/10.1142/S0219498815500747>
16. T. C. Quynh and P. H. Tin, “Modules satisfying extension conditions under monomorphism of their closed submodules,” *Asian-Eur. J. Math.* **5**, 1250041 (2012). <https://doi.org/10.1142/S1793557112500416>
17. R. Wisbauer, *Foundations of Module and Ring Theory* (Gordon and Breach Reading, London, 1991).

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