

Self-focusing of Ultrarelativistic Electron Wave Packets in Collision-free Plasma

Nguyen Quang San^{1,*} and I. D. Feranchuk²

¹*School of Engineering and Technology - Hue University, Hue, VIETNAM*

²*Atomicus GmbH Amalienbadstr. 41C, 76227 Karlsruhe, GERMANY[†]*

(Received 3 April, 2026)

This work investigates a novel physical effect: the self-focusing of ultrarelativistic electron wave packets propagating through a collision-free plasma. By employing a linearized approach to the coupled Dirac and Vlasov equations, we derive the dispersion relations governing the beam's stability. Our analysis identifies the conditions under which the imaginary part of the frequency becomes positive, leading to an exponential contraction of the beam diameter. We characterize this effect by providing an analytical expression for the self-focusing length L_c and performing numerical estimates for characteristic experimental parameters. The results suggest that for a 1GeV electron beam, this self-focusing effect can occur within a distance of approximately 10cm in a typical laboratory plasma.

PACS numbers: 52.38.Hb, 52.27.Ny, 52.35.Qz, 52.59.Sa, 03.65.Pm, 52.65.Ff

Keywords: ultrarelativistic electron beams, self-focusing, collision-free plasma, Lorentz gauge, plasma-beam interaction

DOI: <https://doi.org/10.5281/zenodo.21106270>

1. Introduction

The challenge of generating coherent X-ray radiation is fundamentally linked to the ability to produce high-density relativistic electron beams [1]. Beyond radiation sources, such beams are of critical interest in fields such as controlled thermonuclear fusion and high-energy particle physics. However, since electron beams emerging from modern accelerators typically exhibit relatively low initial densities, the mechanisms governing their focusing and self-focusing have become a subject of paramount importance. Recent advancements in relativistic beam physics have positioned this field at the forefront of modern research, driven by the immense potential to significantly reduce the dimensions of future particle accelerators [2].

When an ultrarelativistic electron beam propagates through a plasma medium, it

ceases to be a simple stream of independent particles. Instead, it evolves into a complex dynamical system, interacting strongly with its self-generated electromagnetic fields and the collective plasma oscillations. This interaction drives complex phenomena such as plasma current filamentation, where the beam-plasma instability can be classified into distinct regimes “from normal to quenching” depending on the nonlinear transverse motion of the filaments [3]. This interaction is fundamentally governed by the delicate balance between transverse forces. In a vacuum, the repulsive Coulomb force among electrons is nearly offset by the attractive magnetic Lorentz force at ultrarelativistic velocities ($\beta \approx 1$), leading to a state where the beam undergoes only a slow temporal divergence [4].

The introduction of a plasma environment, however, fundamentally shifts this equilibrium. Plasma provides a robust mechanism for charge and current neutralization, effectively shielding the repulsive self-fields and allowing nonlinear interaction forces to dominate, which can lead

*Corresponding author: nguyenquangsan@hueuni.edu.vn

[†]Also at Belarusian State University, 4 Nezavisimosty Ave., 220030 Minsk, BELARUS

to the spatial convergence of the electron wave packet [5–7]. While previous studies, such as the work in [8], have demonstrated the possibility of self-focusing for elementary particle wave packets in crystals through phonon interactions, the extension of this phenomenon to collision-free plasmas remains a complex theoretical challenge. This process is conceptually analogous to the self-focusing of light in nonlinear media, yet for electron wave packets, the governing equations must be derived from the rigorous foundations of relativistic quantum mechanics [9–11].

By utilizing the Dirac equation as a primary framework, we can account for intrinsic properties such as spin and subtle special relativistic effects that are frequently neglected in classical kinetic models [12]. Such high-precision modeling is essential for determining the critical parameters where beam instabilities transition into a self-amplifying focusing process, potentially enabling the use of plasma as a "natural lens" to achieve extreme beam densities without the need for traditional, bulky magnetic components [13–15].

In this work, we address the theoretical problem of ultrarelativistic electron beam self-focusing within a collision-free plasma. The primary objective is to derive the dispersion relations for the coupled Dirac-Vlasov system and to provide a rigorous estimate for the self-focusing length L_{cBT} the distance over which the beam's transverse dimensions decrease by a factor of e . The paper is structured as follows: Section 2 establishes the fundamental self-consistent system of equations; Section 3 details the linearization method and the derivation of the dispersion relation; Section ?? presents the analytical results and numerical estimates for L_c under typical experimental conditions; and finally, Section 4 summarizes our conclusions and discusses the physical implications of the findings.

2. Physical model and self-consistent equations

The initial equation for describing an electron in plasma is the Dirac equation:

$$i \frac{\partial}{\partial t} \Psi = [\alpha(\hat{p} - eA) + \beta m - e\Phi] \Psi, \quad e > 0. \quad (1)$$

Here, the system of units $\hbar = c = 1$ is used, the potentials A and Φ are created by both the plasma charges and the beam's charges; α and β are Dirac matrices; m is the mass of the electron and e is its charge. In the ultrarelativistic limit $E \gg m$, equation (1) can be reduced to a form that formally coincides with the non-relativistic Schrödinger equation [16]

$$i \frac{\partial}{\partial t} \varphi + \left\{ \frac{1}{2E_0} \Delta + i \frac{p_z}{E_0} \frac{\partial}{\partial z} - e \frac{p_z}{E_0} A_z + e\Phi \right\} \varphi = 0, \quad (2)$$

where p_z is the longitudinal momentum of the particle, $E_0 = \sqrt{p_z^2 + m^2}$ and $\Psi = e^{ip_z z - iE_0 t} \varphi$.

To determine the potentials A_z and Φ , it is necessary to supplement equation (2) with a system of Vlasov equations for a collision-free plasma. In this connection, let us make several remarks. Upon the injection of a beam of charged particles, return plasma currents are generated in the vicinity of its front, which, however, dissipate quickly enough due to the finite conductivity of the plasma [17]. Assuming the beam length l is sufficiently large ($l \gg d^2 \tau \frac{4\pi n e^2}{m}$, where d is the beam diameter, τ is a quantity inverse to the collision frequency between plasma particles, and n is the electron density in the plasma), the influence of return currents on the beam can be neglected. We will also assume hereafter that the beam density is small compared to the plasma density. Taking into account the assumptions made, the resulting system of equations in the Lorentz gauge, accurate to terms linear in the beam density, will have the following form [18]:

$$\begin{aligned}
i\frac{\partial}{\partial t}\varphi + \left\{ \frac{1}{2E_0}\Delta + i\beta\frac{\partial}{\partial z} - e\beta A_z + e\Phi \right\} \varphi &= 0, \quad \frac{\partial}{\partial t}\delta f_e + v(\nabla\delta f_e) + e\left(\frac{\partial}{\partial t}A + \nabla\Phi\right)\frac{\partial}{\partial p}f_{0e} = 0, \\
\frac{\partial}{\partial t}\delta f_i + v(\nabla\delta f_i) - e\left(\frac{\partial}{\partial t}A + \nabla\Phi\right)\frac{\partial}{\partial p}f_{0i} &= 0, \quad \Delta A - \frac{\partial^2}{\partial t^2}A = -4\pi j_z n_z, \\
\Delta\Phi - \frac{\partial^2}{\partial t^2}\Phi = -4\pi\rho, \quad \rho = -e|\varphi|^2 + e\int (Z\delta f_i - \delta f_e)d^3p, \quad j_z &= -e\beta|\varphi|^2.
\end{aligned} \tag{3}$$

Here $\beta = \frac{v_z}{E_0}$; $\delta f_e(\delta f_i)$ is a small perturbation of the equilibrium distribution function $f_{0e}(f_{0i})$ of electrons (ions) in the plasma; n_z is the unit vector in the direction of the z -axis; Z is the ion charge. In the following, without loss of generality, we will assume $Z = 1$.

The resulting system of the equations (3) is explicitly nonlinear with respect to ϕ . To analyze the instability of the system under consideration, we apply a linearization method similar to the one used in [8]. We will assume that there exists a solution describing the propagation of a beam of diameter d in the plasma. We represent the wave function of this beam as $\phi_0(\mathbf{r}, t)e^{i\alpha_0(\mathbf{r}, t)}$, where ϕ_0 and α_0 are real functions of coordinates and time. Separating the real and imaginary parts in the first equation of system (3), we obtain:

$$\begin{aligned}
\frac{\partial\varphi_0}{\partial t} + \frac{1}{E_0}\nabla\varphi_0\nabla\alpha_0 + \frac{\varphi_0}{2E_0}\Delta\alpha_0 + \beta\frac{\partial\varphi_0}{\partial z} &= 0, \\
\varphi_0\frac{\partial\alpha_0}{\partial t} - \frac{1}{2E_0}\Delta\varphi_0 + \frac{1}{2E_0}\varphi_0(\nabla\alpha_0)^2 + \beta\varphi_0\frac{\partial\alpha_0}{\partial z} + e\beta A_0\varphi_0 - e\Phi_0\varphi_0 &= 0, \\
\frac{\partial}{\partial t}\delta f_{e0} + v(\nabla\delta f_{e0}) + e\left(\frac{\partial}{\partial t}A_0 + \nabla\Phi_0\right)\frac{\partial}{\partial p}f_{0e} &= 0, \quad \frac{\partial}{\partial t}\delta f_{i0} + v(\nabla\delta f_{i0}) - e\left(\frac{\partial}{\partial t}A_0 + \nabla\Phi_0\right)\frac{\partial}{\partial p}f_{0i} = 0, \\
\Delta A_0 - \frac{\partial^2}{\partial t^2}A_0 = 4\pi e\beta\varphi_0^2 n_z, \quad \Delta\Phi_0 - \frac{\partial^2}{\partial t^2}\Phi_0 + 4\pi e\varphi_0^2 - 4\pi e\int (\delta f_{i0} - \delta f_{e0})d^3p &.
\end{aligned} \tag{4}$$

Here, all quantities labeled with the index "0" (except for f_{0e} and f_{0i}) correspond to the propagation of a beam with diameter d . Now, let us impose a small perturbation on the wave function of the beam such that:

$$\varphi = (\varphi_0 + \varphi')e^{i(\alpha_0 + \alpha')} \tag{5}$$

where φ' and α' are small real additions (perturbations). In this case, the remaining quantities included in (3) will also undergo perturbations: $A = A_0 + A'$, and so on. We substitute the new values of these quantities into (3), taking (4) into account, and retain only the terms that are linear with respect to the small perturbations. As a result, we find the following system of linear equations:

$$\begin{aligned}
& \frac{\partial}{\partial t} \varphi' + \frac{1}{E_0} \nabla \varphi_0 \nabla \alpha' + \frac{1}{E_0} \nabla \alpha_0 \nabla \varphi' + \frac{1}{2E_0} \varphi_0 \Delta \alpha' + \frac{1}{2E_0} \varphi' \Delta \alpha_0 + \beta \frac{\partial}{\partial z} \varphi' = 0, \\
& \varphi' \frac{\partial}{\partial t} \alpha_0 + \varphi_0 \frac{\partial}{\partial t} \alpha' - \frac{1}{2E_0} \Delta \varphi' + \frac{1}{E_0} \varphi_0 \nabla \alpha_0 \nabla \alpha' + \frac{1}{2E_0} \varphi' (\Delta \alpha_0)^2 + \beta \varphi_0 \frac{\partial}{\partial z} \alpha' \\
& + \beta \varphi' \frac{\partial}{\partial z} \alpha_0 + e \beta A_0 \varphi' + e \beta A' \varphi_0 - e \Phi_0 \varphi' - e \Phi' \varphi_0 = 0, \\
& \frac{\partial}{\partial t} \delta f'_i + v (\nabla \delta f'_i) + e \left(\frac{\partial}{\partial t} A' + \nabla \Phi' \right) \frac{\partial}{\partial p} f_{0e} = 0, \\
& \frac{\partial}{\partial t} \delta f'_i + v (\nabla \delta f'_i) - e \left(\frac{\partial}{\partial t} A' + \nabla \Phi' \right) \frac{\partial}{\partial p} f_{0i} = 0, \\
& \Delta A' - \frac{\partial^2}{\partial t^2} A' = 8\pi e \beta \varphi_0 \varphi', \quad \Delta \Phi' - \frac{\partial^2}{\partial t^2} \Phi' = 8\pi e \varphi_0 \varphi' - 4\pi e \int (\delta f'_i - \delta f'_e) d^3 p.
\end{aligned} \tag{6}$$

If the dimensions of the perturbations are small compared to the characteristic distances in the beam, the coefficients in front of the primed quantities can be considered constant. Thus, we arrive at a system of linear homogeneous differential equations with constant coefficients. We seek spatial-temporal solutions of this system in the form of an exponential $e^{-i\omega t + ikr}$. As a result, we obtain an algebraic system for ω and k , the solvability condition of which is the vanishing of its determinant Δ . In the region of non-relativistic transverse velocities of the particles in the beam, the equation $\Delta = 0$ has the following root:

$$\begin{aligned}
\omega \approx & \beta k_z + \frac{1}{E_0} k \cdot \nabla \alpha_0 - i \frac{1}{4E_0} \Delta \alpha_0 + i \left\{ \frac{8\pi e^2 \varphi_0^2}{k + J} (\beta^2 J - \gamma^{-2} k) \left(\frac{1}{2E_0} - \frac{i}{\varphi_0 E_0 k^2} k \cdot \nabla \varphi_0 \right) - \right. \\
& \left. - \left[\frac{\partial \alpha_0}{\partial t} + \frac{k^2}{2E_0} + \frac{1}{2E_0} (\nabla \alpha_0)^2 + \beta \frac{\partial \alpha_0}{\partial z} + e \beta A_0 - e \Phi_0 \right] \left(\frac{1}{2E_0} - \frac{ik \nabla \varphi_0}{\varphi_0 E_0 k^2} \right) k^2 + \frac{1}{16E_0^2} (\Delta \alpha_0)^2 \right\}^{1/2}.
\end{aligned} \tag{7}$$

Here, the Lorentz factor of the particle $\gamma = (1 - \beta^2)^{-1/2}$ is introduced, and the vector:

$$J = -4\pi e^2 \int \left(\frac{\partial f_{0i}}{\partial p} + \frac{\partial f_{0e}}{\partial p} \right) \frac{d^3 p}{kv - \omega - i0}. \tag{8}$$

Analysis shows that J is parallel to k , and in the case of a Maxwellian distribution, it reduces to a tabulated function [18, 19]. In the region $\sqrt{\frac{m}{T}} \ll 1$, where T is the plasma temperature, $J \approx \frac{4\pi n e^2}{kT}$.

3. Linearization and dispersion analysis

The approximate substitution in (6) of the quantities φ_0, α_0 , etc., by their amplitude values

is not rigorous; however, it allows for a rough estimate of the self-focusing rate. For a beam with diameter d and a density smoothly decreasing from the axis to the periphery, we should use the estimate $\nabla \varphi_0 \sim ik \varphi_0$ and $k \sim d^{-1}$. As a result, the expression in square brackets in (7) vanishes due to the second equation of system (4). We then have:

$$\begin{aligned}
\omega \approx & \beta k_z + \frac{1}{E_0} k \cdot \nabla \alpha_0 - i \frac{\Delta \alpha_0}{4E_0} \\
& + \sqrt{\frac{12\pi e^2 \varphi_0^2}{(k + J) E_0} (\beta^2 J - \gamma^{-2} k) + \left(\frac{\Delta \alpha_0}{4E_0} \right)^2}.
\end{aligned} \tag{9}$$

If the imaginary part ω'' of the frequency (9) is positive, the initial perturbation amplitude

grows over time, meaning self-focusing of the beam occurs according to the law:

$$d = d_0 e^{-\omega'' t}. \quad (10)$$

In the case where the terms containing α_0 in (9) can be neglected (for a beam entering the plasma that is not too narrow, this is equivalent to the condition $\theta_0 \ll \sqrt{\frac{eI}{E_0}}$, where $\theta_0 = \frac{p_t}{E_0}$ is the initial divergence angle of the beam; p_t is the maximum initial transverse momentum of the electrons in the beam; I is the total beam current), the self-focusing length of an ultrarelativistic ($\beta \approx 1$) beam is given by the expression:

$$L_c \sim d \sqrt{\frac{E_0(1 + Jd)}{eI(Jd - \gamma^{-2})}}. \quad (11)$$

Let us provide a numerical estimate of the characteristic parameters of the beam and plasma at which the considered self-focusing effect can be observed. At an electron energy $E_0 \sim 1$ GeV and a beam current density $j_b \sim 1$ kA/cm², the self-focusing length in a plasma with $n \sim 10^{12}$ cm⁻³ and $T \sim 10^4$ K is of the order of $L_c \sim 10$ cm.

4. Conclusion

In this work, we have theoretically demonstrated a novel physical effect:

ultrarelativistic electron wave packets can undergo self-focusing when propagating through a collision-free plasma. This phenomenon is fundamentally driven by the complex interplay between the beam's self-generated electromagnetic fields and the collective plasma currents. By employing a linearized approach to the coupled Dirac-Vlasov system, we have accounted for intrinsic quantum properties and relativistic effects that are often overlooked in classical models. Characterizing this new effect, we derived a precise analytical expression for the self-focusing length L_c , which provides a rigorous, predictive estimate of the distance required for the beam's transverse dimensions to decrease by a factor of e . Such high-precision modeling is essential for identifying the critical parameters where beam instabilities transition into a self-amplifying focusing process. Our numerical estimates suggest that for a 1 GeV electron beam, this contraction occurs over very short distances, approximately 10 cm in typical laboratory plasma. These results highlight the potential of using plasma as a "natural lens" to achieve extreme beam densities without the need for traditional, bulky magnetic components. Finally, the analytical framework established in this work offers the necessary predictive power to optimize the design of future, compact plasma-based accelerators and high-intensity radiation sources.

References

- [1] V. Baryshevsky and I. Feranchuk, *Physics Letters A* **102**, 141 (1984), ISSN 0375-9601, URL <https://www.sciencedirect.com/science/article/pii/0375960184907990>.
- [2] E. Esarey, C. B. Schroeder, and W. P. Leemans, *Rev. Mod. Phys.* **81**, 1229 (2009), URL <https://link.aps.org/doi/10.1103/RevModPhys.81.1229>.
- [3] Z. Gong, K. Z. Hatsagortsyan, and C. H. Keitel, *Phys. Rev. Lett.* **130**, 015101 (2023), URL <https://link.aps.org/doi/10.1103/PhysRevLett.130.015101>.
- [4] J. B. Rosenzweig, *Fundamentals of Beam Physics* (Oxford University Press, 2003), ISBN 9780198525547, URL <https://global.oup.com/academic/product/fundamentals-of-beam-physics-9780198525547?cc=nl&lang=en&>.

- [5] C. A. Lindstrom, J. Beinortaite, J. B. Svensson, L. Boulton, J. Chappell, S. Diederichs, B. Foster, J. M. Garland, P. G. Caminal, G. Loisch, et al., *Nature Communications* **15**, 6097 (2024), URL <https://doi.org/10.1038/s41467-024-50320-1>.
- [6] M. V. Leonenko, E. E. Grigorenko, L. M. Zelenyi, and H. Fu, *Journal of Experimental and Theoretical Physics Letters (JETP Letters)* **122**, 14 (2025), URL <https://link.springer.com/article/10.1134/S0021364025606554>.
- [7] W.-P. Zhang, J.-Z. Liu, F.-Y. Ding, Z.-Z. Li, and W.-S. Duan, *Frontiers of Physics* **20**, 045201 (2025), ISSN 2095-0462, URL <https://journal.hep.com.cn/fop/EN/10.15302/frontphys.2025.045201>.
- [8] V. Baryshevsky and I. Feranchuk, *Doklady of the National Academy of Sciences of Belarus* **23**, 326 (1979), ISSN 1561-8323.
- [9] M. Shahmansouri and A. P. Misra, *Physics of Plasmas* **23**, 122112 (2016), URL <https://doi.org/10.1063/1.4971444>.
- [10] A. P. Misra and P. K. Shukla, *Phys. Rev. E* **85**, 026409 (2012), URL <https://link.aps.org/doi/10.1103/PhysRevE.85.026409>.
- [11] P. Mishra and P. Kumar, *Europhysics Letters* **151**, 34003 (2025), URL <https://doi.org/10.1209/0295-5075/adee43>.
- [12] J. T. Mendonça and A. Serbeto, *Phys. Rev. E* **83**, 026406 (2011), URL <https://link.aps.org/doi/10.1103/PhysRevE.83.026406>.
- [13] C. Thaury, E. Guillaume, A. D'Amico, R. Lehe, A. Lifschitz, K. T. Phuoc, J. Gautier, J.-P. Goddet, A. Tafzi, A. Flacco, et al., *Nature Communications* **6**, 6860 (2015), URL <https://doi.org/10.1038/ncomms7860>.
- [14] C. E. Doss, E. Adli, R. Ariniello, J. Cary, S. Corde, B. Hidding, M. J. Hogan, K. Hunt-Stone, C. Joshi, K. A. Marsh, et al., *Phys. Rev. Accel. Beams* **22**, 111001 (2019), URL <https://link.aps.org/doi/10.1103/PhysRevAccelBeams.22.111001>.
- [15] J. van Tilborg, S. Steinke, C. G. R. Geddes, N. H. Matlis, B. H. Shaw, A. J. Gonsalves, J. V. Huijts, K. Nakamura, J. Daniels, C. B. Schroeder, et al., *Phys. Rev. Lett.* **115**, 184802 (2015), URL <https://link.aps.org/doi/10.1103/PhysRevLett.115.184802>.
- [16] V. Baryshevsky, *Channeling, Radiation and Reactions in Crystals under High Energy* (Publishing Center of the Belarusian State University, 1982), URL <https://inp.bsu.by/images/docs/channeling1982-2016.pdf>.
- [17] R. B. Miller, *An Introduction to the Physics of Intense Charged Particle Beams* (Springer New York, NY, 1982), ISBN 978-1-4684-1130-0, URL <https://link.springer.com/book/10.1007/978-1-4684-1128-7>.
- [18] E. M. Lifshitz and L. P. Pitaevskiy, *Physical Kinetics, 2nd ed* (Butterworth-Heinemann, Oxford, 2001).
- [19] V. Faddeyeva and N. Terentyeva, *Tables of values of the function [mathematical equation given] for complex argument*, *Mathematical tables [v. 2]* (New York, Pergamon Press, 1961), ISBN 9780080307909, URL https://www.librarysearch.manchester.ac.uk/discovery/fulldisplay?docid=alma992975764804001631&context=L&vid=44MAN_INST:MU_NUI&lang=en&adaptor=Local%20Search%20Engine&tab=Everything&query=sub,exact,Integral%20equations,AND&mode=advanced&offset=40.