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Enhanced quantum features of non-Gaussian states by applying a two-mode squeezing operator to two generalized single-mode squeezed vacuum states

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Abstract

This paper investigates the quantum features of non-Gaussian states generated by applying a two-mode squeezing operator $\hat{S}_{ab}(s)$ to two generalized single-mode squeezed vacuum states (SVSs), each formed by the superposition of an SVS and a single-photon-subtracted SVS. The results show that, in the new states, the two-mode squeezing effect, the Einstein–Podolsky–Rosen (EPR) correlation, the Hillery–Zubairy entanglement, and the linear entropy are all enhanced as the two-mode squeezing parameter s increases. Moreover, both the degrees of squeezing and entanglement are improved and exceed those of the two-mode squeezed vacuum state (TMSVS) for stronger single-mode squeezing parameters. The optimal conditions for achieving such enhancements are identified for different regimes, depending on whether the generalized SVSs are purely Gaussian, maximally non-Gaussian, or intermediate states. In an application, the quantum Cramér–Rao bound (QCRB) is evaluated when the non-Gaussian states are employed as inputs to a Mach–Zehnder interferometer (MZI). The results further demonstrate that the enhanced phase sensitivity in the MZI originates from both the squeezing strengths and the intrinsic photon-subtraction operation. These findings highlight the potential of the new non-Gaussian states as promising resources for quantum metrology and quantum sensing.

1. Introduction

Single-mode nonclassical states are widely employed in various quantum tasks, such as gravitational wave detection [1, 2], quantum key distribution [3, 4], quantum metrology [5], quantum transduction [6], and tests of the fundamental laws of quantum mechanics [7], among others. Within this category, the classes of single-mode squeezed states such as the SVS, photon-added SVS, photon-subtracted SVS [8, 9], and superpositions of photon-added and/or photon-subtracted SVS [10, 11] play an important role in quantum optics. However, these states can be expanded solely in terms of either even or odd photon-number components, indicating that their photon-number distributions are incomplete (or gapped). Such incompleteness may limit their applicability in quantum information processing. To overcome this limitation, a coherent superposition of the even- and odd-parity squeezed states has recently been proposed, yielding a complete photon-number distribution. This configuration, referred to as the qubit-superposed single-mode SVSs (qSSVS), was introduced in [12]. This state exhibits enhanced squeezing strength over the entire range of the squeezing parameter compared with the original SVS. Nevertheless, neither the individual qSSVS nor the other aforementioned single-mode states have yet to be utilized in two-mode or multimode quantum protocols that rely on intermode correlations, such as entanglement.

Continuous-variable entangled states are essential physical resources for quantum information protocols, including quantum teleportation [13], quantum key distribution [14], superdense coding [15], quantum metrology [16], and quantum imaging and sensing [17, 18]. Various schemes have been proposed to generate two-mode entangled states derived from SVSs. For example, Liu [19] introduced a coherent superposition of photon-subtracted SVSs, while Zhang [20] proposed generating entanglement by splitting an SVS using a beam splitter. More recently, Song [21] has demonstrated strong entanglement in states obtained via photon subtraction from two SVSs. Superpositions of squeezed states mediated by cross-Kerr nonlinearity have also been investigated by Azuma [22]. Furthermore, entangled resources have been realized through superradiance and rotating quantum fluids of light [23]. In addition, several methods for generating different entangled states from SVSs have been proposed very recently [24, 25]. Collectively, these schemes yield highly entangled states, particularly effective for quantum applications in the regime of small squeezing parameters. However, extending such schemes to incorporate qSSVS could potentially offer richer two-mode squeezing and stronger entanglement over a broader parameter range, motivating the present study.

In this work, we propose and investigate a new family of non-Gaussian states obtained by applying the two-mode squeezing operator to two generalized single-mode SVSs, specifically two distinct qSSVSs. These new non-Gaussian states, referred to as two-mode-squeezed qSSVSs (TMS-qSSVSs), can be regarded as a generalized form that encompasses several previously known states, including the SVS, the single-photon-subtracted SVS, and the TMSVS. The quantum features of the TMS-qSSVSs are verified through the two-mode squeezing effect, the EPR correlation, the Hillery–Zubairy entanglement, and the linear entropy. In addition, we discuss the enhanced degree of entanglement in the TMS-qSSVSs compared with the states generated by the interference of two SVSs at a single or double beam splitter [26, 27]. Furthermore, when the TMS-qSSVSs are employed as input resources of the MZI, an enhancement of phase sensitivity is demonstrated.

The paper is organized as follows. Section 2 introduces the TMS-qSSVSs. Section 3 focuses on the quantum properties of the TMS-qSSVSs, and section 4 discusses the enhancement of phase sensitivity in the MZI. The main conclusions are summarized in section 5.

2. The TMS-qSSVSs

We begin with a brief review of the SVS, which is a typical Gaussian state. This state is generated by applying the single-mode squeezing operator $\hat{s}(r) = e^{-r(\hat{a}^{\dagger 2} - \hat{a}^2)/2}$ to the vacuum state, i.e., $|r\rangle = \hat{s}(r)|0\rangle$. Here, $\hat{a}^{\dagger}$ and $\hat{a}$ denote the photon creation and annihilation operators, respectively, and r is the single-mode squeezing parameter, which is taken to be real for simplicity. In the Fock basis, the SVS is expressed as

$$|r\rangle = \sqrt{\text{sech } r} \sum_{n=0}^{\infty} \left(-\frac{\tanh r}{2} \right)^n \frac{\sqrt{(2n)!}}{n!} |2n\rangle. \quad (1)$$

As shown in equation (1), the SVS is a superposition of only even-photon-number states, i.e., $|0\rangle, |2\rangle, |4\rangle$, and so on. To generate a squeezed state containing only odd photon-number components, researchers have investigated single-photon subtraction from an SVS by applying the annihilation operator, which represents a typical non-Gaussian operation. This operation yields the single-photon-subtracted SVS, defined as $|r, 1\rangle = \hat{a}|r\rangle$, which is expressed in the normalized form as follows

$$|r, 1\rangle = \frac{\sqrt{\text{sech } r}}{\sinh r} \sum_{n=0}^{\infty} \left(-\frac{\tanh r}{2} \right)^n \frac{\sqrt{(2n)!(2n)}}{n!} |2n-1\rangle. \quad (2)$$

The single-photon-subtracted SVS can be expanded in terms of odd-photon-number states, such as $|1\rangle, |3\rangle, |5\rangle$, and so forth. Therefore, the SVS and the single-photon-subtracted SVS are mutually orthogonal. Recently, owing to its advantageous nonclassical properties, a superposition of an SVS and a single-photon-subtracted SVS, hereafter referred to as the qSSVS, has been proposed and investigated [12]. This state is given by

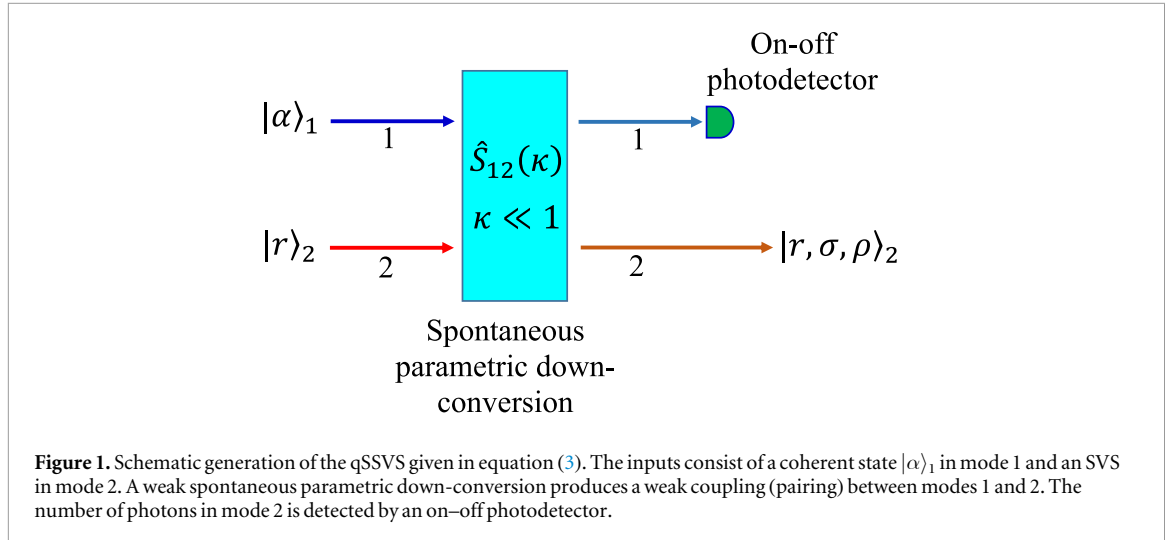
$$|r, \sigma, \rho\rangle = \sigma|r\rangle + \rho|r, 1\rangle, \quad (3)$$

where σ and ρ are assumed to be real coefficients. The qSSVS is normalized as

$$|r, \sigma, \rho\rangle = \mathcal{N}(\sigma + \rho\hat{a})|r\rangle, \quad (4)$$

where $\mathcal{N} = (\sigma^2 + \rho^2 \sinh^2 r)^{-1/2}$. The qSSVS exhibits notable nonclassical properties, such as enhanced squeezing over the entire range of the squeezing parameter r , non-Gaussianity, and others. It is noted that when $\rho = 0$ or $\sigma = 0$, the qSSVS reduces to the SVS or the single-photon-subtracted SVS, respectively. In each case, the remaining coefficient (σ or ρ) vanishes from both the state and its normalization factor.

In the following, we propose the TMS-qSSVSs of two modes a and b by applying the two-mode squeezing operator $\hat{S}_{ab}(s) = e^{s(\hat{a}^{\dagger}\hat{b}^{\dagger} - \hat{a}\hat{b})}$ to two distinct qSSVSs, as follows



$$|s, r_1, r_2\rangle_{ab} = \hat{S}_{ab}(s)|r_1, \sigma_1, \rho_1\rangle_a \otimes |r_2, \sigma_2, \rho_2\rangle_b. \quad (5)$$

Here, s is referred to as the two-mode squeezing parameter, which characterizes the coupling strength between modes a and b , and we choose $s > 0$ for simplicity. The states $|r_1, \sigma_1, \rho_1\rangle_a$ and $|r_2, \sigma_2, \rho_2\rangle_b$ are defined in the same way as in equation (4), with the parameter r and the coefficients σ and ρ replaced by r_1, σ_1, ρ_1 and r_2, σ_2, ρ_2 , respectively, corresponding to modes a and b . The two-mode squeezing operator satisfies $\hat{S}_{ab}^\dagger(s)\hat{S}_{ab}(s) = \hat{S}_{ab}(s)\hat{S}_{ab}^\dagger(s) = \mathbf{I}$; thus, the TMS-qSSVSs in equation (5) are self-normalized. When the states $|r_1, \sigma_1, \rho_1\rangle_a$ and $|r_2, \sigma_2, \rho_2\rangle_b$ reduce to vacuum states, corresponding to $r_1 = r_2 = 0$, the TMS-qSSVSs become identical to the TMSVSs.

In the case $\rho_1 = \rho_2 = 0$, the operator $\hat{S}_{ab}(s)$ acts on two SVSs; thus, the TMS-qSSVSs are purely Gaussian. In addition, when $\sigma_1 = \sigma_2 = 0$, the second special case corresponds to $\hat{S}_{ab}(s)$ acting on two single-photon-subtracted SVSs; that is, the input states are non-Gaussian, and consequently, the TMS-qSSVSs are also non-Gaussian (here, the term ‘input states’ refers to the states before the action of $\hat{S}_{ab}(s)$). In the remaining cases, the input qSSVSs exhibit tunable non-Gaussianity, which can be adjusted through the coefficients $\sigma_1, \sigma_2, \rho_1$, and ρ_2 . Therefore, the TMS-qSSVSs display strong (weak) non-Gaussianity when $\sigma_j \ll \rho_j$ ($\sigma_j \gg \rho_j$), respectively, with $j \in \{1, 2\}$. In the following analyses, we choose the values of these coefficients, as well as the single-mode and two-mode squeezing parameters, such that the quantum features are as distinct and illustrative as possible. It is worth noting that, in the cases of $\sigma_1 = \sigma_2 = 0$ or $\rho_1 = \rho_2 = 0$, the TMS-qSSVSs become independent of the remaining parameters, i.e., ρ_1, ρ_2 or σ_1, σ_2 , respectively. As a result, their quantum features become invariant with respect to these coefficients.

Throughout this paper, we analyze the quantum characteristics of the TMS-qSSVSs from two complementary perspectives. First, we clarify how these properties are either enhanced or suppressed by varying the single-mode and two-mode squeezing parameters, while keeping the coefficients σ_j and ρ_j fixed (with $j \in \{1, 2\}$). Second, by fixing the single-mode and two-mode squeezing parameters, we explore the dependence of these quantum characteristics on the coefficients σ_j and ρ_j . In this direction, our discussion focuses on three representative types of input states: the SVSs (with $\rho_j = 0$), the single-photon-subtracted SVSs (with $\sigma_j = 0$), and the qSSVSs, which interpolate between the two. Based on this analysis, we identify the input states that optimize the quantum properties under consideration—namely, the purely Gaussian, maximally non-Gaussian, and intermediate qSSVS regimes.

As is well known, the SVS and the single-photon-subtracted SVS have already been generated experimentally. Moreover, the qSSVS can be produced using available optical devices such as beam splitters and on-off photodetectors [12]. In addition, we find that the qSSVS can also be generated via a simple scheme consisting of a laser beam, a weak spontaneous parametric down-conversion (SPDC), and an on-off photodetector, as sketched in figure 1. In this scheme, a laser beam characterized by a coherent state $|\alpha\rangle_1$ of mode 1 and an SVS of mode 2 are injected into the two input ports of a weak SPDC characterized by the two-mode squeezing operator $\hat{S}_{12}(\kappa)$, with $\kappa \ll 1$. The state after the interaction in the SPDC is determined as

$$|\psi\rangle_{12} = \hat{S}_{12}(\kappa)|\alpha\rangle_1|r\rangle_2, \quad (6)$$

where $|\alpha\rangle_1$ is a coherent state, i.e., the eigenstate of the annihilation operator $\hat{a}_1$, satisfying $\hat{a}_1|\alpha\rangle_1 = \alpha|\alpha\rangle_1$, and $|r\rangle_2$ is the SVS given in equation (1). The two-mode squeezing operator is written as $\hat{S}_{12}(\kappa) = e^{\kappa(\hat{a}_1^\dagger \hat{a}_2^\dagger - \hat{a}_1 \hat{a}_2)}$. For $\kappa \ll 1$, one can approximate $\hat{S}_{12}(\kappa) \approx 1 + \kappa(\hat{a}_1^\dagger \hat{a}_2^\dagger - \hat{a}_1 \hat{a}_2)$. Therefore, the state in equation (6) can be expressed as

$$\begin{aligned}
|\psi\rangle_{12} &\approx 1 + \kappa(\hat{a}_1^\dagger \hat{a}_2^\dagger - \hat{a}_1 \hat{a}_2)|\alpha\rangle_1 |r\rangle_2 \\
&= |\alpha\rangle_1 |r\rangle_2 - \kappa\alpha|\alpha\rangle_1 \hat{a}_2 |r\rangle_2 + \kappa\hat{a}_1^\dagger \hat{a}_2^\dagger |\alpha\rangle_1 |r\rangle_2.
\end{aligned} \tag{7}$$

At the end of the propagation path of mode 1, a photodetector is placed to detect zero photons in this mode (corresponding to the ‘off’ outcome of the on-off photodetector). The final state is then given by

$$\begin{aligned}
|\psi\rangle_2 &= {}_1\langle 0|\alpha\rangle_1 |r\rangle_2 - \kappa\alpha {}_1\langle 0|\alpha\rangle_1 \hat{a}_2 |r\rangle_2 + \kappa {}_1\langle 0|\hat{a}_1^\dagger |\alpha\rangle_1 \hat{a}_2^\dagger |r\rangle_2 \\
&= e^{-|\alpha|^2/2} |r\rangle_1 - \kappa\alpha e^{-|\alpha|^2/2} \hat{a}_2 |r\rangle_2.
\end{aligned} \tag{8}$$

By setting $\sigma = e^{-|\alpha|^2/2}$ and $\rho = -\kappa\alpha e^{-|\alpha|^2/2}$, the state in equation (8) reduces to the qSSVS given in equation (3). To generate the TMS-qSSVSs, it suffices to inject two distinct qSSVSs into the two input ports of another SPDC. Therefore, the generation of the TMS-qSSVSs is completely feasible.

3. Quantum features of the TMS-qSSVSs

3.1. Two-mode squeezing property

The existence of two-mode squeezing in a state ensures that it belongs to the class of nonclassical states. The two-mode squeezing effect is utilized in many quantum tasks, including quantum information [28, 29] and quantum sensing [30, 31]. Considering the quadrature-phase operators corresponding to two modes a and b as $\hat{X}_x = (\hat{x} + \hat{x}^\dagger)/\sqrt{2}$ and $\hat{P}_x = (\hat{x} - \hat{x}^\dagger)/(i\sqrt{2})$, where $\hat{x}$ ($\hat{x}^\dagger$) denotes the annihilation (creation) operator of mode x , $x = a, b$. The two-mode quadrature operators are defined as $\hat{X} = (\hat{X}_a + \hat{X}_b)/\sqrt{2}$ and $\hat{P} = (\hat{P}_a + \hat{P}_b)/\sqrt{2}$, and they satisfy the uncertainty relation $\Delta^2 \hat{X} \Delta^2 \hat{P} \geq 1/4$, where $\Delta^2 \hat{z} = \langle \hat{z}^2 \rangle - \langle \hat{z} \rangle^2$ (with $z = X, P$). A state is squeezed in the operator $\hat{X}$ (or $\hat{P}$) if $\Delta^2 \hat{X} < 1/2$ (or $\Delta^2 \hat{P} < 1/2$) [32].

For the TMS-qSSVSs, the variances of the two operators, $\hat{X}$ and $\hat{P}$, are given by

$$\begin{aligned}
\Delta^2 \hat{X} &= \frac{\langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{b}^\dagger \hat{b} \rangle + 2\langle \hat{a}^\dagger \hat{b} \rangle + 2\langle \hat{a} \hat{b}^\dagger \rangle + \langle \hat{a}^2 \rangle + \langle \hat{b}^2 \rangle + 1}{2} \\
&\quad - (\langle \hat{a} \rangle + \langle \hat{b} \rangle)^2,
\end{aligned} \tag{9}$$

$$\Delta^2 \hat{P} = \frac{\langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{b}^\dagger \hat{b} \rangle + 2\langle \hat{a}^\dagger \hat{b} \rangle - 2\langle \hat{a} \hat{b}^\dagger \rangle - \langle \hat{a}^2 \rangle - \langle \hat{b}^2 \rangle + 1}{2}. \tag{10}$$

For convenience, we introduce the following shorthand notations for the quantum expectation values:

$N_x = \langle \hat{x}^\dagger \hat{x} \rangle$, $K_{jx} = \langle \hat{x}^j \rangle$, $x = a, b$; $j \in \{1, 2\}$, and $K_{1ab} = \langle \hat{a}^\dagger \hat{b} \rangle$, $K_{2ab} = \langle \hat{a} \hat{b}^\dagger \rangle$. In terms of these quantities, the variances can then be rewritten as

$$\Delta^2 \hat{X} = \frac{N_a + N_b + 2K_{1ab} + 2K_{2ab} + K_{2a} + K_{2b} + 1}{2} - (K_{1a} + K_{1b})^2, \tag{11}$$

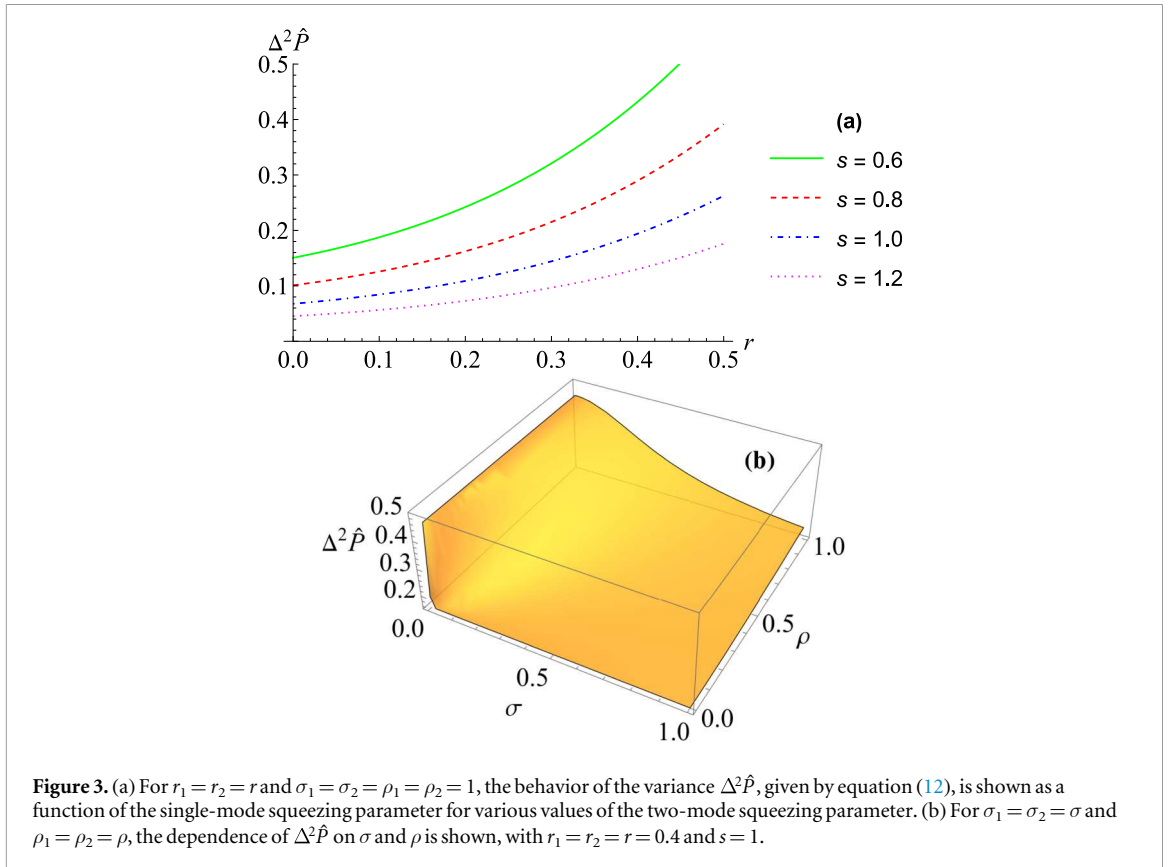
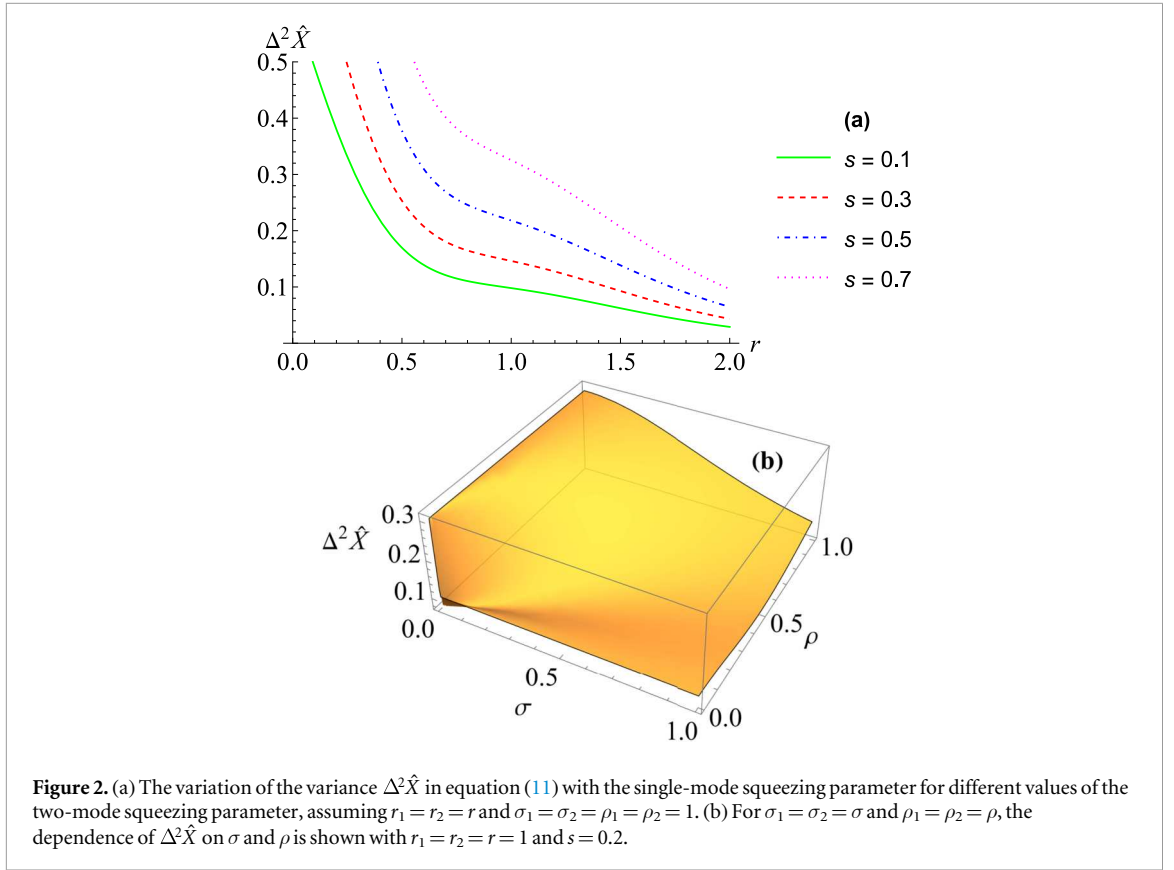
$$\Delta^2 \hat{P} = \frac{N_a + N_b + 2K_{1ab} - 2K_{2ab} - K_{2a} - K_{2b} + 1}{2}. \tag{12}$$

The corresponding expressions for the quantum expectation values N_x , K_{jx} , K_{1ab} , and K_{2ab} are provided in equations (A32), (A35)–(A39), (A42), and (A43) of the appendix.

Firstly, using the result in equation (11), we investigate the two-mode squeezing effect associated with the operator $\hat{X}$. Figure 2(a) presents the variance $\Delta^2 \hat{X}$ as a function of the single-mode squeezing parameter for several values of the two-mode squeezing parameter s , where the input states of modes a and b are taken to be identical symmetric qSSVSs. The squeezing effect becomes more pronounced, i.e., the values of $\Delta^2 \hat{X}$ decrease, as the single-mode squeezing parameter ($r_1 = r_2 = r$) increases. In addition, the variance $\Delta^2 \hat{X}$ increases with larger values of s .

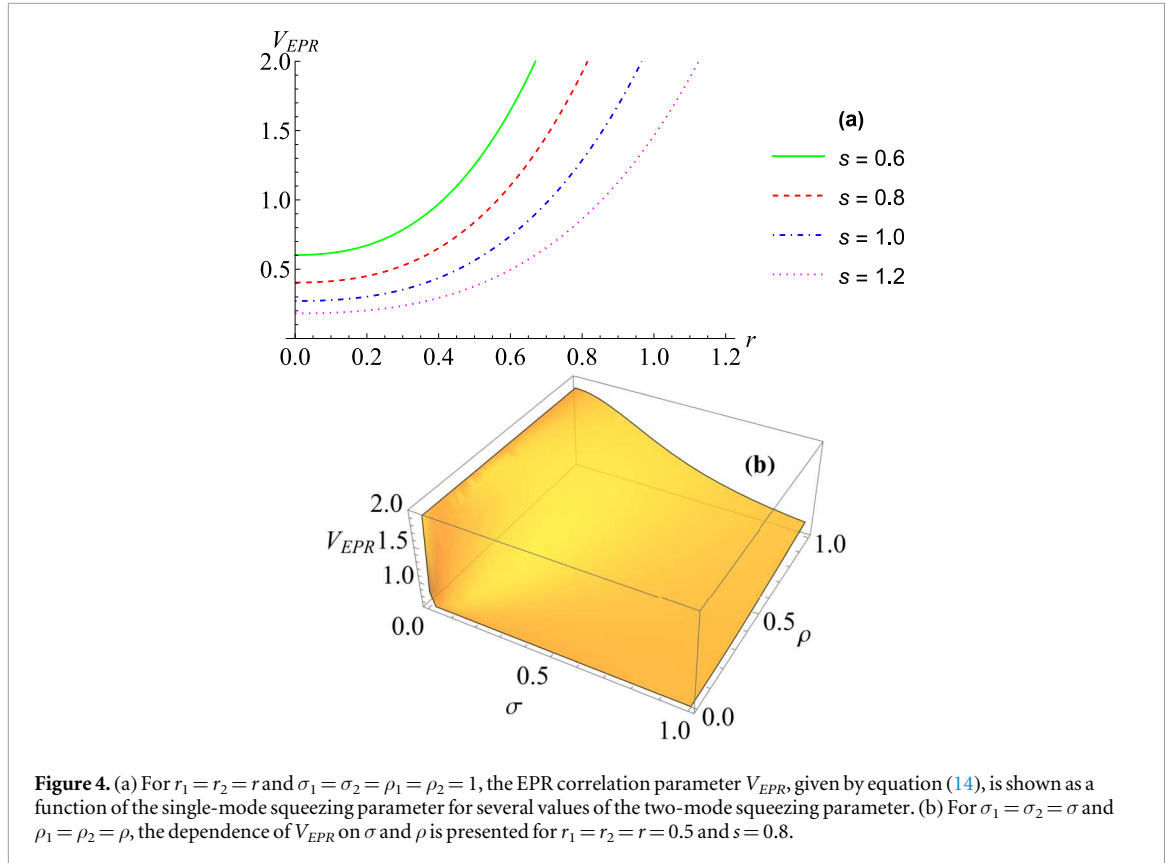
For the case where the input states of modes a and b are identical, figure 2(b) shows the behavior of the variance $\Delta^2 \hat{X}$ as a function of the coefficients $\sigma_1 = \sigma_2 = \sigma$ and $\rho_1 = \rho_2 = \rho$. It should be remembered that when $\rho = 0$ (or $\sigma = 0$), the qSSVSs reduce to the SVSs (or the single-photon-subtracted SVSs). An interesting feature of the two-mode squeezing effect associated with the operator $\hat{X}$ is that the variance $\Delta^2 \hat{X}$ does not reach a minimum at $\rho = 0$ or $\sigma = 0$. This occurs because, in qSSVS, the degree of squeezing is enhanced when $\sigma = 1$ and ρ takes a moderately small value [12]. It is also noted that the degree of two-mode squeezing in the TMS-qSSVSs is higher than that in the TMSVSs.

Next, we examine the two-mode squeezing effect associated with the operator $\hat{P}$. In figure 3(a), the variance $\Delta^2 \hat{P}$ is plotted as a function of the single-mode squeezing parameter ($r_1 = r_2 = r$) for increasing values of s , with $\sigma_1 = \sigma_2 = \rho_1 = \rho_2 = 1$ kept fixed. The results show that the variance $\Delta^2 \hat{P}$ diminishes as r decreases or s increases. In figure 3(b), with the single-mode and two-mode squeezing parameters held constant, and for $\sigma_1 = \sigma_2 = \sigma$ and $\rho_1 = \rho_2 = \rho$, the variance $\Delta^2 \hat{P}$ attains its minimum value at $\rho = 0$, corresponding to the input states exhibiting the strongest Gaussianity.



3.2. The EPR correlation

The EPR correlation is employed in many quantum protocols, such as quantum teleportation [33] and quantum key distribution [34]. In these protocols, enhancing the EPR correlation improves their performance.



To test the EPR correlation in the TMS-qSSVSs, we use the operators $\hat{X}_x = (\hat{x} + \hat{x}^\dagger)/\sqrt{2}$ and $\hat{P}_x = (\hat{x} - \hat{x}^\dagger)/(i\sqrt{2})$, ($x = a, b$). The EPR correlation parameter is defined as [32]

$$V_{EPR} = \Delta^2(\hat{X}_a - \hat{X}_b) + \Delta^2(\hat{P}_a + \hat{P}_b). \quad (13)$$

A two-mode state exhibits the EPR correlation when the parameter $V_{EPR} < 2$. For the TMS-qSSVSs, the parameter V_{EPR} is calculated as

$$\begin{aligned} V_{EPR} &= 2(\langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{b}^\dagger \hat{b} \rangle - 2\langle \hat{a}^\dagger \hat{b}^\dagger \rangle) - 2(\langle \hat{a} \rangle - \langle \hat{b} \rangle)^2 + 2 \\ &= 2(N_a + N_b - 2K_{2ab}) - 2(K_{1a} - K_{1b})^2 + 2. \end{aligned} \quad (14)$$

Here, K_{1a} , N_a , K_{1b} , N_b , and K_{2ab} are defined in section 3.1, and the corresponding expressions are shown, respectively, in equations (A32), (A35)–(A37), and (A43) of the appendix.

Based on equation (14), to clarify the EPR correlation in the TMS-qSSVSs, we also focus on two main aspects as follows. First, when the input states are two identical qSSVSs in a symmetric configuration, the EPR correlation is examined as a function of the single-mode squeezing parameter at fixed values of the two-mode squeezing parameter. Second, when the inputs are the two identical qSSVSs, we investigate the EPR correlation with respect to the coefficients σ and ρ . These coefficients allow us to flexibly adjust the nature of the input states, such as the SVSs, the single-photon-subtracted SVSs, or the qSSVSs with varying the degree of non-Gaussianity. In the first case, figure 4(a) shows that the EPR correlation emerges in the TMS-qSSVSs and strongly depends on both the single-mode and two-mode squeezing parameters; that is, the value of V_{EPR} decreases with increasing s or decreasing r . This implies that the EPR correlation is enhanced by reducing the single-mode squeezing parameter or increasing the two-mode squeezing parameter.

In the second case, as shown in figure 4(b), the highest EPR correlation is obtained for $\rho = 0$, i.e., the input states exhibit the strongest Gaussian character. This behavior resembles the two-mode squeezing effect in the operator $\hat{P}$. This is consistent, since for the enhanced TMSVSs, the squeezing in $\hat{P}$ and V_{EPR} are proportional to each other [32]. Compared with the TMSVSs, the EPR correlation in the TMS-qSSVSs is slightly reduced.

3.3. Application of the Hillery-Zubairy entanglement criterion

The Hillery-Zubairy (HZ) inequality for the class of separable states is expressed as follows [35]

$$\sqrt{\langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{b}^\dagger \hat{b} \rangle} \geq |\langle \hat{a} \hat{b} \rangle|. \quad (15)$$

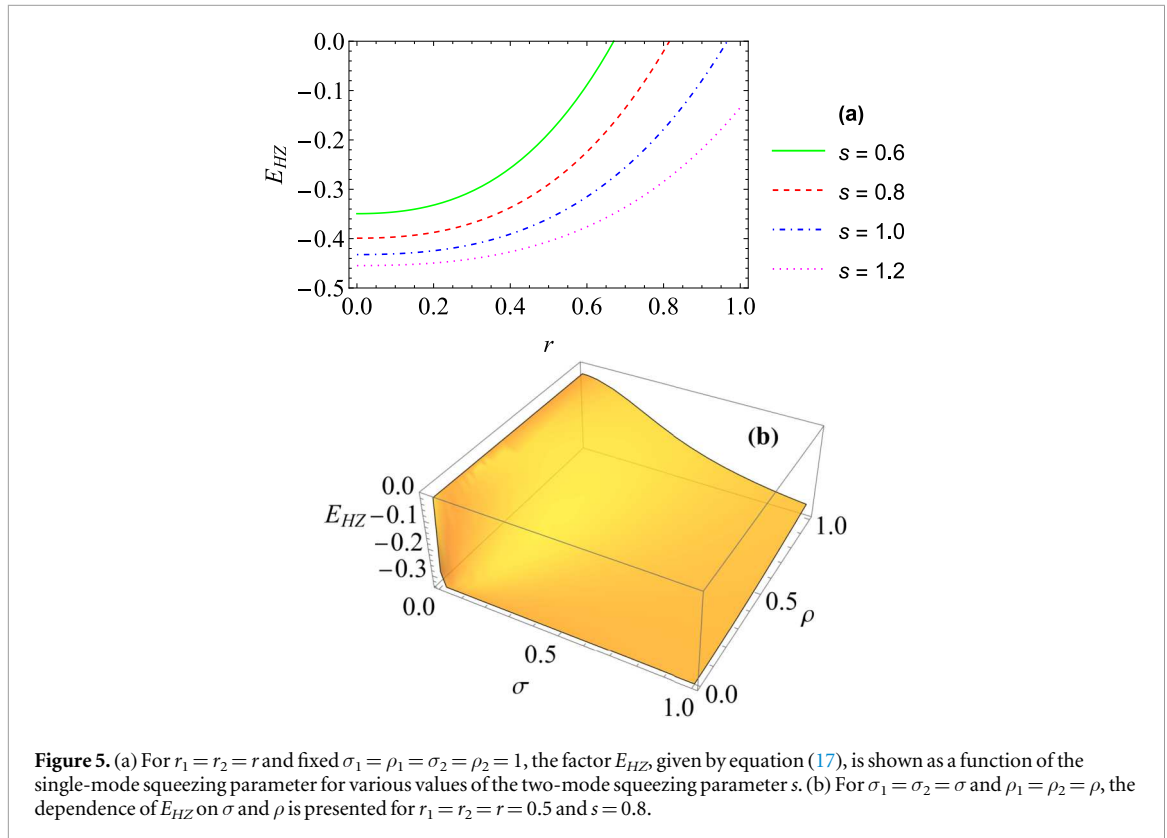


Figure 5. (a) For $r_1 = r_2 = r$ and fixed $\sigma_1 = \rho_1 = \sigma_2 = \rho_2 = 1$, the factor E_{HZ} , given by equation (17), is shown as a function of the single-mode squeezing parameter for various values of the two-mode squeezing parameter s . (b) For $\sigma_1 = \sigma_2 = \sigma$ and $\rho_1 = \rho_2 = \rho$, the dependence of E_{HZ} on σ and ρ is presented for $r_1 = r_2 = r = 0.5$ and $s = 0.8$.

A two-mode state is entangled if it satisfies the following condition

$$E_{HZ} = \sqrt{\langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{b}^\dagger \hat{b} \rangle} - |\langle \hat{a} \hat{b} \rangle| < 0. \quad (16)$$

Here, we introduce the factor E_{HZ} as a parameter for evaluating the degree of violation of the inequality in equation (15). For the TMS-qSSVs, the factor E_{HZ} is written as

$$E_{HZ} = \sqrt{N_a N_b} - |K_{2ab}|, \quad (17)$$

where the corresponding expressions for N_a , N_b , and K_{2ab} are listed, respectively, in equations (A35), (A37), and (A43) of the appendix.

In figure 5(a), the factor E_{HZ} is plotted for the two identical symmetric qSSVs as the squeezing parameters increase. It is evident that the TMS-qSSVs are entangled according to the Hillery–Zubairy criterion. The violation of the inequality in equation (15) becomes more pronounced when the single-mode squeezing parameter is smaller or the two-mode squeezing parameter is larger. In contrast, figure 5(b) shows that the satisfaction of the Hillery–Zubairy criterion in the TMS-qSSVs is most pronounced when the input states are the SVs, i.e., $\rho = 0$. The above results exhibit a similar trend to the variation of the EPR correlation presented in the previous subsection.

We have begun with the analysis of the two-mode squeezing effect and subsequently evaluated the entanglement properties via the Hillery–Zubairy criterion for the TMS-qSSVs. Among the studied properties, only the two-mode squeezing is found to be enhanced compared with that in the TMSVs. Although both the EPR correlation and the entanglement (according to the Hillery–Zubairy criterion) are observed in the TMS-qSSVs, their manifestations become weaker as the single-mode squeezing parameters increase.

Among the three types of input states, the SVs ($\rho_1 = \rho_2 = 0$), the single-photon-subtracted SVs ($\sigma_1 = \sigma_2 = 0$), and the qSSVs (where σ_j and ρ_j , with $j \in \{1, 2\}$, are nonzero), the nonlocal properties are most pronounced in the first case (the strongest Gaussianity) and most ambiguous in the second (the strongest non-Gaussianity). This behavior is rather unusual for non-Gaussian operations such as photon subtraction, as this process typically leads to an enhancement of the EPR correlations and a more pronounced entanglement [32, 36]. Why does the opposite occur in the TMS-qSSVs? The reason lies in the order of operations: in the TMS-qSSVs, the non-Gaussian operations are applied to the separable modes a and b , i.e., before the two-mode squeezing operator acts on them. In contrast, in the systems considered in [32, 36], the photon

subtraction is performed on the states that have already been entangled. To further clarify the entanglement characteristics of the TMS-qSSVSs, we subsequently compute the linear entropy of these states.

3.4. The degree of entanglement

By examining the entanglement properties of the TMS-qSSVSs using the Hillery–Zubairy entanglement criterion, we observe that the manifestation of entanglement in these states almost disappears when the single-mode squeezing parameters become sufficiently large. However, this criterion provides only a method for detecting entanglement, that is, a necessary but not sufficient condition. Therefore, to further clarify the degree of entanglement in the TMS-qSSVSs, we investigate their linear entropy. The linear entropy of a two-mode state, corresponding to modes a and b , is defined as follows [37].

$$E_L = 1 - \text{Tr}_x(\hat{\rho}_x)^2, \quad (18)$$

where Tr_x denotes the partial trace over mode x (with $x = a, b$), and $\hat{\rho}_x$ is the reduced density operator of mode x . The state is entangled if $E_L > 0$, and maximal entanglement is achieved when $E_L = 1$. By using the transformation relations among the single-mode and two-mode squeezing operators and the creation and annihilation operators [see equations (A1)–(A4) and (A28)–(A31) in the appendix], the TMS-qSSVSs can be expressed in terms of these operators as

$$|s, r_1, r_2\rangle_{ab} = \mathcal{N}_1 \mathcal{N}_2 \hat{s}_a(r_1) \hat{s}_b(r_2) \hat{S}_{ab}(s) (\sigma_1 + \rho_1 \cosh r_1 \hat{a} - \rho_1 \sinh r_1 \hat{a}^\dagger) \\ \otimes (\sigma_2 + \rho_2 \cosh r_2 \hat{b} - \rho_2 \sinh r_2 \hat{b}^\dagger) |0, 0\rangle_{ab}. \quad (19)$$

In the equation (19), the two single-mode squeezing operators $\hat{s}_a(r_1)$ and $\hat{s}_b(r_2)$ act as local operations on each mode, and can therefore be omitted in the calculation of the linear entropy. Hence, we only need to consider the state

$$|s, r\rangle_{ab} = \mathcal{N}^2 \hat{S}_{ab}(s) (\sigma + \rho \cosh r \hat{a} - \rho \sinh r \hat{a}^\dagger) \\ \otimes (\sigma + \rho \cosh r \hat{b} - \rho \sinh r \hat{b}^\dagger) |0, 0\rangle_{ab}, \quad (20)$$

where, for simplicity in calculating the linear entropy, we have assumed $r_1 = r_2 = r$, $\sigma_1 = \sigma_2 = \sigma$, and $\rho_1 = \rho_2 = \rho$.

By moving the two-mode squeezing operator $\hat{S}_{ab}(s)$ after the creation and annihilation operators of modes a and b , and then expanding the TMSVS in the Fock basis, we obtain

$$|s, r\rangle_{ab} = \mathcal{N}^2 \sum_{n=0}^{\infty} (a_{1,n} |n, n\rangle_{ab} + a_{2,n} |n, n+1\rangle_{ab} + a_{2,n} |n, n+1\rangle_{ab}). \quad (21)$$

Here

$$a_{1,n} = \frac{\mathcal{N}^2 \tanh^n s}{\cosh s} \left[\sigma^2 + \frac{\rho^2 \sinh^2 r (n - \sinh^2 s)}{\sinh s \cosh s} \right], \quad (22)$$

$$a_{2,n} = -\frac{\mathcal{N}^2 \sigma \rho \sinh r \tanh^n s}{\cosh^2 s} \sqrt{n+1}. \quad (23)$$

For the TMS-qSSVSs, the reduced density operators, defined as $\hat{\rho}_a = \text{Tr}_b(\hat{\rho}_{ab})$ and $\hat{\rho}_b = \text{Tr}_a(\hat{\rho}_{ab})$, can be expressed as

$$\hat{\rho}_a = \hat{\rho}_b = \text{Tr}_a(\hat{\rho}_{ab}) = \text{Tr}_a(|s, r\rangle_{ab} \langle s, r|). \quad (24)$$

After the calculations, the linear entropy of the TMS-qSSVSs is given by

$$E_L = 1 - \sum_{n=0}^{\infty} (b_{1,n}^2 + 2b_{2,n}^2 + b_{3,n}^2 + 2b_{4,n}^2 + 2b_{3,n}b_{1,n+1}), \quad (25)$$

where

$$b_{1,n} = a_{1,n}^2 + a_{2,n}^2, \quad b_{2,n} = a_{1,n}a_{2,n} + a_{1,n+1}a_{2,n}, \quad b_{3,n} = a_{2,n}^2, \quad (26)$$

$$b_{4,n} = a_{2,n}a_{2,n+1}. \quad (27)$$

Figure 6(a) shows the linear entropy of the TMS-qSSVSs as a function of the single-mode squeezing parameter r for several values of the two-mode squeezing parameter s , while figure 6(b) illustrates its dependence on both σ and ρ . The TMS-qSSVSs are found to be entangled in all cases considered, and the linear entropy increases with either s or r , as expected. This result further confirms that the TMS-qSSVSs exhibit a higher degree of

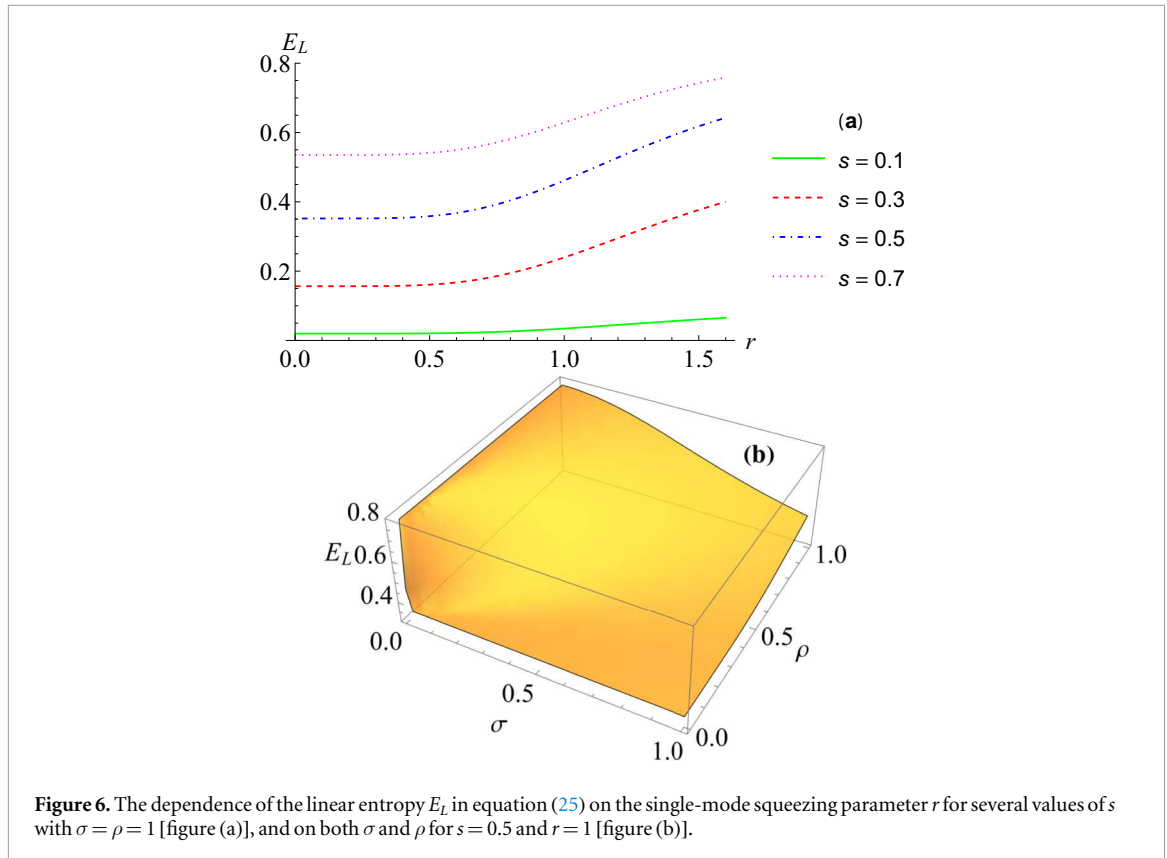


Figure 6. The dependence of the linear entropy E_L in equation (25) on the single-mode squeezing parameter r for several values of s with $\sigma = \rho = 1$ [figure (a)], and on both σ and ρ for $s = 0.5$ and $r = 1$ [figure (b)].

entanglement than the TMSVS. Moreover, among the three types of input states, the single-photon-subtracted SVSs lead to the strongest degree of entanglement, whereas the SVSs yield the weakest. In other words, a higher non-Gaussianity of the input states gives rise to enhanced entanglement strength. This provides further evidence that non-Gaussian operations play a crucial role in enhancing the degree of entanglement [38, 39].

When two SVSs are injected into a beam splitter [26] or into a two-beam-splitter cascade [27], the resulting state can become entangled. In the single beam splitter configuration, the degree of entanglement is maximized at the TMSVS. Thus, this configuration does not surpass the degree of entanglement achievable with the TMS-qSSVSs. In the cascade beam splitter scheme, the maximal entanglement entropy does not exceed that of the single-beam-splitter case (for the same squeezing strengths); in particular, when the two input SVSs are identical, the output becomes disentangled. Therefore, among these beam-splitter-based schemes, the TMS-qSSVSs exhibit the highest degree of entanglement. Moreover, the entanglement of the TMS-qSSVSs can be further enhanced by increasing the two-mode squeezing parameter s .

The enhancement in the degree of entanglement is not necessarily accompanied by an improvement in the EPR correlation [32, 36]. A similar behavior is also observed in the TMS-qSSVSs. Nevertheless, a high degree of entanglement still plays an important role in improving the performance of quantum tasks. A typical example is the enhancement of phase sensitivity in a quantum MZI, which will be clarified in the following section.

4. Enhancement of phase sensitivity in the TMS-qSSVSs

Enhancing phase sensitivity in quantum measurement plays an important role in quantum metrology [40, 41]. One commonly used device is the optical MZI, which allows the detection of tiny phase shifts along the paths of optical beams [42–44]. Measuring such small phase changes enables the detection of extremely weak signals that are inaccessible to classical measurements. As is well known, when two modes, a and b , of a state $|\Psi\rangle_{ab}$ are fed into an MZI, the ultimate accuracy in measuring the phase shift φ is determined by the QCRB [45]

$$\Delta\varphi_Q = \frac{1}{\sqrt{F_Q(\varphi)}}, \quad (28)$$

where $F_Q(\varphi)$ denotes the quantum Fisher information (QFI). The QFI can be calculated using the formula

$$F_Q(\varphi) = 4[\langle\Phi'(\varphi)|\Phi'(\varphi)\rangle_{ab} - |\langle\Phi'(\varphi)|\Phi(\varphi)\rangle_{ab}|^2], \quad (29)$$

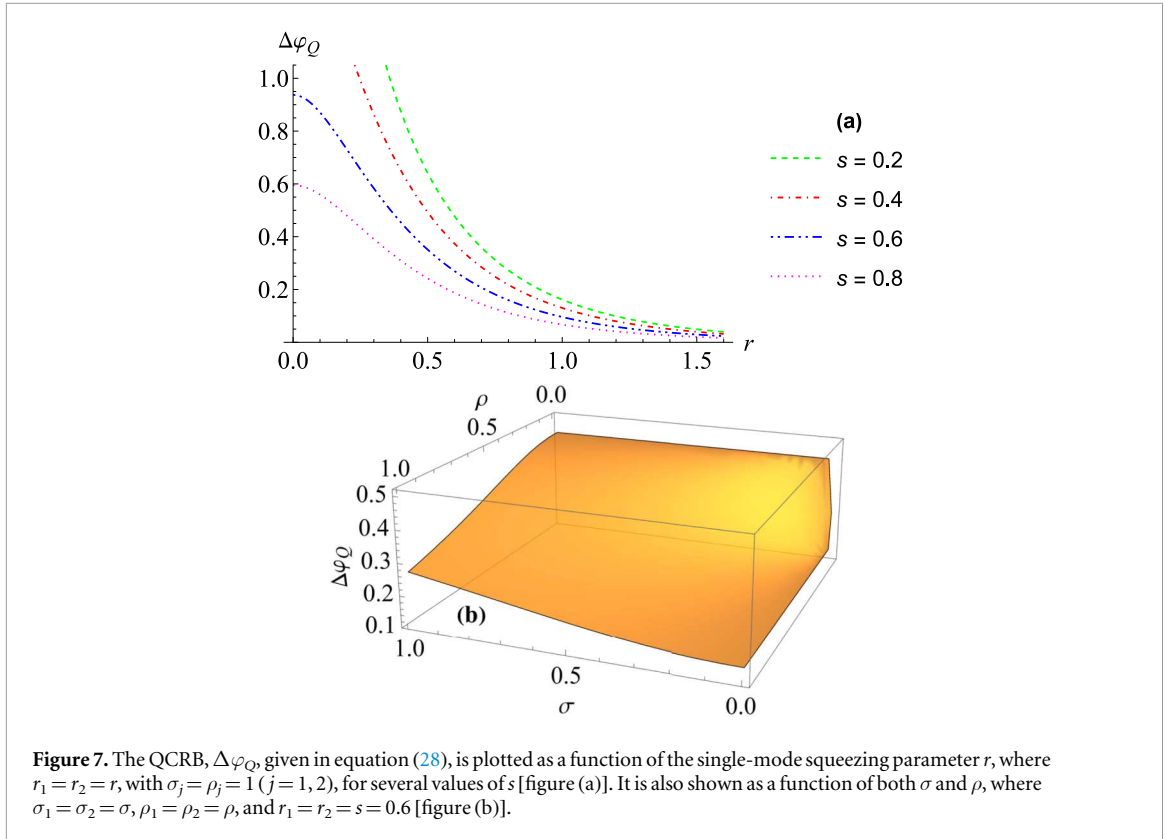


Figure 7. The QCRB, $\Delta\varphi_Q$, given in equation (28), is plotted as a function of the single-mode squeezing parameter r , where $r_1 = r_2 = r$, with $\sigma_j = \rho_j = 1$ ($j = 1, 2$), for several values of s [figure (a)]. It is also shown as a function of both σ and ρ , where $\sigma_1 = \sigma_2 = \sigma$, $\rho_1 = \rho_2 = \rho$, and $r_1 = r_2 = s = 0.6$ [figure (b)].

Here, $|\Phi(\varphi)\rangle_{ab} = e^{-i\varphi\hat{f}_3} e^{-i\pi\hat{f}_1/2} |\Psi_i\rangle_{ab}$, with $\hat{f}_1 = (\hat{a}^\dagger\hat{b} + \hat{a}\hat{b}^\dagger)/2$, $\hat{f}_3 = (\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})/2$, and $|\Phi'(\varphi)\rangle_{ab} = \partial|\Phi(\varphi)\rangle_{ab}/\partial\varphi$. For pure states, the QFI can be expressed in terms of creation and annihilation operators as [46]

$$F_Q(\varphi) = 2\langle\hat{a}^\dagger\hat{a}\rangle\langle\hat{b}^\dagger\hat{b}\rangle + \langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{b}^\dagger\hat{b}\rangle - 2|\langle\hat{a}\rangle|^2|\langle\hat{b}\rangle|^2 - 2\Re(\langle\hat{a}^2\rangle\langle\hat{b}^{\dagger 2}\rangle - \langle\hat{a}^{\dagger 2}\rangle\langle\hat{b}^2\rangle), \quad (30)$$

where $\Re(z)$ denotes the real part of the complex number z . A smaller QCRB corresponds to higher phase sensitivity. For the TMS-qSSVs, we use the QCRB in equation (28) to evaluate the enhancement of phase sensitivity in the MZI by expressing the QFI in terms of quantum expectation values as

$$F_Q(\varphi) = 2N_a N_b + N_a + N_b - 2K_{1a}^2 K_{1b}^2. \quad (31)$$

The values of K_{1a} , N_a , K_{1b} , and N_b can be found in equations (A32), (A35), (A36), and (A37) of the [appendix](#).

In figure 7, the QCRB of the TMS-qSSVs is shown for two cases. In the first case, the two-mode squeezing parameter is fixed at $s = 0.2, 0.4, 0.6$, and 0.8 , while $\sigma_j = \rho_j = 1$ (with $j \in \{1, 2\}$), and the single-mode squeezing parameters are set as $r_1 = r_2 = r$ [see figure 7(a)]. As expected, the TMS-qSSVs enhance the phase sensitivity of the MZI, as indicated by the reduction of the QCRB with increasing the single-mode and/or two-mode squeezing parameters.

In the second case, with $\sigma_1 = \sigma_2 = \sigma$, $\rho_1 = \rho_2 = \rho$, and $r_1 = r_2 = s = 0.6$, figure 7(b) shows the QCRB as a function of both σ and ρ . Among the three types of input states, the single-photon-subtracted SVs yield the highest enhancement of phase sensitivity, whereas the SVs lead to the lowest. This means that the QCRB attains its minimum value when the input states possess the strongest non-Gaussianity, corresponding to $\sigma_1 = \sigma_2 = 0$, whereas it becomes maximal for the purely Gaussian states, corresponding to $\rho_1 = \rho_2 = 0$. These results are consistent with the behavior of the linear entropy variation in the TMS-qSSVs.

When the TMS-qSSVs are employed as inputs to the MZI, the phase sensitivity is enhanced and can be further improved by increasing the squeezing parameters. The improvement in phase sensitivity primarily arises from the increase in the mean photon number $N = N_a + N_b$ [see figure 8(a)], where $N_a = N_b$, and $\sigma_j = \rho_j = 1$, $j \in \{1, 2\}$, are fixed. It is worth noting that the variation of the QCRB with respect to N in the cases $\sigma_j \neq \rho_j$ is completely analogous to that in the above case. An increase in the squeezing parameters also leads to a higher mean photon number, as illustrated in figure 8(b). However, even when the squeezing parameters are fixed, a noticeable enhancement can still be observed, as shown in figure 7(c). As seen in figure 8(c), the mean

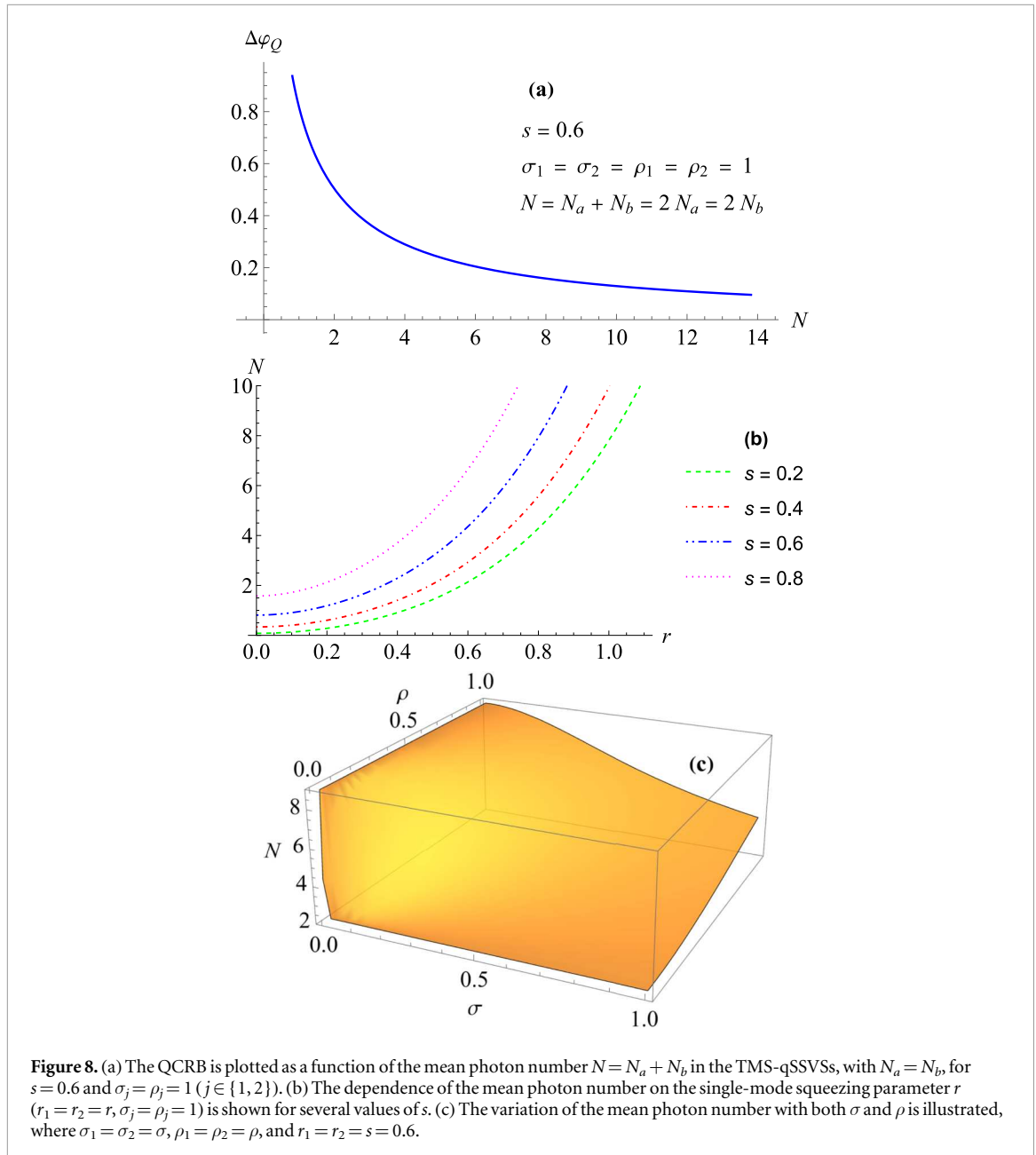
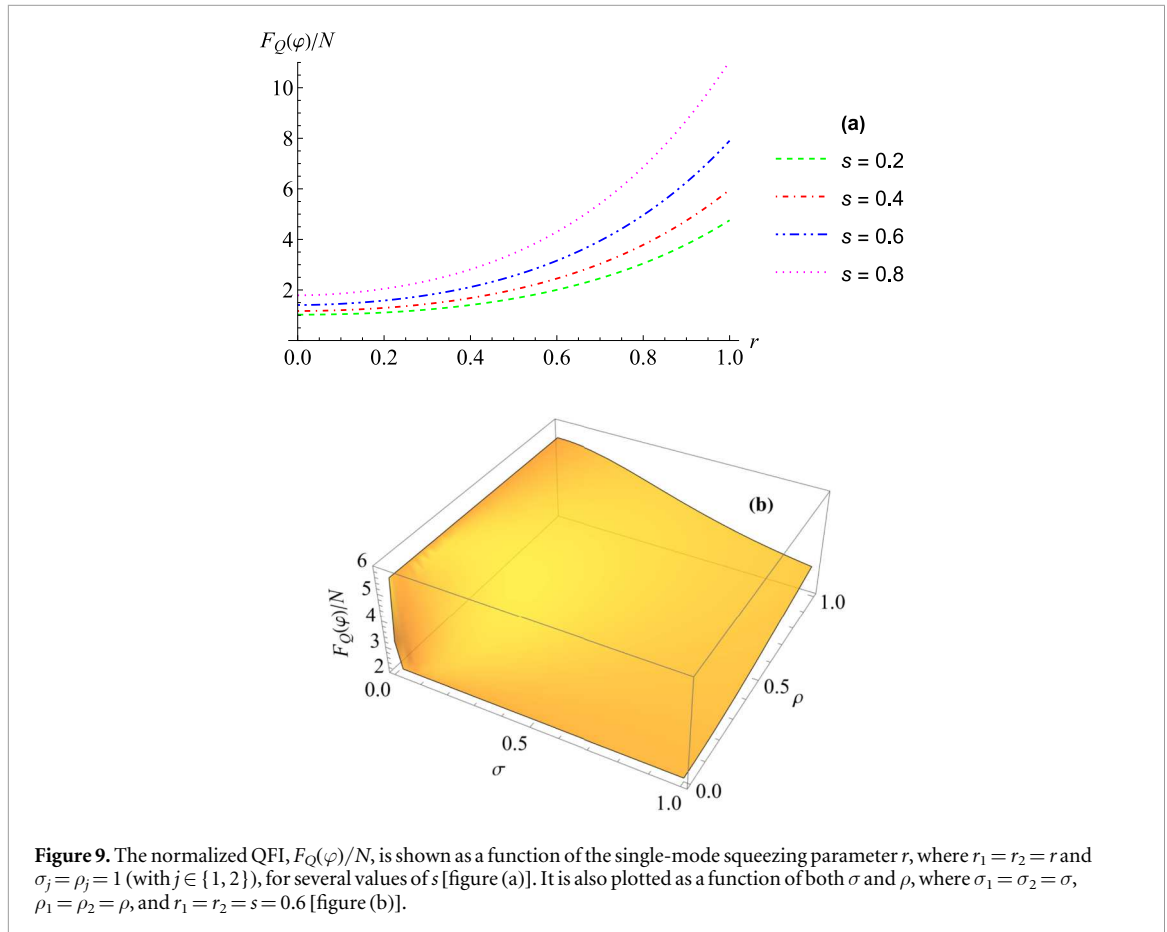


Figure 8. (a) The QCRB is plotted as a function of the mean photon number $N = N_a + N_b$ in the TMS-qSSVSs, with $N_a = N_b$, for $s = 0.6$ and $\sigma_j = \rho_j = 1$ ($j \in \{1, 2\}$). (b) The dependence of the mean photon number on the single-mode squeezing parameter r ($r_1 = r_2 = r$, $\sigma_j = \rho_j = 1$) is shown for several values of s . (c) The variation of the mean photon number with both σ and ρ is illustrated, where $\sigma_1 = \sigma_2 = \sigma$, $\rho_1 = \rho_2 = \rho$, and $r_1 = r_2 = s = 0.6$.

photon number reaches its maximum at $\sigma = 0$, i.e., when the input states exhibit the strongest non-Gaussianity. In this case, the QCRB attains its minimum value. Therefore, the enhancement of phase sensitivity is largely attributed to the photon-subtraction operation in the TMS-qSSVSs.

To examine the improvement in phase sensitivity per photon for the TMS-qSSVSs, we evaluate the normalized quantum Fisher information, $F_Q(\varphi)/N$, which quantifies the Fisher information per particle. An increase in the normalized QFI serves as a witness to the enhancement of phase sensitivity per photon. As shown in figure 9, $F_Q(\varphi)/N$ increases for the TMS-qSSVSs as the squeezing parameters are raised [see figure 9(a)]. This behavior is consistent with the simultaneous increase in both the degree of entanglement and the mean photon number when the squeezing amplitudes grow. Moreover, $F_Q(\varphi)/N$ is further enhanced by increasing ρ , and gets maximum when $\sigma = 0$ [see figure 9(b)], confirming that photon subtraction improves the phase sensitivity per photon in the considered regimes.

In the TMS-qSSVSs, both the total phase sensitivity and the phase sensitivity per photon are enhanced not only by increasing the squeezing strengths but also by the photon-subtraction operation, which leads to a higher mean photon number. This non-Gaussian operation plays a major role in enhancing the phase sensitivity in the MZI. Accordingly, the challenge of further improving phase sensitivity lies not only in increasing the squeezing amplitudes but also in managing the corresponding rise in the average photon number of the generated continuous-variable states. The latter effect can make the system more susceptible to photon loss,



decoherence, interference at the beam splitters of the MZI, and other experimental imperfections, thereby posing a practical limitation for realistic implementations.

5. Conclusion

In this paper, we have introduced a new class of two-mode non-Gaussian states, referred to as the TMS-qSSVs, which are obtained by applying the two-mode squeezing operator $\hat{S}_{ab}(s)$ to two distinct qSSVs. The TMS-qSSVs exhibit the strongest non-Gaussianity in case the qSSVs reduce to the single-photon-subtracted SVs. In particular, when $r_1 = r_2 = 0$, the TMS-qSSVs reduce to the well-known TMSVs.

By analyzing the two-mode squeezing and entanglement properties of the TMS-qSSVs, we first found that the single-mode squeezing parameter in the input enhances the two-mode squeezing effect associated with $\hat{X}$, increases the linear entropy, and also improves the phase sensitivity when these states are employed as input resources in the MZI. Secondly, the two-mode squeezing parameter enhances the two-mode squeezing effect associated with $\hat{P}$, strengthens the EPR correlation, amplifies the Hillery–Zubairy entanglement, and further improves both the linear entropy and the phase sensitivity in the MZI.

In case $\rho_1 = \rho_2 = 0$, the TMS-qSSVs reduce to the SVs corresponding to purely Gaussian states. Under this condition, the TMS-qSSVs demonstrate optimal effectiveness in the two-mode squeezing associated with $\hat{P}$, the EPR correlation, and the Hillery–Zubairy entanglement. When the qSSVs transform into the single-photon-subtracted SVs (i.e., the strongest non-Gaussian states, obtained by setting $\sigma_1 = \sigma_2 = 0$), the TMS-qSSVs exhibit optimal linear entropy and phase sensitivity in the MZI. Additionally, for the qSSVs with $\sigma_j = 1$ and moderate ρ_j ($j \in \{1, 2\}$), the two-mode squeezing effect associated with $\hat{X}$ is further enhanced.

Compared with the Gaussian TMSVs, the non-Gaussian TMS-qSSVs show superior two-mode squeezing associated with $\hat{X}$, a higher degree of entanglement, and enhanced phase estimation precision. Notably, the enhanced phase sensitivity in the MZI arises not only from the squeezing strengths but also from the increased mean photon number resulting from photon subtraction in the SVs. These results suggest that TMS-qSSVs are promising candidates for applications in quantum metrology and sensing.

Acknowledgments

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix. Calculation of quantum averages in the TMS-qSSVs

In the SVS, to calculate the quantum averages of operators, we employ the transformation relations between the single-mode squeezing operators and the photon creation and annihilation operators for modes a and b , as follows

$$\hat{s}_a^\dagger(r_1)\hat{a}\hat{s}_a(r_1) = \hat{a} \cosh r_1 - \hat{a}^\dagger \sinh r_1, \quad (\text{A1})$$

$$\hat{s}_a^\dagger(r_1)\hat{a}^\dagger\hat{s}_a(r_1) = \hat{a}^\dagger \cosh r_1 - \hat{a} \sinh r_1, \quad (\text{A2})$$

$$\hat{s}_b^\dagger(r_2)\hat{b}\hat{s}_b(r_2) = \hat{b} \cosh r_2 - \hat{b}^\dagger \sinh r_2, \quad (\text{A3})$$

$$\hat{s}_b^\dagger(r_2)\hat{b}^\dagger\hat{s}_b(r_2) = \hat{b}^\dagger \cosh r_2 - \hat{b} \sinh r_2. \quad (\text{A4})$$

Moreover, we make use of the commutation relation between the creation and annihilation operators, such as

$$[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1, \quad (\text{A5})$$

$$[\hat{b}, \hat{b}^\dagger] = \hat{b}\hat{b}^\dagger - \hat{b}^\dagger\hat{b} = 1. \quad (\text{A6})$$

Therefore, the quantum averages for mode a in the SVS can be obtained as follows

$$A_1 = {}_a\langle r_1|\hat{a}^\dagger\hat{a}|r_1\rangle_a = \sinh^2 r_1, \quad (\text{A7})$$

$$A_2 = {}_a\langle r_1|\hat{a}^{\dagger 2}\hat{a}^2|r_1\rangle_a = \frac{[3 \cosh(2r_1) - 1] \sinh^2 r_1}{2}, \quad (\text{A8})$$

$$A_3 = {}_a\langle r_1|\hat{a}^{\dagger 3}\hat{a}^3|r_1\rangle_a = \frac{3[5 \cosh(2r_1) + 1] \sinh^4 r_1}{2}, \quad (\text{A9})$$

$$A_4 = {}_a\langle r_1|\hat{a}^2|r_1\rangle_a = \langle r_1|\hat{a}^{\dagger 2}|r_1\rangle = -\sinh r_1 \cosh r_1, \quad (\text{A10})$$

$$A_5 = {}_a\langle r_1|\hat{a}^{\dagger 3}\hat{a}|r_1\rangle_a = {}_a\langle r_1|\hat{a}^\dagger\hat{a}^3|r_1\rangle_a = -3 \sinh^3 r_1 \cosh r_1. \quad (\text{A11})$$

For mode b , the corresponding results can be obtained by replacing the notations a and r_1 with b and r_2 , respectively, in equations (A7)–(A10). In addition, the symbols A_k ($k = 1, 2, \dots, 5$) are replaced by the corresponding B_k .

The computation of the higher-order moments above is rather lengthy; however, it can be significantly simplified by employing an alternative method based on the action of the creation and annihilation operators on Fock states, i.e.,

$$\hat{a}|n\rangle_a = \sqrt{n}|n-1\rangle_a, \quad \hat{b}|n\rangle_b = \sqrt{n}|n-1\rangle_b, \quad (\text{A12})$$

$$\hat{a}^\dagger|n\rangle_a = \sqrt{n+1}|n+1\rangle_a, \quad \hat{b}^\dagger|n\rangle_b = \sqrt{n+1}|n+1\rangle_b. \quad (\text{A13})$$

The following formulae can be derived from the above relation

$$\hat{a}^k|n\rangle_a = \frac{\sqrt{n!}}{\sqrt{(n-k)!}}|n-k\rangle_a, \quad \hat{b}^k|n\rangle_b = \frac{\sqrt{n!}}{\sqrt{(n-k)!}}|n-k\rangle_b, \quad k \leq n, \quad (\text{A14})$$

$$\hat{a}^{\dagger k}|n\rangle_a = \frac{\sqrt{(n+k)!}}{\sqrt{n!}}|n+k\rangle_a, \quad \hat{b}^{\dagger k}|n\rangle_b = \frac{\sqrt{(n+k)!}}{\sqrt{n!}}|n+k\rangle_b. \quad (\text{A15})$$

Using the expression of the SVS in the Fock basis given in equation (1), the expectation values of the relevant operators can be computed. For example,

$$\begin{aligned}
M^{(k)} &= {}_a\langle \hat{a}^{\dagger k} \hat{a}^k | \hat{r}_1 \rangle_a \\
&= \text{sech } r_1 \sum_{m,n=0}^{\infty} \left(-\frac{\tanh r_1}{2} \right)^{m+n} \frac{\sqrt{(2m)!(2n)!}}{m!n!} {}_a\langle 2m | \hat{a}^{\dagger k} \hat{a}^k | 2n \rangle_a \\
&= \text{sech } r_1 \sum_{m,n=0}^{\infty} \left(-\frac{\tanh r_1}{2} \right)^{m+n} \frac{\sqrt{(2m)!(2n)!}}{m!n!} \\
&\quad \times \frac{\sqrt{(2m)!(2n)!}}{\sqrt{(2m-k)!(2n-k)!}} {}_a\langle 2m-k | 2n-k \rangle_a \\
&= \text{sech } r_1 \sum_{n=0}^{\infty} \left(\frac{\tanh r_1}{2} \right)^{2n} \frac{[(2n)!]^2}{(n!)^2(2n-k)!}.
\end{aligned} \tag{A16}$$

By using professional computational software, such as *Mathematica*, and selecting appropriate numerical values, the moment $M^{(k)}$ in equation (A16) can be simplified. For example, for $k=3$, the value of $M^{(3)}$ is given as follows:

$$M^{(3)} = \frac{3 \sinh^3 r_1 \tanh r_1 (3 + 2 \tanh^2 r_1)}{(\text{sech}^2 r_1)^{3/2}}. \tag{A17}$$

The quantity A_3 in equation (A8) represents the simplified form of the above expression.

For the qSSVS, utilizing the results in equations (A7)–(A10) and applying equation (4) together with the transformations given in equations (A5) and (A6), the quantum averages can be derived as presented below.

$$C_1 = {}_a\langle \hat{a} \rangle_a = {}_a\langle \hat{a}^{\dagger} \rangle_a = \mathcal{N}_1^2 \sigma_1 \rho_1 (A_1 + A_4), \tag{A18}$$

$$C_2 = {}_a\langle \hat{a}^2 \rangle_a = {}_a\langle \hat{a}^{\dagger 2} \rangle_a = \mathcal{N}_1^2 (\sigma_1^2 A_4 + \rho_1^2 A_5), \tag{A19}$$

$$C_3 = {}_a\langle \hat{a}^{\dagger} \hat{a} \rangle_a = \mathcal{N}_1^2 (\sigma_1^2 A_1 + \rho_1^2 A_2), \tag{A20}$$

$$C_4 = {}_a\langle \hat{a}^{\dagger 2} \hat{a}^2 \rangle_a = \mathcal{N}_1^2 (\sigma_1^2 A_2 + \rho_1^2 A_3), \tag{A21}$$

$$C_5 = {}_a\langle \hat{a}^{\dagger 2} \hat{a} \rangle_a = {}_a\langle \hat{a}^{\dagger} \hat{a}^2 \rangle_a = \mathcal{N}_1^2 \sigma_1 \rho_1 (A_2 + A_5). \tag{A22}$$

Here, the notation ${}_a\langle \cdots \rangle_a$ refers to the expectation value with respect to the state ${}_a\langle r_1, \sigma_1, \rho_1 | \cdots | r_1, \sigma_1, \rho_1 \rangle_a$, and the normalization factor is given by $\mathcal{N}_1 = (\sigma_1^2 + \rho_1^2 \sinh^2 r_1)^{-1/2}$. By replacing $a \rightarrow b$, $A \rightarrow B$, and the corresponding subscripts, we similarly obtain

$$D_1 = {}_b\langle \hat{b} \rangle_b = {}_b\langle \hat{b}^{\dagger} \rangle_b = \mathcal{N}_2^2 \sigma_2 \rho_2 (B_1 + B_4), \tag{A23}$$

$$D_2 = {}_b\langle \hat{b}^2 \rangle_b = {}_b\langle \hat{b}^{\dagger 2} \rangle_b = \mathcal{N}_2^2 (\sigma_2^2 B_4 + \rho_2^2 B_5), \tag{A24}$$

$$D_3 = {}_b\langle \hat{b}^{\dagger} \hat{b} \rangle_b = \mathcal{N}_2^2 (\sigma_2^2 B_1 + \rho_2^2 B_2), \tag{A25}$$

$$D_4 = {}_b\langle \hat{b}^{\dagger 2} \hat{b}^2 \rangle_b = \mathcal{N}_2^2 (\sigma_2^2 B_2 + \rho_2^2 B_3), \tag{A26}$$

$$D_5 = {}_b\langle \hat{b}^{\dagger 2} \hat{b} \rangle_b = {}_b\langle \hat{b}^{\dagger} \hat{b}^2 \rangle_b = \mathcal{N}_2^2 \sigma_2 \rho_2 (B_2 + B_5). \tag{A27}$$

In this context, the notation ${}_b\langle \cdots \rangle_b$ refers to the expectation value with respect to the state ${}_b\langle r_2, \sigma_2, \rho_2 | \cdots | r_2, \sigma_2, \rho_2 \rangle_b$, and the normalization factor is $\mathcal{N}_2 = (\sigma_2^2 + \rho_2^2 \sinh^2 r_2)^{-1/2}$. It should be noted that, in evaluating the quantum averages in equations (A18)–(A21), calculations performed in the Fock basis with the aid of *Mathematica* generally yield more cumbersome expressions.

For the two-mode squeezing operator, the following transformation relations hold

$$\hat{S}_{ab}^{\dagger}(s) \hat{a} \hat{S}_{ab}(s) = \hat{a} \cosh s + \hat{b}^{\dagger} \sinh s, \tag{A28}$$

$$\hat{S}_{ab}^{\dagger}(s) \hat{a}^{\dagger} \hat{S}_{ab}(s) = \hat{a}^{\dagger} \cosh s + \hat{b} \sinh s, \tag{A29}$$

$$\hat{S}_{ab}^{\dagger}(s) \hat{b} \hat{S}_{ab}(s) = \hat{b} \cosh s + \hat{a}^{\dagger} \sinh s, \tag{A30}$$

$$\hat{S}_{ab}^{\dagger}(s) \hat{b}^{\dagger} \hat{S}_{ab}(s) = \hat{b}^{\dagger} \cosh s + \hat{a} \sinh s. \tag{A31}$$

Based on these relations, the expectation values of relevant operators in the TMS-qSSVSs $|s, r_1, r_2\rangle_{ab}$, as defined in equation (5), can be evaluated. For example,

$$\begin{aligned}
K_{1a} &= \langle \hat{a}^{\dagger} \rangle = {}_{ab}\langle s, r_1, r_2 | \hat{a}^{\dagger} | s, r_1, r_2 \rangle_{ab} \\
&= {}_b\langle r_2, \sigma_2, \rho_2 | {}_a\langle r_1, \sigma_1, \rho_1 | \hat{S}_{ab}^{\dagger}(s) \hat{a}^{\dagger} \hat{S}_{ab}(s) | r_1, \sigma_1, \rho_1 \rangle_a | r_2, \sigma_2, \rho_2 \rangle_b \\
&= {}_b\langle r_2, \sigma_2, \rho_2 | {}_a\langle r_1, \sigma_1, \rho_1 | \hat{a}^{\dagger} \cosh s + \hat{b} \sinh s | r_1, \sigma_1, \rho_1 \rangle_a | r_2, \sigma_2, \rho_2 \rangle_b \\
&= \cosh s {}_a\langle \hat{a}^{\dagger} \rangle_a + \sinh s {}_b\langle \hat{b} \rangle_b = C_1 \cosh s + D_1 \sinh s = \langle \hat{a} \rangle,
\end{aligned} \tag{A32}$$

where C_1 and D_1 are given by equations (A18) and (A23), respectively. Another case is given below

$$\begin{aligned} N_a &= \langle \hat{a}^\dagger \hat{a} \rangle = {}_{ab} \langle s, r_1, r_2 | \hat{a}^\dagger \hat{a} | s, r_1, r_2 \rangle_{ab} \\ &= {}_b \langle r_2, \sigma_2, \rho_2 | {}_a \langle r_1, \sigma_1, \rho_1 | \hat{S}_{ab}^\dagger(s) \hat{a}^\dagger \hat{a} \hat{S}_{ab}(s) | r_1, \sigma_1, \rho_1 \rangle_a | r_2, \sigma_2, \rho_2 \rangle_b. \end{aligned} \quad (\text{A33})$$

Using the identity $\hat{S}_{ab}(s) \hat{S}_{ab}^\dagger(s) = \mathbf{1}$, we insert $\hat{S}_{ab}(s) \hat{S}_{ab}^\dagger(s)$ between the two operators $\hat{a}^\dagger \hat{a}$, which leads to

$$\begin{aligned} N_a &= {}_b \langle r_2, \sigma_2, \rho_2 | {}_a \langle r_1, \sigma_1, \rho_1 | \hat{S}_{ab}^\dagger(s) \hat{a}^\dagger \hat{S}_{ab}(s) \hat{S}_{ab}^\dagger(s) \hat{a} \hat{S}_{ab}(s) \\ &\quad \times | r_1, \sigma_1, \rho_1 \rangle_a | r_2, \sigma_2, \rho_2 \rangle_b \\ &= {}_b \langle r_2, \sigma_2, \rho_2 | {}_a \langle r_1, \sigma_1, \rho_1 | (\hat{a}^\dagger \cosh s + \hat{b} \sinh s) \\ &\quad \times (\hat{a} \cosh s + \hat{b}^\dagger \sinh s) | r_1, \sigma_1, \rho_1 \rangle_a | r_2, \sigma_2, \rho_2 \rangle_b. \end{aligned} \quad (\text{A34})$$

Performing straightforward transformations, and noting that $\hat{b} \hat{b}^\dagger = \mathbf{1} + \hat{b}^\dagger \hat{b}$ [see equation (A6)], yields the expectation value of the photon number in mode a for the TMS-qSSVs as

$$N_a = C_3 \cosh^2 s + D_3 \sinh^2 s + 2C_1 D_1 \sinh s \cosh s + \sinh^2 s. \quad (\text{A35})$$

In a completely analogous manner, several quantum averages in the TMS-qSSVs are given by

$$K_{1b} = \langle \hat{b}^\dagger \rangle = \langle \hat{b} \rangle = D_1 \cosh s + C_1 \sinh s, \quad (\text{A36})$$

$$\begin{aligned} N_b &= \langle \hat{b}^\dagger \hat{b} \rangle \\ &= D_3 \cosh^2 s + C_3 \sinh^2 s + 2C_1 D_1 \sinh s \cosh s + \sinh^2 s, \end{aligned} \quad (\text{A37})$$

$$K_{2a} = \langle \hat{a}^{\dagger 2} \rangle = \langle \hat{a}^2 \rangle = C_2 \cosh^2 s + D_2 \sinh^2 s + 2C_1 D_1 \sinh s \cosh s, \quad (\text{A38})$$

$$K_{2b} = \langle \hat{b}^{\dagger 2} \rangle = \langle \hat{b}^2 \rangle = D_2 \cosh^2 s + C_2 \sinh^2 s + 2C_1 D_1 \sinh s \cosh s, \quad (\text{A39})$$

$$\begin{aligned} K_{3a} &= \langle \hat{a}^{\dagger 2} \hat{a}^2 \rangle = C_4 \cosh^4 s + 4C_5 D_1 \sinh s \cosh^3 s \\ &\quad + 4C_1 (2D_1 + D_5) \sinh^3 s \cosh s + (2 + 4D_3 + D_4) \sinh^4 s \\ &\quad + 2[C_2 D_2 + 2C_3 (1 + D_3)] \sinh^2 s \cosh^2 s, \end{aligned} \quad (\text{A40})$$

$$\begin{aligned} K_{3b} &= \langle \hat{b}^{\dagger 2} \hat{b}^2 \rangle = D_4 \cosh^4 s + 4C_1 D_5 \sinh s \cosh^3 s \\ &\quad + 4D_1 (2C_1 + C_5) \sinh^3 s \cosh s + (2 + 4C_3 + C_4) \sinh^4 s \\ &\quad + 2[C_2 D_2 + 2D_3 (1 + C_3)] \sinh^2 s \cosh^2 s. \end{aligned} \quad (\text{A41})$$

The above results concern the expectation values of the individual modes (a or b) in the TMS-qSSVs. However, the two-mode correlations are more explicitly manifested through the quantum averages of operator pairs involving both modes, such as $\hat{a}^\dagger \hat{b}$, $\hat{a}^\dagger \hat{b}^\dagger$, and related combinations. Using the same computational approach as above, the expectation values of these operators are obtained as follows

$$K_{1ab} = \langle \hat{a}^\dagger \hat{b} \rangle = \langle \hat{a} \hat{b}^\dagger \rangle = (C_2 + D_2) \sinh s \cosh s + C_1 D_1 (\cosh^2 s + \sinh^2 s), \quad (\text{A42})$$

$$K_{2ab} = \langle \hat{a}^\dagger \hat{b}^\dagger \rangle = \langle \hat{a} \hat{b} \rangle = (C_3 + D_3 + 1) \sinh s \cosh s + C_1 D_1 (\cosh^2 s + \sinh^2 s). \quad (\text{A43})$$

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