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Some Connections of Weak Convergence with the Convergence of the Dispersion Functions*

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Abstract. The aim of this note is to establish some connections of weak convergence of $\mathcal{L}^1$- random variables with the convergence of the dispersion functions.

1. Introduction

In this note, let $(\Omega, \mathcal{A}, P)$ denote a probability space, and let $\mathcal{L}^1$ denote the set of all $\mathcal{L}^1$- random variables. Let $X \in \mathcal{L}^1$ and F_X be the distribution function of X. We define the dispersion function of X by

$$D_X(u) := E \mid X - u \mid \quad \text{for every} \quad u \in \mathbb{R} = (-\infty, +\infty),$$

that is, the absolute moment of order $r = 1$ of the random variable X with respect to u, for all $u \in \mathbb{R}$.

In recent years some results concerning the dispersion function $D_X(u)$ have been investigated by Munoz-Perez and Sanchez-Gomez in [1, 2], Thu and Turkan in [3], Thu and Hung in [4 - 6], Hung in [7].

It is worth pointing out that the dispersion function $D_X(u)$ can be considered as a generalization of the mean absolute deviation and the median absolute deviation (see for more details [3 - 6]).

The dispersion function has the following elementary properties (see [1, 2, 4 - 7] for the complete bibliography).

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a. The function $D_X(u)$ is almost everywhere differentiable on $\mathbb{R}$ and its derivative has at most a countable number of discontinuity points.
b. The function $D_X(u)$ is convex on $\mathbb{R}$.
c. $\lim\limits_{u\to+\infty} D'_X(u) = 1$ and $\lim\limits_{u\to-\infty} D'_X(u) = -1$.
d. $\lim\limits_{u\to+\infty} (D_X(u) - u) = -EX$ and $\lim\limits_{u\to-\infty} (D_X(u) + u) = EX$.
e. $F_X(u) = \frac{1}{2}(D'_X(u) + 1)$, for all $u \in C_F$, where $D'_X(u)$ is derivative of the function $D_X(u)$ and C_F denotes a set of continuity points of F_X.
f. $D_X(u) = \int\limits_{-\infty}^{+\infty} \mid F_X(x) - F_u(x) \mid dx$, where $F_u(x)$ is the distribution function of the degenerate random variable at u.
g. $\int\limits_{-\infty}^{+\infty} \mid D_X(u) - D_{EX}(u) \mid du = \sigma^2$.

The main purpose of this note is to establish some connections between the weak convergence of the $\mathcal{L}^1$- random variables (denoted by $\Rightarrow$) and the convergence of the dispersion functions in order to outline the relation between the dispersion functions and the distribution functions. The following theorems are main results of this paper.

Theorem 1. *Let $X, X_1, X_2, \ldots, X_n, \ldots \in \mathcal{L}^1$. If there exists a $p > 1$, such that*

$$\sup_{n\in\mathbb{N}} E \mid X_n \mid^p < +\infty \tag{A}$$

and if

$$F_{X_n} \Rightarrow F_X, \quad as \quad n \to +\infty,$$

then

$$D_{X_n}(u) \to D_X(u) \quad as \quad n \to \infty, \quad for\ every \quad u \in \mathbb{R}.$$

Theorem 2. *Let $\{X_n, n \geq 1\}$ be a sequence of $\mathcal{L}^1$- random variables and let $\{D_n, n \geq 1\}$ be the corresponding sequence of dispersion functions. Suppose that $D_n(u) \to D(u)$ as $n \to \infty$, for every $u \in \mathbb{R}$. Then*
a. *$D(u)$ is a convex function in $\mathbb{R}$.*
b. *Let $\mathcal{H}$ be the set of all points $u \in \mathbb{R}$, such that $D'(u), D'_n(u), n \in \mathbb{N}$ exist. Then $\mathcal{H}$ is a dense set in $\mathbb{R}$ and*

$$\lim_{n\to\infty} D'_n(u) = D'(u), \quad \forall u \in \mathcal{H}.$$

c. *Let $\mathcal{H}_0$ be a set of all points such that $D'(u)$ exists. Then for every $u \in \mathcal{H}_0$,*

$$\lim_{u\to+\infty} D'(u) = 1, \quad and \quad \lim_{u\to-\infty} D'(u) = -1.$$

Theorem 3. *Let $X, X_1, X_2, \ldots X_n, \ldots \in \mathcal{L}^1$ and let $D, D_1, D_2, \ldots D_n, \ldots$ be the corresponding dispersion functions. Assume that $D_n(u) \to D(u)$ as $n \to \infty$, for every $u \in \mathbb{R}$. Then*

a. $F_n \Rightarrow F_X$.
b. $\lim_{n\to\infty} \sup_{u\in\mathbb{R}} | D_n(u) - D(u) | = 0$.

Theorem 4. *Let $\{X_n, n \geq 1\}$ be a sequence of $\mathcal{L}^1$- random variables and let $\{D_n, n \geq 1\}$ be the corresponding sequence of dispersion functions. Let assumption* (A) *of Theorem* 1 *hold, and $D_n(u) \to D(u)$, as $n \to \infty$ for every $u \in \mathbb{R}$. Then there exists a distribution function F such that $D(u) = \int_{\mathbb{R}} | x - u | \, dF(x)$.*

The results obtained are intended to show that a connection between the weak convergence of $\mathcal{L}^1$- random variables and the convergence of the dispersion measures based on the dispersion functions can be used in studying weak limit theorems.

2. Proofs.

Proof of Theorem 1. By assumption (A) of Theorem 1 and since $X_n \in \mathcal{L}^1$, for all $n \in \mathbb{N}$, we see that $(X_n)_{n\in\mathbb{N}}$ are uniformly integrable. We thus get

$$\lim_{n\to\infty} \int_{\mathbb{R}} | x | \, dF_{X_n}(x) = \int_{\mathbb{R}} | x | \, dF_X(x).$$

It follows easily that the function $g_u(x) =| x - u | - | x |, x \in \mathbb{R}$, is continuous and bounded on $\mathbb{R}$, so we have

$$\lim_{n\to\infty} \int_{\mathbb{R}} (| x - u | - | x |) dF_{X_n}(x) = \int_{\mathbb{R}} (| x - u | - | x |) dF_X(x)$$

or

$$\lim_{n\to\infty} \int_{\mathbb{R}} | x - u | \, dF_n(x) = \int_{\mathbb{R}} | x - u | \, dF(x).$$

The proof is complete. ∎

Remark to Theorem 1. It should be noted that Theorem 1 does not hold without the assumption (A).

Let $\{X_n, n \geq 1\}$ be a sequence of random variables with the distribution functions

$$F_n(x) = \begin{cases} 0, & \text{if } \ x \leq 0, \\ 1 - \exp(-x), & \text{if } \ 0 < x \leq n, \\ \dfrac{1}{\exp(2n) - n\exp(n)}(x - n) + 1 - \exp(-n), & \text{if } \ n < x \leq \exp(n), \\ 1, & \text{if } \ \exp(n) < x. \end{cases}$$

and let X be a random variable with the distribution function

$$F(x) = \begin{cases} 0, & \text{if } \ x \leq 0, \\ 1 - \exp(-x), & \text{if } \ 0 < x. \end{cases}$$

Then, although

$$\sup_{n\in\mathbb{N}} E \mid X_n \mid < +\infty,$$

and

$$\lim_{n\to\infty} F_n(x) = F(x), \quad \forall x \in \mathbb{R}$$

but $D_{X_n}(u) \nrightarrow D_X(u)$ as $n \to +\infty$.

Lemma 1. *The family of $(D_X)_{X\in\mathcal{L}^1}$ is equally continuous on $\mathbb{R}$.*

Proof. Let $X \in \mathcal{L}^1$. For every $u \in \mathbb{R}$, we have

$$-1 \le D_X'^{-}(u) \le D_X'^{+}(u) \le 1, \quad \forall u \in \mathbb{R}.$$

Consequently, with $h > 0$

$$-1 \le D_X'^{+}(u) \le \frac{D_X(u+h) - D_X(u)}{h} \le D_X'^{-}(u+h) \le 1,$$

it follows that

$$\mid D_X(u+h) - D_X(u) \mid \le h.$$

The proof of Lemma 1 is complete. ∎

Proof of Theorem 2.
a. For every $\alpha, u, v \in \mathbb{R}, 0 \le \alpha \le 1$, we have

$$D_n(\alpha u + (1-\alpha)v) \le \alpha D_n(u) + (1-\alpha)D_n(v), \quad \forall n \in \mathbb{N}.$$

It follows that

$$\lim_{n\to\infty} D_n(\alpha u + (1-\alpha)v) \le \lim_{n\to\infty} (\alpha D_n(u) + (1-\alpha)D_n(v)).$$

Consequently,

$$D(\alpha u + (1-\alpha)v) \le \alpha D(u) + (1-\alpha)D(v).$$

Therefore $D(u)$ is convex on $\mathbb{R}$.
b. Since a convex function $\mathbb{R}$ has at most a countable number of undifferentiable points, it follows that $\mathbb{R} \setminus \mathcal{H}$ has at most a countaible number, too. Thus, the set $\mathcal{H}$ is dense in $\mathbb{R}$.

To complete the proof it remains to show that

$$\lim_{n\to\infty} D'_{X_n}(u) = D'_X(u), \quad \forall u \in \mathcal{H}. \tag{2.1}$$

Let $a, b \in \mathbb{R}, a < b$. Since the sequence $(D_n)_{n\in\mathbb{N}}$ is equally continuous on $[a, b]$ (see Lemma 1), it follows that the sequence $(D_n)_{n\in\mathbb{N}}$ converges to D at every point in $[a, b]$. Thus we get

$$\lim_{n\to\infty} \sup_{u\in[a,b]} \mid D_n(u) - D(u) \mid = 0.$$

Assume that $\delta > 0$, then there exists a n_0 such that

$$D(u) - \delta < D_n(u) < D(u) + \delta, \quad \forall n > n_0.$$

Then, we can show that for every $u \in \mathbb{R}$,

$$D'^{-}(u) \leq \liminf_{n\to\infty} D_n'^{-}(u) \leq \limsup_{n\to\infty} D_n'^{+}(u) \leq D'^{+}(u),$$

and consequently the proof follows.
c. Since $\mathcal{H} \subset \mathcal{H}_0$ and $\mathcal{H}$ is a dense set in $\mathbb{R}$, and $D'(u)$ is not a decreasing function on $\mathcal{H}_0$, from (2.1), we deduce that $-1 \leq D'(u) \leq 1$.

On the other hand

$$| E(X_n) | \leq \int_{\mathbb{R}} | x | \, dF_n(x),$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}} | x | \, dF_n(x) = \lim_{n\to\infty} D_n(0) = D(0).$$

Thus $(E(X_n))_{n\in\mathbb{N}}$ are bounded, i.e. there exists $M > 0$ such that $0 \leq | E(X_n) | < M$, for all $n \in \mathbb{N}$.

Since $D_n(u) \geq | E(X_n) - u |, \quad$ for all $u \in \mathbb{R}, n \in \mathbb{N}$, it follows that

$$D_n(u) - u \geq -E(X_n) > -M, \quad \forall u \in \mathbb{R}, n \in \mathbb{N},$$

and

$$D_n(u) + u \geq E(X_n) > -M, \quad \forall u \in \mathbb{R}, n \in \mathbb{N}.$$

Therefore

$$\lim_{n\to+\infty} D_n(u) - u = D(u) - u \geq -M, \quad \forall u \in \mathbb{R},$$

and

$$\lim_{n\to+\infty} D_n(u) + u = D(u) + u \geq -M, \quad \forall u \in \mathbb{R}.$$

Finally, the functions $f(u) = D(u) - u$ and $g(u) = D(u) + u$ are convex on $\mathbb{R}$, where $f'(u) \leq 0$, and $g'(u) \geq 0$. We thus conclude that

$$\lim_{u\to+\infty} f'(u) = 0$$

and

$$\lim_{u\to-\infty} g'(u) = 0,$$

thus, for $u \in \mathcal{H}_0$,

$$\lim_{u\to+\infty} D'(u) = 1, \quad \lim_{u\to-\infty} D'(u) = -1.$$

■

Proof of Theorem 3.

a. The proof is immediate from the part b) of Theorem 2 and the properties of the dispersion function from Sec. 1.

b. We have

$$D_n(u) - u \geq -E(X_n), \quad D_n(u) + u \geq E(X_n), \quad \forall n \in \mathbb{N},$$

and

$$D(u) - u \geq -E(X), \quad D(u) + u \geq E(X).$$

By virtue of the properties of dispersion functions and remark above, it follows that for all $\epsilon > 0$, there exist u_0 and n_0, such that

$$-E(X) \leq D(u) - u \leq -E(X) + \frac{\epsilon}{2}, \quad \forall u \geq u_0,$$

and

$$D(u_0) - u_0 + \frac{\epsilon}{2} \geq D_n(u_0) - u_0, \quad \forall n > n_0.$$

In short, we have

$$-E(X) + \epsilon \geq D(u_0) - u_0 + \frac{\epsilon}{2} \geq D_n(u_0) - u_0 \geq -E(X_n), \quad \forall n > n_0, u \geq u_0,$$

or for all $\epsilon > 0$, there exists n_0 such that

$$E(X) \leq E(X_n) + \epsilon, \quad \forall n > n_0.$$

In the same manner we can see that for all $\epsilon > 0$, there exists n_1 such that

$$E(X) \leq E(X_n) - \epsilon, \quad \forall n > n_1.$$

Consequently,

$$\lim_{n\to\infty} E(X_n) = E(X).$$

According to properties of the dispersion function, one finds that

$$\sup_{u\in\mathbb{R}} | D_n(u) - D(u) | < +\infty.$$

It remains to prove that

$$\lim_{n\to\infty} \sup_{u\in\mathbb{R}} | D_n(u) - D(u) | = 0. \tag{2.2}$$

Let (2.2) be not true, i.e. there exists $\epsilon > 0$ such that for all n_0, there exists $n > n_0$ such that

$$\sup_{u\in\mathbb{R}} | D_n(u) - D(u) | > \epsilon.$$

For every $k \in \mathbb{N}$, let us put

$$n_k = \inf\{n \in \mathbb{N} \mid n > n_1 + k, \sup_{u\in\mathbb{R}} | D_n(u) - D(u) | > \epsilon\}, \tag{2.3}$$

where n_1 is a positive integer number satisfying $\sup_{u\in\mathbb{R}} | D_{n_1}(u) - D(u) | > \epsilon$.

It follows that, for every $k \in \mathbb{N}$, there are u_{n_k} such that

$$| D_{n_k}(u_{n_k}) - D(u_{n_k}) | > \epsilon. \tag{2.4}$$

According to Theorem 2, the sequence $(u_{n_k})_{k\in\mathbb{N}}$ is not bounded. Thus we can extract from $(u_{n_k})_{k\in\mathbb{N}}$ a subsequence convergent to $+\infty$ or to $-\infty$. Without loss of generality we can assume

$$\lim_{k\to\infty} u_{n_k} = +\infty.$$

Then we have

$$\lim_{m\to\infty}\lim_{k\to\infty}(D_m(u_{n_k})-u_{n_k}) = \lim_{m\to\infty}(-EX_m) = -E(X),$$
$$\lim_{k\to\infty}\lim_{m\to\infty}(D_m(u_{n_k})-u_{n_k}) = \lim_{k\to\infty}(D(u_{n_k})-u_{n_k}) = -E(X).$$

Thus

$$\lim_{k\to\infty}(D_{n_k}(u_{n_k})-u_{n_k}) = -E(X). \tag{2.5}$$

On the other hand, it is obvious that

$$\lim_{k\to\infty}(D(u_{n_k})-u_{n_k}) = -E(X). \tag{2.6}$$

But this shows that (2.3) - (2.6) are in contradictions, and we have the desired proof. ■

Proof of Theorem 4. According to the Theorem 3, if we put

$$F(x) = \begin{cases} \frac{1}{2}(D^{'}(x)+1), & \text{if } x \in \mathcal{H}, \\ \lim_{x_n\to x} F(x_n) & \text{if } x \notin \mathcal{H}(x_n < x, x_n \in \mathcal{H}), \end{cases}$$

then $F(x)$ is a distribution function and $F_n \Rightarrow F$.

On the other hand, for all $a, b \in \mathcal{H}, a < b$, we have

$$\lim_{n\to\infty}\int_a^b |\, x \,|\, dF_n(x) = \int_a^b |\, x \,|\, dF(x).$$

Since $\int_{\mathbb{R}} |\, x \,|\, dF_n(x) < M$, it shows that $\int_{\mathbb{R}} |\, x \,|\, dF(x) < M$, thus we conclude that F is a distribution function of a random variable with finite mean.

Let $D^\star(u)$ be a corresponding dispersion function of the function F. By virtue of the Theorem 1 we have

$$\lim_{n\to\infty} D_n(u) = D^\star(u), \quad \forall u \in \mathbb{R}.$$

It follows that $D^\star(u) = D(u)$, for all $u \in \mathbb{R}$, and this completes the proof of the theorem. ■

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