Journal of Algebra and Its Applications
Vol. 8, No. 3 (2009) 379–387


# ON SMALL INJECTIVE RINGS AND MODULES*

LE VAN THUYET† and TRUONG CONG QUYNH‡
*Department of Mathematics*
*Hue University, Vietnam*
†*lvthuyethue@gmail.com*
‡*tcquynh@dce.udn.vn*

Received 26 April 2007
Accepted 19 November 2008

Communicated by D. V. Huynh

A right $R$-module $M_R$ is called *small injective* if every homomorphism from a small right ideal to $M_R$ can be extended to an $R$-homomorphism from $R_R$ to $M_R$. A ring $R$ is called right small injective, if the right $R$-module $R_R$ is small injective. We prove that $R$ is semiprimitive if and only if every simple right (or left) $R$-module is small injective. Further we show that the Jacobson radical $J$ of a ring $R$ is a noetherian right $R$-module if and only if, for every small injective module $E_R$, $E^{(\mathbb{N})}$ is small injective.

*Keywords*: Small injective rings (modules); PF rings; QF rings.

2000 Mathematics Subject Classification: 16D50, 16D70, 16D80

## 1. Introduction

Throughout the paper, $R$ represents an associative ring with identity $1 \neq 0$ and all modules are unitary $R$-modules. We write $M_R$ (resp., ${}_RM$) to indicate that $M$ is a right (resp., left) $R$-module. By $J$ (resp., $Z_r$, $S_r$) we denote the Jacobson radical (resp., the right singular ideal, the right socle) of $R$. For a module $M_R, E(M_R)$ stands for its injective hull. If $X$ is a subset of $R$, the right (resp., left) annihilator of $X$ in $R$ is denoted by $r_R(X)$ (resp., $l_R(X)$) or simply $r(X)$ (resp., $l(X)$) if no confusion appears. If $N$ is a submodule of $M$ (resp., proper submodule) we denote it by $N \leq M$ (resp., $N < M$). Moreover, we write $N \leq^e M, N \ll M$, $N \leq^{\oplus} M$ and $N \leq^{\max} M$ to indicate that $N$ is an essential submodule, a small submodule, a direct summand and a maximal submodule of $M$, respectively. A module $M$ is called uniform if $M \neq 0$ and every nonzero submodule of $M$ is essential in $M$.

Recall that a ring $R$ is called *right mininjective* if every homomorphism from a minimal right ideal to $R$ is given by left multiplication by an element of $R$. $R$ is called

*The work was supported by the Natural Science Council of Vietnam.

*right Kasch* if every simple right $R$-module embeds in $R$; or equivalently, $l(I) \neq 0$ for every maximal right ideal $I$ of $R$. A ring $R$ is called quasi-Frobenius (briefly, *QF-ring*) if it is right (or left) artinian and right (or left) self-injective. $R$ is said to be right pseudo-Frobenius (briefly, *PF-ring*) if $R_R$ is an injective cogenerator in the category of right $R$-modules.

Yousif and Zhou [13] proved that, for a semiperfect ring $R$ with an essential right socle, $R$ is right self-injective if and only if $R$ is right small injective. In [10], Shen and Chen proved that if $R$ is semilocal, then $R$ is right self-injective if and only if $R$ is right small injective. They also gave some new characterizations of QF rings and right PF rings in terms of small injectivity.

In this paper, we show that $R$ is PF if and only if $R$ is right small injective, right Kasch and left min-CS. We also prove that if every simple right (resp., left) $R$-module is small injective, then $R$ is semiprimitive. Finally, using the small injectivity, we give some characterizations for the Jacobson radical of a ring $R$ to be a noetherian right $R$-module.

General background materials can be found in [2, 4, 8, 12].

## 2. Results

A module $M_R$ is called *small injective* if every homomorphism from a small right ideal to $M_R$ can be extended to an $R$-homomorphism from $R_R$ to $M_R$. A ring $R$ is called right small injective if $R_R$ is small injective.

**Examples.** (i) Let $R = \mathbb{Z}$ be the ring of integers, then $R$ is small injective but not self-injective.

(ii) Let $R = \{\left(\begin{smallmatrix} n & x \\ 0 & n \end{smallmatrix}\right) | n \in \mathbb{Z}, x \in \mathbb{Z}_2\}$ (see [13, Example 1.6]). Then $R$ is a commutative ring and $J = S_r = \{\left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right) | x \in \mathbb{Z}_2\}$. Therefore $R$ is small injective.

We claim that $R$ is not injective. Let $I = \{\left(\begin{smallmatrix} 2n & 0 \\ 0 & 2n \end{smallmatrix}\right) | n \in \mathbb{Z}\}$ be an ideal of $R$ and $g : I \to R$ with $f(\left(\begin{smallmatrix} 2n & 0 \\ 0 & 2n \end{smallmatrix}\right)) = \left(\begin{smallmatrix} 0 & \bar{n} \\ 0 & 0 \end{smallmatrix}\right)$, then $g$ is a homomorphism. Assume that $\bar{g} : R \to R$ be a homomorphism that extends $g$. There exists $\left(\begin{smallmatrix} n_1 & x_1 \\ 0 & n_1 \end{smallmatrix}\right) \in R$ such that $\bar{g}(\left(\begin{smallmatrix} n & x \\ 0 & n \end{smallmatrix}\right)) = \left(\begin{smallmatrix} n_1 & x_1 \\ 0 & n_1 \end{smallmatrix}\right)\left(\begin{smallmatrix} n & x \\ 0 & n \end{smallmatrix}\right)$ for all $\left(\begin{smallmatrix} n & x \\ 0 & n \end{smallmatrix}\right) \in R$. Thus $f(\left(\begin{smallmatrix} 2n & 0 \\ 0 & 2n \end{smallmatrix}\right)) = \left(\begin{smallmatrix} n_1 & x_1 \\ 0 & n_1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 2n & 0 \\ 0 & 2n \end{smallmatrix}\right)$. It implies that $\left(\begin{smallmatrix} 0 & \bar{n} \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2nx_1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. This is a contradiction.

A ring $R$ is defined to be *right minsymmetric* (cf. [8]) if, for $k \in R$, whenever $kR$ is a minimal right ideal of $R$ then $Rk$ is a minimal left ideal of $R$.

Next, we show the following lemmas.

**Lemma 2.1 (McCoy's Lemma).** *Let $R$ be a ring and $a, c \in R$. If $b = a$-$aca$ is a regular element of $R$, then so is $a$.*

**Proof.** This follows easily from the definition. □

**Lemma 2.2.** *Let $R$ be a right minsymmetric ring with $S_r \leq^e R_R$. If the ascending chain $r(a_1) \leq r(a_2a_1) \leq \cdots r(a_na_{n-1}\cdots a_1) \leq \cdots$ terminates for every infinite sequence $a_1, a_2, \ldots$ in $R$, then $R$ is right perfect.*

**Proof.** Since $R$ is right minsymmetric, $S_r \leq S_l$. Then $J \leq l(S_r) = Z_r$ since $S_r \leq^e R_R$. Now we prove that $Z_r$ is right $T$-nilpotent. In fact that, let $x_1, x_2, \ldots, x_n, \ldots \in Z_r$. We have

$$r(x_1) \leq r(x_2x_1) \leq \cdots.$$

By hypothesis, there exists $k \in \mathbb{N}$ such that $r(x_k \cdots x_2x_1) = r(x_{k+1}x_k \cdots x_2x_1)$. If $x_k \cdots x_2x_1 \neq 0$, then $x_k \cdots x_2x_1R \cap r(x_{k+1}) \neq 0$ (since $x_{k+1} \in Z_r$). There exists $r \in R$ such that $0 \neq x_k \cdots x_2x_1r \in r(x_{k+1})$ and so $x_{k+1}x_k \cdots x_2x_1r = 0$. That means $r \in r(x_{k+1}x_k \cdots x_2x_1) = r(x_k \cdots x_2x_1)$ or $x_k \cdots x_2x_1r = 0$, this is a contradiction. Hence $x_k \cdots x_2x_1 = 0$. Thus $Z_r$ is right $T$-nilpotent. It implies that $Z_r \leq J$. Thus $J = l(S_r) = Z_r$. And then $J$ is also right $T$-nilpotent.

Now we prove that $R/J$ is a von Neumann regular ring. Let $a_1 \notin J$, then $r(a_1)$ is not an essential right ideal of $R$. Hence there exists $I \leq R_R$ such that $I \neq 0$ and $I \cap r(a_1) = 0$. But $S_r \leq^e R_R$, there exists a simple right ideal $bR$ of $R$ such that $bR \leq I$. Then $bR \cap r(a_1) = 0$, that is $a_1b \neq 0$. Therefore $R = l(bR \cap r(a_1)) = l(b) + Ra_1$ by [8, Proposition 2.26], write $1 = c_1a_1 + t$, where $t \in l(b), c_1 \in R$. So $b = c_1a_1b$. It is easy to see that $0 \neq b \in r(a_1 - a_1c_1a_1)\backslash r(a_1)$, $r(a_1) < r(a_1 - a_1c_1a_1)$. Put $a_2 = a_1 - a_1c_1a_1$. We denote by $\bar{a} = a + J \in R/J$. If $a_2 \in J$, then we have $\bar{a}_1 = \bar{a}_1\bar{c}_1\bar{a}_1$, i.e., $\bar{a}_1$ is a regular element of $R/J$. If $a_2 \notin J$, there exists $a_3 \in R$ such that $r(a_2) < r(a_3)$ with $a_3 = a_2 - a_2c_2a_2$ for some $c_2 \in R$ by the preceding proof. Repeating the above-mentioned process, we get a strictly ascending chain $r(a_1) < r(a_2) < \cdots$, where $a_{i+1} = a_i - a_ic_ia_i$ for some $c_i \in R$, $i = 1, 2, \ldots$. Let $b_1 = a_1, b_2 = 1 - a_1c_1, \ldots, b_{i+1} = 1 - a_ic_i, \ldots$, then $a_1 = b_1, a_2 = b_2b_1, \ldots, a_{i+1} = b_{i+1}b_i \cdots b_2b_1, \ldots$, whence we have the following strictly ascending chain $r(b_1) < r(b_2b_1) < \cdots$, which contradicts the hypothesis. So there exists a positive integer $m$ such that $a_{m+1} \in J$, i.e., $a_m - a_mc_ma_m \in J$. This shows that $\bar{a}_m$ is a regular element of $R/J$, and hence $\bar{a}_{m-1}, \bar{a}_{m-2}, \ldots, \bar{a}_1$ are regular elements of $R/J$ by Lemma 2.1, i.e., $R/J$ is von Neumann regular.

To show that $R/J$ is semisimple, by [7, Corollary 2.16], we only need to prove that $R/J$ contains no infinite sets of nonzero orthogonal idempotents. This can be proved by arguing as [3, p. 2107]. □

From above lemma, we give some properties of PF and QF rings.

**Corollary 2.3.** *Let $R$ be a right small injective ring with $S_r \leq^e R_R$. If the ascending chain $r(a_1) \leq r(a_2a_1) \leq \cdots r(a_na_{n-1}\cdots a_1) \leq \cdots$ terminates for every infinite sequence $a_1, a_2, \ldots$ in $R$, then $R$ is right PF.*

**Proof.** By Lemma 2.2, $R$ is right semiperfect. But since $R$ is right small injective, $R$ is right self-injective by [10, Theorem 3.16]. Thus $R$ is right self-injective, semiperfect and $S_r \leq^e R_R$; i.e., $R$ is right PF by [6, Theorem 2.1]. □

**Corollary 2.4.** *A ring $R$ is QF if and only if $R$ is right small injective with $S_r \leq^e R_R$, $l(J^2)$ is countable generated as a left ideal and the ascending chain $r(a_1) \leq r(a_2a_1) \leq \cdots \leq r(a_na_{n-1}\cdots a_1) \leq \cdots$ terminates for every infinite sequence $a_1, a_2, \ldots$ in $R$.*

**Proof.** By Lemma 2.2, [9, Lemma 2.2] and [13, Theorem 2.18]. □

With a left and right small injective ring, we have:

**Proposition 2.5.** *Let $R$ be a ring. Then the following conditions are equivalent*:

(1) *$R$ is QF.*
(2) *$R$ is a right and left small injective ring with $S_r \leq^e R_R$ and the ascending chain $r(a_1) \leq r(a_2a_1) \leq \cdots \leq r(a_na_{n-1}\cdots a_1) \leq \cdots$ terminates for every infinite sequence $a_1, a_2, \ldots$ in $R$.*

**Proof.** (1) ⇒ (2) is clear.

(2) ⇒ (1). By Lemma 2.2, $R$ is right perfect. But $R$ is also left small injective which yields $R$ is left self-injective. Thus $R$ is QF by [6, Corollary 2.3]. □

**Remark.** The condition "$S_r \leq^e R_R$" in Proposition 2.5 can be not omitted. Let $R = \mathbb{Z}$ be the ring of integer numbers, then $R$ is small injective, noetherian but $R$ is not QF.

A ring $R$ is called *left CS* (resp., *left min-CS*) if every left ideal (resp., minimal left ideal) is essential in a direct summand of ${}_RR$. It is well-known that $R$ is right PF if and only if $R$ is right self-injective, right Kasch. But it is unknow whether a right small injective, right Kasch ring is right PF.

**Theorem 2.6.** *Let $R$ be a ring. Then the following conditions are equivalent*:

(1) *$R$ is right PF.*
(2) *$R$ is right small injective, right Kasch and left min-CS.*
(3) *$R$ is right small injective, right Kasch and $lr(a)$ is essential in a direct summand of ${}_RR$ for every simple right (resp., left) ideal $aR$ (resp., $Ra$) of $R$.*

**Proof.** (1) ⇒ (2) is clear.

(2) ⇒ (3). Assume that $Ra$ is a simple left ideal. If $(Ra)^2 \neq 0$ then $Ra \leq^{\oplus} {}_RR$ which implies that $lr(a) = Ra$. Otherwise, $a^2 = 0$ and so $a \in J$. Since $R$ is right small injective, $Ra = lr(a)$ is a simple left ideal. Then by (2), $lr(a)$ is essential in a direct summand of ${}_RR$. On the other hand, if $bR$ is a simple right ideal then $Rb$ is

a simple left ideal (because $R$ is right minsymmetric). It is easy to see that $lr(b)$ is essential in a direct summand of ${}_RR$.

(3) $\Rightarrow$ (1). Let $T$ be a maximal right ideal of $R$. Since $R$ is right Kasch, $l(T) \neq 0$. There exists $0 \neq a \in l(T)$ or $T \leq r(a)$ which yields $T = r(a)$. But $aR \cong R/r(a)$ and so $aR$ is a right simple ideal. Therefore $l(T) = lr(a) \leq^e Re$ for some $e^2 = e \in R$ by hypothesis. Thus $R$ is semiperfect by [8, Lemma 4.1]. This implies that $R$ is right self-injective and so right PF by [8, Corollary 7.32]. □

**Corollary 2.7.** *If $R$ is right small injective, right Kasch and left CS, then $R$ is right PF.*

Recall that a ring $R$ is called *semiprimitive* if $J = 0$.

**Theorem 2.8.** *Let $R$ be a ring. Then the following conditions are equivalent:*

(1) *$R$ is semiprimitive.*
(2) *Every right (or left) $R$-module is small injective.*
(3) *Every simple right (or left) $R$-module is small injective.*

**Proof.** It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1). Suppose that $J \neq 0$, let $0 \neq a \in J$, that is $aR \ll R_R$. If $J + r(a) < R$, then we take a maximal right ideal $I$ of $R$ such that $J + r(a) \leq I$. Then $R/I$ is small injective by (3). We define $\varphi : aR \to R/I$ by $\varphi(ar) = r + I$. Then $\varphi$ is a well-defined $R$-homomorphism. So there exists $c \in R$ such that $1 + I = ca + I$ and then $1 - ca \in I$. But $ca \in J \leq I$ which yields $1 \in I$, a contradiction. Therefore $J + r(a) = R$ and so $r(a) = R$ (because $J \ll R_R$). So $a = 0$, which is a contradiction. □

**Proposition 2.9.** *If every simple singular right $R$-module is small injective, then for every $a \in J$, $r(a) \leq^{\oplus} R_R$ and $aR$ is projective.*

**Proof.** For every $a \in J$, let $L = RaR + r(a)$. There exists $K_R \leq R_R$ such that $L \oplus K \leq^e R_R$. Assume that $L \oplus K \neq R$, then there exists a maximal right ideal $I$ of $R$ such that $L \oplus K \leq I$ and so $I \leq^e R_R$. Therefore $R/I$ is small injective by hypothesis. We define $\varphi : aR \to R/I$ by $\varphi(ar) = r + I$. Then $\varphi$ is a well-defined $R$-homomorphism. So there exists $c \in R$ such that $1 + I = ca + I$ and then $1 - ca \in I$. But $ca \in RaR \leq I$ which yields $1 \in I$, a contradiction. Thus $L \oplus K = R$ or $RaR + (r(a) \oplus K) = R$ which implies that $r(a) \oplus K = R$ (since $RaR \ll R_R$). Then $r(a) = (1 - e)R$ for some $e^2 = e \in R$ and it follows that $a_s x = a'_s x$. Let $\psi : eR \to aeR$ be defined by $\psi(er) = aer$ for all $r \in R$. Then $\psi$ is a well-defined $R$-epimorphism. It is easy to see that $\mathrm{Ker}(\psi) = eR \cap r(a) = 0$. Hence $\psi$ is an isomorphism and then $aR = aeR$ is projective. □

**Corollary 2.10.** *If every simple singular right $R$-module is small injective, then $Z_r \cap J = 0$.*

Recall that a ring $R$ is called *zero insertive* (*briefly ZI*) [11], if for $a, b \in R, ab = 0$ implies $aRb = 0$. Note that if $R$ is a ZI ring, then every idempotent in $R$ is central and $r(a), l(a)$ are two-sided ideals with $a \in R$.

**Lemma 2.11.** *If $R$ is a ZI ring, then $RaR + r(a)$ is an essential right ideal of $R$, for every $a \in R$.*

**Proof.** Given $a \in R$ and assume that $(RaR + r(a)) \cap I = 0$, where $I \leq R_R$. Then $aI \leq I \cap RaR = 0$. Hence $I \leq r(a)$; whence $I = 0$. □

**Proposition 2.12.** *Let $R$ be a ZI ring. If every simple singular right* (*or left*) *$R$-module is small injective, then $R$ is semiprimitive.*

**Proof.** Suppose that there exists $0 \neq a \in J$; whence $RaR \ll R_R$. If $RaR + r(a) < R$, then there exists a maximal right ideal $I$ of $R$ such that $RaR + r(a) \leq I$. By Lemma 2.11, $I$ is an essential right ideal of $R$. Therefore $R/I$ is small injective by hypothesis. Let $\varphi : aR \to R/I$ be defined via $\varphi(ar) = r + I$ for all $r \in R$. It is easy to see that $\varphi$ is a well-defined $R$-homomorphism. Since $R/I$ is small injective, there exists $c \in R$ such that $1 + I = \varphi(a) = ca + I$. But $ca \in RaR \leq I$ which yields $1 \in I$, a contradiction. Therefore $RaR + r(a) = R$ which implies that $r(a) = R$ (since $RaR \ll R_R$). So $a = 0$, which is a contradiction. □

Finally we consider the direct-sum representation of small injective module.

Let $M$ be a right $R$-module. We denote that $r_J(N) = \{a \in J | Na = 0\}$ and $l_M(I) = \{m \in M | mI = 0\}$ where $N \subseteq M$ and $I \subseteq J$. Then $r_J(X) \leq J_R$ and $l_M(I) \leq {}_S M$ where $S = \text{End}(M_R)$.

**Lemma 2.13.** *The following conditions are equivalent for a right $R$-module $M$:*

(1) *$R$ satisfies the ACC for right ideals of form the $r_J(X)$, where $X \subseteq M$.*
(2) *For each right ideal $I_R \leq J_R$ there corresponds a finitely generated right ideal $I_1 \leq I_R$ such that $l_M(I) = l_M(I_1)$.*

**Proof.** (1) $\Rightarrow$ (2). The condition that $R$ satisfies the ACC for right ideals of the form $r_J(X)$, where $X \subseteq M$ is equivalent to that $R$ satisfies the DCC for $l_M(A)$ where $A \subseteq J$. Let $I_1$ be a finitely generated right ideal of $I_R$ such that $l_M(I_1)$ is minimal in the set

$$\Omega = \{l_M(K) | K_R \text{ is finitely generated and } K_R \leq I_R\}.$$

If $x \in I$, then $H = I_1 + xR \leq I_R$ is finitely generated and $l_M(H) \leq l_M(I_1)$. By the choice of $I_1$, we have $l_M(H) = l_M(I_1)$, so $l_M(I_1)x = 0$. It implies that $l_M(I_1)I = 0$, that is $l_M(I_1) \leq l_M(I)$. But $I_1 \leq I$ implies $l_M(I_1) \geq l_M(I)$, so $l_M(I) = l_M(I_1)$.

(2) $\Rightarrow$ (1). Let $I_1 \leq I_2 \leq \cdots \leq I_n \cdots$ be a chain of right ideals, where $I_i = r_J(M_i)$ and $M_i \subseteq M$ for each $i$, let $X_i = l_M(I_i)$ for each $i = 1, 2, \ldots,$ and

$I = \bigcup_{i=1}^{\infty} I_i$, then $I \leq J_R$. By (2), there exists a finitely generated right ideal $I_1$ of $I$ such that $l_M(I) = l_M(I_1)$. Since $I_1$ is finitely generated, there is an integer $k$ such that $I_1 \leq I_m$ for all $m \geq k$, that is $l_M(I_1) \geq l_M(I_m) = X_m$ for all $m \geq k$. But $l_M(I_1) = \bigcap_{i=1}^{\infty} l_M(I_i) = \bigcap_{i=1}^{\infty} X_i$, that is $l_M(I_1) = X_m$ for all $m \geq k$. Then $I_m = r_J(X_m) = I_k$ for all $m \geq k$, proving (1). □

Now an argument of the proof of Faith [5, Proposition 3], we have:

**Proposition 2.14.** *The following conditions on a small injective module $E_R$ are equivalent*:

(1) *$E^{(\mathbb{N})}$ is small injective.*
(2) *$R$ satisfies the ACC for right ideals of form $r_J(X)$, where $X \subseteq E$.*
(3) *$E^{(S)}$ is small injective for any index set $S$.*

**Proof.** (3) ⇒ (1) is clear.

(1) ⇒ (2). Assume that $r_J(X_1) < r_J(X_2) < \cdots < r_J(X_n) < \cdots, X_i \subseteq E$. Then

$$l_E r_J(X_1) > l_E r_J(X_2) > \cdots > l_E r_J(X_n) > \cdots$$

because $r_J l_E r_J(X) = r_J(X)$ for any $X \subseteq E$. For each $i \geq 1$, choose $m_i \in l_E r_J(X_i) \backslash l_E r_J(X_{i+1})$. Hence there exists $a_{i+1} \in r_J(X_{i+1})$ such that $m_i a_{i+1} \neq 0$. Let $T = \bigcup_{i=1}^{\infty} r_J(X_i)$ which yields $T \ll R_R$. Then, for all $t \in T$ there exists $n_t \geq 1$ such that $t \in r_J(X_i)$ for all $i \geq n_t$. Then $m_i t = 0$ for all $i \geq n_t$, and if $\bar{m} = (m_i)_i$, then $\bar{m}t \in E^{(\mathbb{N})}$ for every $t \in T$. Hence $\varphi_{\bar{m}} : T \to E^{(\mathbb{N})}$ is well-defined by $\varphi_{\bar{m}}(t) = \bar{m}t$. Since $E^{(\mathbb{N})}$ is small injective by hypothesis, $\varphi_{\bar{m}}$ extends to $\psi : R \to E^{(\mathbb{N})}$. So $\varphi_{\bar{m}}(t) = \bar{m}t = \psi(t) = \psi(1)t$ for all $t \in T$. But $\psi(1) \in E^{(\mathbb{N})}$, so there exists $k \geq 1$ such that $m_i t = 0$ for all $i \geq k$ and all $t \in T$. In particular, $m_i a_{i+1} = 0$ for all $i > k$, which is a contradiction.

(2) ⇒ (3). Let $I$ be a right ideal and $I_R \leq J_R$. By Lemma 2.13, there is $I_1 = r_1 R + r_2 R + \cdots r_n R \leq I_R$ such that $l_M(I) = l_M(I_1)$. Let $\varphi : I \to E^{(S)}$ be an $R$-homomorphism. Since $E^S$ is small injective by [10, Proposition 3.5], there exists an element $a \in E^S$ such that $\varphi(r) = ar$ for all $r \in I$. In particular $\varphi(r_i) = ar_i \in E^{(S)}, i = 1, 2, \ldots,$ there exists an element $a' \in E^{(S)}$ such that $a_s r_i = a'_s r_i$ for all $s \in S, i = 1, 2, \ldots,$ where $g_s$ is the $s$ th-coordinate of any $g \in E^S$. Since $\{r_1, r_2, \ldots, r_n\}$ generates $I_1$, this implies that $ar = a'r$ for all $r \in I_1$, whence $(a_s - a'_s) \in l_M(I_1)$ for all $s \in S$. Since $l_M(I) = l_M(I_1)$, it follows that $a_s x = a'_s x$ for all $s \in S, x \in I$, that is $ax = a'x$ for all $x \in I$. Thus $\varphi(x) = a'x$ for all $x \in I$ with $a' \in E^{(S)}$, so $E^{(S)}$ is small injective. □

**Lemma 2.15 ([1]).** *Let $M$ be a right $R$-module. Then* $\mathrm{Rad}(M)$ *is noetherian if and only if $M$ has ACC on small submodules.*

**Lemma 2.16.** *Every direct summand of a small injective module is small injective.*

**Proof.** It is clear. □

It is well-know that $R$ is right noetherian if and only if $E^{(\mathbb{N})}$ is injective for every injective module $E_R$. We also have:

**Theorem 2.17.** *For a ring $R$, the following conditions are equivalent*:

(1) *$J$ is noetherian as a right $R$-module.*
(2) *Every direct sum of small injective right $R$-modules is small injective.*
(3) *If $M_1, M_2, \ldots$ are simple right modules then $\bigoplus_{i=1}^{\infty} E(M_i)$ is small injective.*
(4) *$E^{(\mathbb{N})}$ is small injective for every small injective module $E_R$.*

**Proof.** (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) are clear.

(1) $\Rightarrow$ (2). Let $E = \bigoplus_{i\in I} E_i$, where each $E_i$ is small injective, $T_R \leq J_R$, and $\varphi : T \to E$ an $R$-homomorphism. Since $T$ is finitely generated by (1), we can write $\varphi : T \to E_1 = \bigoplus_{i\in I_1} E_i$, where $I_1 \subseteq I$ is a finite subset. Since $E_1$ is small injective, let $\bar{\varphi} : R \to E_1$ extends $\varphi$. Then $\iota\bar{\varphi}$ extends $\varphi$, where $\iota : E_1 \to E$ is the inclusion.

(3) $\Rightarrow$ (1). Let $I_0 < I_1 < \cdots$ be a strictly ascending chain of small finitely generated right ideals of $R$. Let $I = \bigcup_{i=0}^{\infty} I_i$, then $I$ is a small right ideal of $R$. For each $i \geq 1$ choose $M_i \leq^{\max} I_i$ such that $I_{i-1} \leq M_i$. Thus $K_i = I_i/M_i$ is a simple right $R$-module. We define $\eta_i : I_i/I_{i-1} \to K_i$ by $\eta_i(x + I_{i-1}) = x + M_i$, and write $\iota_i : K_i \to E(K_i)$ for the inclusion. Since $E(K_i)$ is injective, let $\varphi_i : I/I_{i-1} \to E(K_i)$ be the homomorphism such that $\varphi_i = \iota_i\eta_i$ (see the following diagram):

$$\begin{array}{ccc}
I_i/I_{i-1} & \hookrightarrow & I/I_{i-1} \\
\eta_i \downarrow & & \\
K_i & \swarrow \varphi_i & \\
\iota_i \downarrow & & \\
E(K_i) & &
\end{array}$$

Since $M_i < I_i$, there exists $c_i \in I_i \backslash M_i$ such that $\varphi_i(c_i + I_{i-1}) \neq 0$. For each $t \in I$, choose $n_t \geq 1$ such that $t \in I_{i-1}$ for all $i \geq n_t$ and so $\varphi_i(t + I_{i-1}) = 0$ for all $i \geq n_t$, so we can define $\alpha : I \to \bigoplus_{i=1}^{\infty} E(K_i)$ by $\alpha(t) = (\varphi_i(t + I_{i-1}))_i$. Since $\bigoplus_{i=1}^{\infty} E(K_i)$ is small injective by (3), $\alpha$ extends to $\bar{\alpha} : R \to \bigoplus_{i=1}^{\infty} E(K_i)$. Write $\bar{\alpha}(1) = (b_i)_i$, so there exists $n \geq 1$ such that $b_i = 0$ for all $i \geq n$. Given any $t \in I$, we have $(\varphi_i(t + I_{i-1}))_i = \alpha(t) = \bar{\alpha}(t) = \bar{\alpha}(1)t = (b_it)_i$. Thus $\varphi_i(t + I_{i-1}) = 0$ for all $i \geq n$ and all $t \in I$. But $\varphi_n(c_n + I_{n-1}) \neq 0$ by the definition of $\varphi_i$, and this contradiction proves (1).

(4) $\Rightarrow$ (1). Let $I_1 \leq I_2 \leq \cdots$ be a chain of small right ideals. For each $i$, let $E_i = E(R/I_i)$, and $E = \bigoplus_{i=1}^{\infty} E_i$. For every $i \geq 1, \prod_{j=1}^{\infty} E_j = E_i \bigoplus(\prod_{j\neq i} E_j)$. Let $M_i = \prod_{j=1}^{\infty} E_j$, then $M_i$ is small injective by [10, Proposition 3.5]. By above

notation, we have

$$\bigoplus_{i=1}^{\infty} M_i = \left(\bigoplus_{i=1}^{\infty} E_i\right) \bigoplus \left(\bigoplus_{i=1}^{\infty} \prod_{j\neq i} E_j\right).$$

By assumption, $\bigoplus_{i=1}^{\infty} M_i$ is small injective. Thus $E$ itself is small injective by Lemma 2.16. Now the $R$-homomorphism $f : \bigcup_{i=1}^{\infty} I_i \to E$ defined by $f(t) = (t+I_i)_i$ extends to $\bar{f} : R \to E$. Let $n \geq 1$ such that $\bar{f}(1) \in \bigoplus_{j=1}^{n} E_j$. Then $f(\bigcup_{i=1}^{\infty} I_i) \leq \bigoplus_{j=1}^{n} E_j$. So, if $t \in \bigcup_{i=1}^{\infty} I_i$ then $t \in I_m$ for all $m > n$, and so $\bigcup_{i=1}^{\infty} I_i = I_{n+1}$ and the chain should terminate. □

**Corollary 2.18.** *The following conditions are equivalent for a ring $R$:*

(1) *$J$ is noetherian as a right $R$-module.*
(2) *Every direct sum of injective right $R$-modules is small injective.*
(3) *$E^{(\mathbb{N})}$ is small injective for every injective modules $E_R$.*


## Acknowledgment

The authors wish to thank the referee for many useful suggestions and comments.


## References


[1] I. Al-Khazzi and P. F. Smith, Modules with chain conditions on superfluous submodules, *Comm. Algebra* **19** (1991) 2331–2351.
[2] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules* (Springer-Verlag, New York, 1974).
[3] J. Chen and N. Ding, On general principally injective rings, *Comm. Algebra* **27** (1999) 2097–2116.
[4] N. V. Dung, D. V. Huynh, P. F. Smith and R. Wisbauer, *Extending Modules* (Pitman, London, 1994).
[5] C. Faith, Rings with ascending chain conditions on annihilators, *Nagoya Math. J.* **27** (1966) 179–191.
[6] C. Faith and D. V. Huynh, When self-injective rings are QF: A report on a problem, *J. Algebra Appl.* **1** (2002) 75–105.
[7] K. R. Goodearl, *Von Neumann Regular Rings* (Pitman, London, 1979).
[8] W. K. Nicholson and M. F. Yousif, *Quasi-Frobenius Rings* (Cambridge University Press, Cambridge, 2003).
[9] T. C. Quynh, On semiperfect F-injective rings, *Int. Electronic J. Algebra* **1** (2007) 18–29.
[10] L. Shen and J. Chen, New characterizations of quasi-Frobenius rings, *Comm. Algebra* **34** (2006) 2157–2165.
[11] G. Shin, Prime ideal and sheaf representation of a pseudo symmetric rings, *Trans. American Math. Soc.* **84** (1973) 43–60.
[12] R. Wisbauer, *Foundations of Module and Ring Theory* (Gordon and Breach, 1991).
[13] M. F. Yousif and Y. Q. Zhou, FP-injective, simple-injective and quasi-Frobenius rings, *Comm. Algebra* **32** (2004) 2273–2285.