Representations of Polyzetas on Transcendence Bases of Shuffle and Stuffle Algebras

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ABSTRACT

By an effective construction of pairs of bases in duality in shuffle and quasi-shuffle algebras, we identify the local coordinates of noncommutative generating series of polyzetas. Algorithms lead to the ideal of homogeneous polynomial relations, in weight, among polyzetas and their explicit representation on irreducible elements.

INTRODUCTION

For any composition, $(s_1,\dots,s_r)$, of positive integers such that $s_1>1$, the polyzetas (also called multiple zeta values), denoted by $\zeta(s_1,\dots,s_r)$, are defined by the following convergent series

$$\zeta(s_1,\dots,s_r) := \sum_{n_1>\dots>n_r>0} \frac{1}{n_1^{s_1}\dots n_r^{s_r}}.$$

ALGEBRAS ON WORDS

Let $Y=\{y_s\}_{s\geq 1}$ be the alphabet with in the order $y_1>y_2>\cdots$, we define the $q-$stuffle product ($q\in\mathbb{Q}$) by induction: $u \uplus_q 1_{Y^*} = 1_{Y^*} \uplus_q u = u$ and for $y_s, y_t \in Y,\ u,v\in Y^*$

$$y_s u \uplus_q y_t v = y_s(u \uplus_q y_t v) + y_t(y_s u \uplus_q v) + q y_{s+t}(u \uplus_q v)$$

This product, together with concatenation product, establish pairs of Hopf algebras in duality [1] ($\mathcal{A} := \mathbb{Q}\langle Y\rangle$)

$$(\mathcal{A}, conc, 1_{Y^*}, \Delta_{\uplus_q}, \varepsilon, \mathcal{S}_q) \leftrightarrows (\mathcal{A}, \uplus_q, 1_{Y^*}, \Delta_{conc}, \varepsilon, \mathcal{S}_q)$$

in which there are respectively two transcendence bases $\{\Pi_l^{(q)}\}_{l\in\mathcal{L}ynY}$ and $\{\Sigma_l^{(q)}\}_{l\in\mathcal{L}ynY}$.

Example.

$$\Pi_{y_2}^{(q)} = y_2 - \tfrac{1}{2}q y_1^2,\ \Pi_{y_3}^{(q)} = y_3 - \tfrac{1}{2}q y_1 y_2 - \tfrac{1}{2}q y_2 y_1 + \tfrac{1}{3}q^2 y_1^3$$
$$\Sigma_{y_2}^{(q)} = y_2,\ \Sigma_{y_3}^{(q)} = y_3,\ \Sigma_{y_2y_1}^{(q)} = y_2y_1 + \tfrac{1}{2}q y_3.$$

ALGORITHM TO REPRESENT A POLYNOMIAL ON THE BASES [2]

INPUT: A homogeneous polynomial P of weight n.
OUTPUT: The representation of P on the basis $\{\Sigma_w^{(q)}\}_{w\in Y^*}$.
Step 1. We choose the leading term of P, assumed $\lambda_1 w_1$, and express $w_1 = \Sigma_{w_1}^{(q)} + \sum_{v<w_1,(v)=n}\alpha_v v$. Then $P = \lambda_{w_1}\Sigma_{w_1}^{(q)} + \sum_{v<w_1,(v)=n}\beta_v v$.
Step 2. We repeat **Step 1**, P now understood as the polynomial $\sum_{v<w_1,(v)=n}\beta_v v$, and so on until the last monomial which admits the smallest word of weight n, y_n, and we really have $y_n = \Sigma_{y_n}^{(q)}$. At last, by re-putting the coefficients, we will obtain the representation of the original

$$P = \sum_{v\leq w_1,(v)=n} \lambda_v \Sigma_v^{(q)}.$$

Example.
With $P := 2y_1y_2 - \frac{1}{2}y_3$, *(corresponding $n=3, q=1$)*

$$\begin{aligned} P &= 2\Sigma_{y_1y_2} - 2\Sigma_{y_2y_1} - \frac{3}{2}\Sigma_{y_3} \\ &= -1/2\Pi_{y_3} - 1/4\Pi_{y_2y_1} + 3/2\Pi_{y_1y_2} + 11/12\Pi_{y_1^3}. \end{aligned}$$

ALGORITHM TO REPRESENT STRUCTURE OF POLYZETAS [2]

INPUT: A positive integer n.
OUTPUT: Representations of polyzetas of weight n in terms of irreducible elements of polyzetas on the transcendence basis $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$ (corresponding $q=1$).
Step 1. We set the list, denoted X_n, of all words of weight n of X^*x_1.
Step 2. For each $w\in X_n$, we take polyzetas at the polynomial $\mathcal{P} := \pi_Y(S_w)$ and thanks to *Algorithm* 1 we represent $\zeta(\mathcal{P})$ in terms of polyzetas of $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$, called $\zeta_\Sigma(\mathcal{P})$. We proceed to establish polynomial relations of weight n:

i) If $w\in\mathcal{L}ynX$ then we store the representation $\zeta_\Sigma(\mathcal{P})$ to the variable $\zeta(S_w)$,

ii) If $w = x_1u, u\in x_0X^*x_1$ then we make the relation $\zeta_\Sigma(\mathcal{P})$ and set a relation $\zeta(\mathcal{P}_\Sigma)=0$.

iii) If $w\in x_0X^*x_1 \setminus \mathcal{L}ynX$, we rewrite w in its form of Lyndon factorization, $w = l_1^{i_1}\dots l_k^{i_k}$, then take $\zeta(S_{l_j}), j=1..k$ from data and make the relation

$$\frac{1}{i_1!\dots i_k!}\zeta(S_{l_1})^{i_1}\dots\zeta(S_{l_k})^{i_k} = \zeta_\Sigma(\mathcal{P})$$

Step 3. Eliminating all these relations to reduce representations of polyzetas in terms of irreducible elements.

The following results are obtained by program [3] on Maple

	on $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$		on $\{S_l\}_{l\in\mathcal{L}ynX}$	
3	$\zeta(\Sigma_{y_2y_1})$	$=\frac{3}{2}\zeta(\Sigma_{y_3})$	$\zeta(S_{x_0x_1^2})$	$=\zeta(S_{x_0^2x_1})$
4	$\zeta(\Sigma_{y_4})$	$=\frac{2}{5}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^3x_1})$	$=\frac{2}{5}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_3y_1})$	$=\frac{3}{10}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^2x_1^2})$	$=\frac{1}{10}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_2y_1^2})$	$=\frac{2}{3}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0x_1^3})$	$=\frac{2}{5}\zeta(S_{x_0x_1})^2$
5	$\zeta(\Sigma_{y_3y_2})$	$=3\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2})-5\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^3x_1^2})$	$=-\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})+2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_4y_1})$	$=-\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2})+\frac{5}{2}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1x_0x_1})$	$=-\frac{3}{2}\zeta(S_{x_0^4x_1})+\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})$
	$\zeta(\Sigma_{y_2^2y_1})$	$=\frac{3}{2}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2})-\frac{25}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1^3})$	$=-\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})+2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_3y_1^2})$	$=\frac{5}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1x_0x_1^2})$	$=\frac{1}{2}\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_2y_1^3})$	$=\frac{1}{4}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2})+\frac{5}{4}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1^4})$	$=\zeta(S_{x_0^4x_1})$

Let's denote Z_n the vector space generated by polyzetas of weight n. These results also verify the Zagier's dimension conjecture: $d_1=0, d_2=d_3=1, d_n=d_{n-2}+d_{n-3}, \forall n\geq 4$.

RELATIONS AMONG POLYZETAS

By considering one more alphabet $X=\{x_0,x_1\}$, $x_0<x_1$ and encoding compositions formed $(s_1,\dots,s_k)$ to words $x_0^{s_1-1}x_1\dots x_0^{s_k-1}x_1 \rightleftarrows_{\pi_X}^{\pi_Y} y_{s_1}\dots y_{s_k}$, we construct polyzetas as morphisms on shuffle and stuffle algebras, respectively, and establish a relation between two non commutative generating series[a]

$$\prod_{\substack{l\in\mathcal{L}ynY\\ l\neq y_1}}^{\searrow} \exp(\zeta(\Sigma_l)\,\Pi_l) = B'(y_1)\pi_Y \prod_{\substack{l\in\mathcal{L}ynX\\ l\notin X}}^{\searrow} \exp(\zeta(S_l)\,P_l)\ (\star)$$

where $B'(y_1) := \exp\left(\sum_{k\geq 2}\frac{(-1)^{k-1}}{k}\zeta(k)y_1^k\right)$.

Proposition 1. *i) For any $v\in Y^*\setminus y_1Y^*$, $\zeta(\Sigma_v) = \zeta(\pi_X\Sigma_v)$.*
ii) For any $v = y_1^k w\in Y^*, k\geq 1, w\in Y^*\setminus y_1Y^*$, $\zeta_{\sqcup\!\sqcup}(\pi_X\Sigma_v) + \sum_{m=2}^k B^{(m)}\zeta_{\sqcup\!\sqcup}(\pi_X\Sigma_{y_1^{k-m}w}) = 0$, where $B^{(m)} := \langle B'(y_1)|y_1^m\rangle$, $m\geq 0$.

Proposition 2. *i) For any $u\in X^*\setminus x_1X^*$, $\zeta(S_u) = \zeta(\pi_Y S_u)$.*
ii) For any $u\in x_1X^*\setminus x_1^2X^*$, $\zeta_{\uplus}(\pi_Y S_u) = 0$.
iii) For any $u = x_1^k w\in X^*, k\geq 2, w\in X^*\setminus x_1X^*$, $B^{(k)}\zeta(S_w) = \zeta_{\uplus}(\pi_Y S_u)$.

Example. With $v = y_1^2y_2$:
$\zeta(S_{x_0x_1^3}) - \zeta(S_{x_0^2x_1^2}) + \frac{1}{2}\zeta(S_{x_0^3x_1}) + B^{(2)} = 0$.
With $u = x_0x_1x_0^2x_1$: $\zeta(S_{x_0x_1x_0^2x_1}) = \zeta(\Sigma_{y_2y_3}) + 2\zeta(\Sigma_{y_3x_2}) + 6\zeta(\Sigma_{y_4x_1}) - 5\zeta(\Sigma_{y_5})$

[a] $\{P_w\}_{w\in X^*}$ denotes the Poincaré-Birkhoff-Witt basis of $(\mathbb{Q}\langle X\rangle, conc, 1_{X^*})$ and , $\{S_w\}_{w\in X^*}$ denotes the Schützenberger basis of $(\mathbb{Q}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*})$.

CONCLUSIONS

In the classical theory of (finite-dimensional) Lie groups, every ordered basis of the Lie algebra provides a system of local coordinates of a suitable neighborhood of unity via an ordered product of one-parameter groups corresponding to the (ordered) basis. Here, we get a perfect analogue of this geometrical picture for the Hausdorff groups (in shuffle and stuffle Hopf algebras) through Schützengerger's factorization, this doesn't depend on the regularization of shuffle and quasi-shuffle. Moreover, through the bridge equation ($\star$) which relates two elements on these groups and an identification of local coordinates of their L.H.S. and R.H.S. which involve only convergent polyzetas as local coordinates, we get a further trust on Zagier's conjecture, up to weight 12, which is not due to the regularized double-shuffle relations.

REFERENCES

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